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ON THE SEARCH FOR BOREL 1 SELECTIONS

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If Φ is a function, called a *multifunction*, from a given topological space X into the space of all non-void subsets of a given topological space Y, then a *selection* for Φ is any function f from X into Y such that $f(x) \in \Phi(x)$ for all $x \in X$. The problem of finding "nice" selections for "nice" multifunctions is an old one and has been extensively studied. However, most of the work has been devoted to finding selections which are measurable with respect to some measure (see Wagner [11] and [12]).

The purpose of this paper is to continue the investigation of the conditions under which a given multifunction admits a Borel 1 selection. We will not only collate the known results for the first time but also present some new ones as well as pose many interesting and challenging open problems.

Our investigation will also, of necessity, cover the search for continuous selections and Borel 2 selections. It will be seen that continuous selections exist only under stringent conditions while Borel 2 selections exist under relatively relaxed conditions. The problem of finding Borel 1 selections occupies an intermediate stage and thus is a fruitful source of many interesting results as well as further problems.

We will focus on the conditions under which a multifunction Φ admits a Borel 1 selection in terms of the topological nature of the values $\Phi(x)$, of the graph of Φ and of the inverse image of open sets under Φ. For the most part we will assume that both X and Y are metric spaces. Thus, we will not be concerned with the problem of ascertaining the minimal topological conditions (weaker than metricity) of X and Y under which a certain kind of multifunction admits a Borel 1 selection (as is done, for example, in Čoban [3] and [4]). We will particularly emphasize simple spaces such as Polish spaces and specifically $\mathbf{R}^n$.

NOTATION AND TERMINOLOGY

We will denote the set of all non-empty subsets of a space Y by 2^Y. By $\mathscr{G}(Y)$, $\mathscr{F}(Y)$, $\mathscr{K}(Y)$ and $\mathscr{A}(Y)$ we mean respectively the classes of all open, closed, convex and ambiguous (i.e., both F_σ and G_δ) members of 2^Y. We will occasionally write $\mathscr{D}$ instead of $\mathscr{D}(Y)$ e.t.c. when there is no danger of consfusion. By $\mathbf{R}$, $\mathbf{N}$, and $\mathbf{Q}$ we mean the set of all real, irrational and rational numbers, respectively. By Π_x and Π_y we mean the projections of $X \times Y$ onto X and Y, respectively.

If Φ is a multifunction from X into 2^Y, written $\Phi: X \to 2^Y$, we say that Φ is *lower semi-continuous* or l. s. c. (*upper semi-continuous* or u. s. c.) if for each open (closed) subset V of Y the set

$$\Phi^{-1}(V) = \{x: \Phi(x) \cap V \neq \emptyset\}$$

is open (closed, respectively) in X. If we impose the condition that the set $\Phi^{-1}(V)$ be of Borel additive class α (multiplicative class α) then we say that Φ is *lower semi-continuous of class* α or l. s. c.(α) (upper semi-continuous of class α or u. s. c.(α), respectively).

Notice that when $\Phi(x) = \{f(x)\}$ for a function $f: X \to Y$ the definitions of l. s. c.(α) or u. s. c (α) and l. c. s. or u. s. c. reduce to the definitions of a Borel α and continuous function, respectively. When each open set in Y (or X) is F_σ then it is easy to see that l. s. c. (1) is implied by u. s. c. (or l. s. c.). A similar assertion is valid for higher classes.

If $\Phi: X \to 2^Y$, then the *graph* of Φ is the set

$$\operatorname{gr} \Phi = \{(x, y): y \in \Phi(x), x \in X\}$$

in the topological space $X \times Y$. It is easily verified that a closed graph with Y compact implies u. s. c. and, moreover, a u. s. c. $\Phi: X \to \mathscr{F}(Y)$ has a closed graph if Y is regular.

1. CONTINUOUS SELECTIONS

Rather stringent conditions must be imposed upon a multifunction to admit a continuous selection. The most significant result in this regard is the following.

Theorem 1 (Michael [10]). *Let $\Phi: X \to \mathscr{K}(\mathbf{R}^n)$. If X is perfectly normal and Φ is l. s. c., then Φ has a continuous selection.*

The following seven examples indicate that this result is the "best" possible in the sense that none of the hypotheses can be weakened in any "nice" way. (In general, we will omit the proofs that the defined multifunction actually satisfies the asserted, conditions.)

Example 1 (Michael [10]). *There exists a l. s. c. and u. s. c. Φ: $[0, 1] \to \mathscr{F}(\mathbf{R}^2)$ with compact graph and arcwise connected values which has no continuous selection.*

Proof. Let $A = \overline{\{(x, \sin 1/x): x \neq 0\}} \cup (\{0\} \times [-1, 1])$ Define $\Phi(0) = \{0\} \times \times [-1, 1]$ and for $x \neq 0$ let $\Phi(x)$ be the closed arc of A between x and $x/2$.

In the next two examples, for a given space S, $l_1(S)$ denotes the set of all functions y from S into $\mathbf{R}$ such that $\|y\| = \sum_{x \in S} |y(x)| < \infty$.

Example 2 (Michael [10]). *There exists a l. s. c. Φ: $[0, 1] \to \mathscr{F}\mathscr{K}(Y)$, where Y is a normed linear space, which admits no continuous selection.*

Proof. Let $\{z_n\}_{n=1}^{\infty}$ be an enumeration of the set $\mathbf{Z} = \mathbf{Q} \cap [0, 1]$. Let $Y = \{y \in l_1(\mathbf{Z})$: $\{x: y(x) \neq 0\}$ is finite$\}$ and $C = \{y \in Y$: $y(x) \geqq 0$ for $x \in \mathbf{Z}\}$. Define $\Phi(x) = C$ if $x \notin \mathbf{Z}$ and $\Phi(z_n) = C \cap \{y \in Y$: $y(z_n) \geqq 1/n\}$.

Example 3 (Michael [10]). *There exists a l. s. c. Φ: $[0, 1] \to \mathscr{G}\mathscr{K}(l_1[0, 1])$ having no continuous selection.*

Proof. Put $\Phi(x) = \{y \in l_1[0, 1]: y(x) > 0\}$.

Example 4. *There exists an u. s. c. Φ: $[0, 1] \to \mathscr{F}\mathscr{K}[0, 1]$ with a compact graph and disjoint values but with no continuous selection.*

Proof. Let f be the Cantor function mapping the Cantor set continuously onto $[0, 1]$. Let g be the linear extension of f and $\Phi(x) = g^{-1}(x)$.

Example 5. *There exists a l. s. c. Φ: $\mathbf{R} \to \mathscr{G}(\mathbf{R})$ having an open graph but no continuous selection.*

Proof. Define

$$\Phi(x) = \begin{cases} (0, 1) & \text{if } x \geqq 1\,, \\ (2, 3) & \text{if } x \leqq 0\,, \\ (0, 1) \cup (2, 3) & \text{if } 0 < x < 1\,. \end{cases}$$

Example 6. *There exists a l. s. c. Φ: $\mathbf{R} \to \mathscr{F}[0, 1]$ having an ambiguous and arcwise connected graph but no continuous selection.*

Proof. Put $\Phi(x) = \{y: (x, y) \in A\}$ where $A = \{(x, y): x + y = 1, 0 \leqq x \leqq 1\} \cup \{(x, y): y = 1, x > -1\} \cup \{(x, y): y = 0, x < 2\}$.

The last example was pointed out to us by G. Domenichini.

Example 7. *There exists a l. s. c. Φ: $\mathbf{R} \to \mathscr{F}(\mathbf{Q})$ with a closed graph but with no continuous selection.*

Proof. Define $\Phi(x) = \{y \in \mathbf{Q}: x \leqq y \leqq x + 1\}$. Obviously gr$\Phi$ is closed since it is homeomorphic to $\mathbf{R} \times (\mathbf{Q} \cap [0, 1])$. Since any continuous function from $\mathbf{R}$ into $\mathbf{Q}$ must be constant, there is no continuous selection.

It follows from the results of the next section that multifunctions satisfying the given conditions of Examples 1, 4, 5, 6 and 7 do have Borel 1 selections. The question remains open for Examples 2 and 3. (See Question 7.)

Examples 1 and 6 show that a l. s. c. multifunction Φ having closed values does not necessarily have a continuous selection even under strong conditions on Y and Φ.

On the other hand if Φ: $[0, 1] \to \mathscr{F}[0, 1]$ is l. s. c. and has a closed graph then it does have a continuous selection by Proposition 1 below. This suggests the following

Question 1. *Can a meaningful characterization be found for those l. s. c. multifunctions from* $\mathbf{R}$ *to* $\mathscr{F}(\mathbf{R})$ *which have a continuous selection?*

In the case when the values of Φ are convex, then there is one such "meaningful" characterization (see Ceder [1]).

Example 1 gives a Φ: $\mathbf{R} \to \mathscr{F}(\mathbf{R}^2)$ which is both l. s. c. and u. s. c. and has a compact graph, yet has no continuous selection. Curiously, this cannot happen if the range space is $\mathbf{R}$ as shown by

Proposition 1. *Let* Φ: $\mathbf{R} \to 2^R$. *If* Φ *is l. s. c. and u. s. c. (resp. l. s. c.* (α) *and u. s. c.* (α)*) and each* $\Phi(x)$ *is compact, then* Φ *has a continuous or a Borel* α *selection, respectively.*

Proof. Put $f(x) = \sup \Phi(x)$ and use $f^{-1}(a, b) = \Phi^{-1}((a, b)) - \Phi^{-1}([b, \infty))$.

Corollary 1. *Let* Φ: $\mathbf{R} \to \mathscr{F}[0, 1]$. *If* Φ *is l. s. c. and has a closed graph, then* Φ *has a continuous selection.*

Proof. It is easy to show that Φ is u. s. c. Then apply Proposition 1.

2. BOREL 1 SELECTIONS

Since either u. s. c. or l. s. c. implies l. s. c. (1) let us first consider multifunctions which are l. s. c.
(1). We begin with the following general result:

Theorem 2 (Debs [5]). *Let* $\Phi: X \to 2^Y$, *where* Y *is a Polish space. If*

(1) $\mathrm{gr}\Phi \in (\mathscr{A} \times \mathscr{G})_{\sigma\delta}$ *and*

(2) Φ *is l. s. c.* (1)

then Φ *has a Borel* 1 *selection.*

Corollary 2. *If* $\mathrm{gr}\Phi$ *is open, then* Φ *has a Borel* 1 *selection.*

Corollary 3. (Debs [5]). *If* Φ *is l. s. c.* (1) *and* $\mathrm{gr}\Phi$ *is* G_δ, *then* Φ *has a Borel* 1 *selection.*

Note that the continuous analogues of these corollaries are not valid as seen from Examples 5 and 6.

The next two examples show that neither hypothesis in Corollary 3 can be dispensed with.

Example 8. *There exists a l. s. c. Φ: $\mathbf{R} \to \mathscr{G}[0, 1]$ having a non-Borelian graph and having no Borel selection.*

Proof. Since the class of all Borel functions from $\mathbf{R}$ to $[0, 1]$ has the same cardinality as $\mathbf{R}$ it is clear that we can choose a function f: $\mathbf{R} \to [0, 1]$ such that f intersects each Borel function from $\mathbf{R}$ to $[0, 1]$. Put $\Phi(x) = [0, 1] - \{f(x)\}$. The graph of Φ is not Borel by Corollary 5 of Debs [5].

Example 9. *There exists a l. s. c.* (2) *and u. s. c.* (1) *Φ: $\mathbf{R} \to \mathscr{G}[0, 1]$ which has a G_δ graph but no Borel* 1 *selection.*

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base of open intervals for $[0, 1]$. Let $\{A_n\}_{n=1}^{\infty}$ be any collection of disjoint, countable, dense subsets of $\mathbf{R}$. Define $\Phi(x) = B_n$ if $x \in A_n$ and $\Phi(x) = [0, 1]$ otherwise. Clearly any selection for Φ must be discontinuous everywhere so that Φ has no Borel 1 selection. Moreover, it is easy to check that Φ is both l. s. c. (2) and u. s. c. (1).

Example 8 shows that a l. s. c. multifunction with open values may fail to have a Borel graph. This is in marked contrast with the case of the u. s. c. multifunctions: it is proved in [7] that a u. s. c. multifunction has a Borel α graph ($\alpha \geqq 2$) if and only if each of its values is a Borel subset of class α. However, if the values are closed a l. s. c. multifunction does have a Borel graph as provided by the following

Proposition 2. *Let Φ: $X \to \mathscr{F}(Y)$ where Y is separable. If Φ is l. s. c. or l. s. c.* (1), *then* grΦ *is G_δ or $F_{\sigma\delta}$, respectively.*

Proof. Let $\{G_n\}_{n=1}^{\infty}$ be a countable base for Y. Then

$$\mathrm{gr}\Phi = (X \times Y) - \bigcap_{n=1}^{\infty} [(X - \Phi^{-1}(G_n)) \times G_n],$$

which implies the conclusion.

Next, we have another general result:

Theorem 3 (Kuratowski, Ryll-Nardzewski [9]). *Let Φ: $X \to \mathscr{F}(Y)$, where X is a metric space and Y is a Polish space. If Φ is l. s. c.* (1), *then Φ has a Borel* 1 *selection.*

The above theorem together with Example 8, Corollary 3 and Proposition 2 suggest several questions, the first of which is

Question 2. *Let Φ: $X \to 2^Y$, where X is metric and Y Polish. If Φ is l. s. c.* (1) *and* grΦ *is $F_{\sigma\delta}$ (or $G_{\delta\sigma}$), does Φ have a Borel* 1 *selection?*

A related question, an affirmative answer to which would give a negative answer to Question 2 in the $G_{\delta\sigma}$ case, is

Question 3. *Does there exist a Borel* 2 *function from $\mathbf{R}$ into $\mathbf{R}$ which intersects each Borel* 1 *function from $\mathbf{R}$ into $\mathbf{R}$?*

An affirmative answer to Question 3 is plausible because there exists a Borel 1 function intersecting each continuous function from $\mathbf{R}$ into $\mathbf{R}$, as shown by simple examples.

Another interesting question is whether or not the "Polish"-ness of Y is essential in Theorem 3. Specifically,

Question 4. *Do there exist metric spaces X and Y and a l. s. c. (1) $\Phi: X \to \mathscr{F}(Y)$ which has no Borel 1 selection?*

If the graph of Φ is closed there may or need not exist a Borel 1 selection depending on the "degree of compactness" present as shown by the next example and theorem.

Example 10. *There exists a $\Phi: [0, 1] \to \mathscr{F}(N)$ having a closed graph but no Borel 1 selection.*

Proof. Since each analytic set in $[0, 1] = I$ is a projection of some closed set in $I \times \mathbf{N}$ we can find a closed set $F_1 \subseteq I \times (\mathbf{N} \cap I)$ and a closed set $F_2 \subseteq I \times \times (\mathbf{N} \cap [2, 3])$ whose respective x-projections are $\mathbf{N} \cap I$ and $I - \mathbf{N}$. Now put $\Phi(x) = \{y: (x, y) \in F_1 \cup F_2\}$. Clearly any selection for Φ is discontinuous everywhere and moreover, grΦ is closed, being equal to $F_1 \cup F_2$.

Example 10 can be strengthened to assert that no Borel selection exists (see Example 11).

Theorem 4. *Let $\Phi: X \to 2^Y$ where X and Y are metric spaces. If Y is σ-compact and* grϕ *is F_σ, then Φ is l. s. c. (1) and Φ has a Borel 1 selection.*

Proof. Clearly we can express grΦ as $\bigcap_{n=1}^{\infty} F_n$ where $\{F_n\}_{n=1}^{\infty}$ is an ascending sequence of closed sets such that $Y_n = \operatorname{cl} \Pi_y(F_n)$ is compact and thus, a Polish space. It is easily verified that each $X_n = \Pi_x(F_n)$ is closed. Define Φ_n in X_n by $\Phi_n(x) = \{y: (x, y) \in \in F_n\}$. Then gr$\Phi_n = F_n$ and $\Phi_n: X_n \to \mathscr{F}(Y_n)$.

Since a multifunction with a closed graph and values in a compact space is u.s.c. each Φ_n is u. s. c. and, hence, l. s. c. (1). Let V be any set in Y. Then $V = \bigcup_{n=1}^{\infty} V_n$ where $\{V_n\}_{n=1}^{\infty}$ is an ascending sequence of closed sets with $V_n \subseteq Y_n$. Since $\Phi(x) = = \bigcup_{n=1}^{\infty} \Phi_m(x)$ we have $\Phi^{-1}(V) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \Phi_m^{-1}(V_n)$. Since each $\Phi_m^{-1}(V_n)$ is closed in the closed subspace X_m it follows that $\Phi^{-1}(V)$ is F_σ in X. Hence, Φ is l. s. c.(1).

Applying Theorem 3 to $\Phi_n: X_n \to \mathscr{F}(Y_n)$ we obtain a Borel 1 selection h_n for Φ_n. Putting $A_n = X_n - \bigcup_{k<n} X_k$ and $h = \bigcup_{n=1}^{\infty} (h_n \urcorner A_n)$ we obtain h as a Borel 1 selection for Φ.

Corollary 4. *Let $\Phi: X \to 2^Y$ where X and Y are metric spaces. If* grΦ *is a K_σ set, then Φ is l. s. c. (1) and Φ has a Borel 1 selection.*

Corollary 5. *Let $\Phi: X \to 2^Y$ where X and Y are metric spaces. If Y is countable and* grΦ *is an F_σ, then Φ is l. s. c. (1) and Φ has a Borel 1 selection.*

It would be interesting to know if we can weaken Theorem 4 by replacing the σ-compactness of Y by the condition that each $\Phi(x)$ is σ-compact. Hence, we have the following question, open even in the case where grΦ is closed.

Question 5. *Let $\Phi: X \to 2^Y$ where X and Y are metric spaces. If each $\Phi(x)$ is σ-compact and* grΦ *is F_σ, does there exist a Borel 1 selection for Φ?*

Next, we have the Borel 1 analogue of Michael's Theorem 1.

Theorem 5. *Let $\Phi: X \to \mathscr{K}(\mathbf{R}^n)$ where X is metric. If Φ is l. s. c. (1), then Φ has a Borel 1 selection.*

Proof. We will prove the theorem for $n = 1$ or 2. The case for general n will then be clear.

Case 1: $n = 1$.

Define $\Phi^*(x) = \overline{\Phi(x)}$. Then Φ^* is also l.s.c. (1) and hence according to [9] has a Borel 1 selection g. Define $\chi(x) = \Phi(x) - g(x)$. Then χ too is l. s. c. (1). It will suffice to find a Borel 1 selection f for χ. Then $f + g$ will be a Borel 1 selection for Φ.

First note that $0 \in \overline{\chi(x)}$ for all x. For n a non-zero integer put $B_n = \{x: 1/n \in \operatorname{int} \chi(x)\}$. Clearly $X - \{x: \chi(x) = \{0\}\} = \bigcup_n B_n$. Also B_n is an F_σ set, say $\bigcup_{m=1}^\infty B_{n,m}$ where each $B_{n,m}$ is closed. Rearranging $\{B_{n,m}: m \geqq 1, n \neq 0\}$ as $\{A_k\}_{k=1}^\infty$ and putting $C_m = A_m - \bigcup_{k=1}^{m-1} A_k$ we obtain a sequence $\{C_m\}_{m=1}^\infty$ of disjoint ambiguous sets whose union is $\bigcup_{m=1}^\infty \bigcup_{n \neq 0} B_{n,m}$. Moreover, for each k there exists $n(k)$ such that $C_k \subseteq B_{n(k)}$.

Now choose a sequence $\{r_m\}_{m=1}^\infty$ tending to 0 such that $0 < r_m\, n(m) < 1$. Define

$$f(x) = \begin{cases} r_m & \text{if } \ x \in C_m\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Since $\bigcup_{m=1}^\infty C_m = X - \{x: \chi(x) = \{0\}\}$, f is a selection for χ. Now let V be open. In case $0 \notin V$ put $A = \{m: r_m \in V\}$. Then $f^{-1}(V) = \bigcup\{C_m: m \in A\}$ which is F_σ. In case $0 \in V$, $B = \{m: r_m \notin V\}$ is finite and $f^{-1}(V) = X - \bigcup\{C_m: m \in B\}$ which is again F_σ. Hence, f is a Borel 1 selection for χ.

Case 2. $n = 2$.

As in Case 1 let g be a Borel 1 selection for Φ^* and put $\chi(x) = \Phi(x) - g(x)$ so that $0 \in \overline{\chi(x)}$ for all x. Again it suffices to find a Borel 1 selection for χ.

Let $\{I_n\}_{n=1}^\infty$ be a countable base of open arcs on the unit circle. Define $W_n = \{\lambda x: x \in I_n, \lambda > 0\}$. Let $T_{n,m}$ be the closed triangle having two of its sides contained in the boundary of W_n and its third side having distance $1/m$ from 0.

Let $L_{n,m}$ denote the interior of that side of $T_{n,m}$ which misses 0 Put $A_{n,m} = W_n - T_{nm}$.

Then A_{nm} is open and $B_{n,m} = \{x: \chi(x) \cap A_{n,m} \neq \emptyset\}$ is an F_σ set. Moreover, $X - \{x: \chi(x) = \{0\}\} = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty B_{nm}$. As in Case 1 we can find a sequence of disjoint ambiguous sets $\{C_k\}_{k=1}^\infty$ such that $\bigcup_{k=1}^\infty C_k = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty B_{nm}$ and such that for each k there exists $n(k)$ and $m(k)$ such that $C_k \subseteq B_{n(k),m(k)}$. Note that $B_{n,m} \subseteq B_{n,l}$ whenever $l \geqq m$. Pick a monotonic sequence $\{r(k)\}_{n=1}^\infty$ of positive integers approaching $+\infty$ such that $m(k) < r(k)$. Then $C_k \subseteq B_{n(k),r(k)}$.

For each k define

$$\Gamma_k(x) = \begin{cases} \chi(x) \cap L_{n(k),r(k)} & \text{if} \quad x \in C_k\,, \\ L \quad \text{otherwise}\,, \end{cases}$$

where L is the line extending $L_{n(k),r(k)}$. Then it is easy to show that Γ_k is l. s. c. (1) from X into the convex subsets of L. Hence, by Case 1 there is a Borel 1 selection h_k for Γ_k. Now define

$$h(x) = \begin{cases} h_k(x) & \text{if} \quad x \in C_k\,, \\ 0 & \text{if} \quad x \notin \bigcup_{k=1}^\infty C_k\,. \end{cases}$$

Then h is a Borel 1 selection for χ.

Recalling Examples 1, 2, and 3 we may ask whether the range space in Theorem 5 can be generalized. Specifically,

Question 6. *Let $\Phi: X \to \mathscr{K}(Y)$ where X is metric and Y is an infinite dimensional normed linear space. Does Φ have a Borel 1 selection when Φ is l. s. c. (1)?*

Question 7. *Let $\Phi: X \to 2^{\mathbf{R}^n}$ where X is a metric space. Does Φ have a Borel 1 selection when Φ is l. s. c. (1) and each $\Phi(x)$ is an arc?**)

If we require that the values of Φ are closed in either Question 7 or 8 then Theorem 3 yields a Borel 1 selection.

Since any l. s. c. or u. s. c. multifunction is also l. s. c. (1) each of the previous results in this section about l. s. c. (1) multifunctions admitting Borel 1 selections apply also to l. s. c. and u. s. c. multifunctions. However, in this case it may be possible to weaken some of the other hypotheses. The next two results are of this kind.

Theorem 6 (Čoban [3]). *Let $\Phi: X \to \mathscr{F}(Y)$ where Y is a metric space. If Φ is l. s. c., then Φ has a Borel 1 selection if any one of the following conditions is met:*

(1) *each $\Phi(x)$ is compact and X is perfectly normal,*

(2) *Y is complete and X is paracompact and Hausdorff,*

(3) *Φ is u. s. c. and Y is complete.*

*) The answer to Question 7 is negative if we only require each $\Phi(x)$ to be arcwise connected, because, following the construction of Example 8, we can obtain a l.s.c. $\Phi: \mathbf{R}^2 \to \mathscr{G}(\mathbf{R}^2)$ whose values are the plane minus a point.

The above result suggests the question of how essential the completeness of Y is. In particular, does the multifunction of Example 2 have a Borel 1 selection, or as a possibly simpler question, we have

Question 8. *Let Φ: $\mathbf{R} \to \mathscr{F}(\mathbf{Q})$. Does Φ have a Borel* 1 *selection when Φ is l. s. c.?*

While Theorem 6 can be considered as a l. s. c. version of Theorem 3, the next result concerns u. s. c. multifunctions.

Theorem 7. *Let $\Phi: X \to \mathscr{F}(Y)$ where X is perfectly normal and Y is a metric space. Suppose Φ is u. s. c.; then Φ has a Borel* 1 *selection in each of the following two cases:*

1) (Engelking [6], Čoban [4]) *X is paracompact, Y is complete and each $\Phi(x)$ is separable;*

2) Čoban [4]) *each $\Phi(x)$ is compact.*

We conclude this section by estalishing a relationship between l. s. c. (1) and l. s. c. This is the multifunction analogue of the fact that a Borel 1 function from one Polish space to another is continuous on a residual G_δ set.

Theorem 8. *Let $\Phi: X \to 2^Y$ where X and Y are Polish spaces. Then*

1) *if Φ is l. s. c.* (1) *there exists a residual G_δ set B such that Φ is l. s. c. at each point of B;*

2) *if* grΦ *is analytic there exists a residual G_δ set D such that $\Phi \mid D$ is l. s. c.*

Proof. Case 1) is a direct translation of the proof that a Borel 1 function from X into Y is continuous on a residual G_δ set (see Kuratowski [8]).

In Case 2), let $\{G_n\}$ be an open base for Y. Since grΦ is analytic, each $\Phi^{-1}(G_n)$ has the Baire property in X. We then conclude the proof as in the case of functions.

In Case 2) in the above result, if Theorem 2 or 3 can be applied to $\Phi \mid D$, there exists a Borel 1 function h from D into Y which is a selection for $\Phi \mid D$. h can be extended to a Borel 1 function on all of X, but, as shown by Example 10, this extension need not be a selection for Φ.

3. BOREL α SELECTIONS, $\alpha \geqq 2$

There are many theorems in the selection theory concluding that there is a Borel measurable selection (see the survey papers of Wagner [11] and [12]). If Y is separable then any such Borel measurable selection becomes a Borel α measurable function for an unspecified α. For example, it should be noted that if X and Y are Polish spaces, the multifunction in question 5 admits a Borel selection, by a theorem of Ščegolkov. In this section we will deal only with results for a specific α.

We begin by citing the following deep theorem:

Theorem 9 (Jayne and Rogers [7]). *Let Φ: $X \to 2^Y$ where X and Y are metric spaces. If Φ is u. s. c., then Φ has a Borel 2 selection.*

It is then natural to ask the following

Question 9. *Does Φ in Theorem 9 admit a Borel 1 selection?*

Theorem 10 (Debs [5]). *Let Φ: $X \to 2^Y$ where Y is Polish. Let $\mathscr{A}_\alpha$ be the collection of all ambiguous sets of class α. If*

(1) $\mathrm{gr}\Phi \in (\mathscr{A}_\alpha \times \mathscr{G})_{\sigma\delta}$ *and*

(2) Φ *is l. s. c.*(α)

then has a Φ Borel α selection.

Corollary 6 (Debs [5]). *Let Φ: $X \to 2^Y$ where Y is Polish. If Φ is l. s. c. (2) and has a G_δ graph, then Φ has a Borel 2 selection.*

It is unknown whether one can weaken the above G_δ requirement.

Specifically,

Question 10. *Let Φ: $X \to 2^Y$ where Y is Polish. Does there exist a Borel 2 selection when* $\mathrm{gr}\Phi$ *is $F_{\sigma\delta}$?*

Theorem 11 (Kuratowski, Ryll-Nardzewski [9]). *Let Φ: $X \to \mathscr{F}(Y)$ where Y is Polish and X metric. If Φ is l s. c. (2), then Φ has a Borel 2 selection.*

Theorem 12. *Let Φ: $X \to \mathscr{K}(\mathbf{R}^n)$ where X is metric. If Φ is l. s. c. (2), then Φ has a Borel 2 selection.*

Proof. An inspection of the proof of Theorem 6 shows that the proof can be generalized to apply to $\alpha \geqq 2$ by using Theorem 11 instead of Theorem 3.

As examples of multifunctions having no Borel 2 selection for a specific $\alpha \geqq 2$ or for a general $\alpha \geqq 2$ we have Example 8 and the stronger version of Example 10, namely Example 11 of the next section.

If the answer to any one of Questions 2, 5, 6, 7, 8, 9, is that there is no Borel 1 selection, then the corresponding question may be asked for Borel 2 selections. All these new questions are open as well, with the exception of 9.

Clearly an unlimited number of other questions on Borel α selections for a specific α can be formulated but since the focus of this paper is on Borel 1 selections we will not attempt such a project here.

4. SELECTIONS FOR MULTIFUNCTIONS OF THE FORM f^{-1}

If f is a function from Y *onto* X, then by defining $\Phi(x) = f^{-1}(x)$ we obtain a multifunction Φ, denoted by f^{-1} in this case, from X into 2^Y. Thus, f^{-1} becomes a special multifunction with the additional property that the values $f^{-1}(x)$ are mutually disjoint.

We can therefore translate the previous theorems on selections for multifunctions and obtain interesting results on f^{-1}. The fact that the values of f^{-1} are disjoint will give us, in general, additional information.

Before listing these analogues let us first list the important relationships between f and f^{-1}, all of which have immediate proofs:

(1) *f^{-1} is l. s. c. iff f is open.*

(2) *f^{-1} is u. s. c. iff f is closed.*

(3) *f^{-1} is l. s. c. (1), iff f maps open sets onto F_σ sets.*

(4) $\mathrm{gr}(f^{-1}) = (\mathrm{gr} f)^{-1}$.

(5) *If X is metric and Y is σ-compact metric and f is continuous, then f^{-1} is l. s. c. (1).*

(6) *If f is a Darboux (in particular, an approximately continuous or a derivative) function from $\mathbf{R}$ onto $\mathbf{R}$, then f^{-1} is l. s. c. (1).*

A Darboux function maps intervals onto intervals. Hence, (6) follows from (3).

Now we translate most of the results of Section 2 into results on f^{-1}.

Theorem 13. *Let $f\colon Y \overset{\text{onto}}{\to} X$, where X and Y are metric spaces with Y Polish. Then $f^{-1}\colon X \to 2^Y$ has a Borel 1 selection if any one of the following conditions is met:*

(1) *$\mathrm{gr} f$ is G_δ and f maps open sets onto F_σ sets;*

(2) *f is Borel 1 and f maps open sets onto F_σ sets;*

(3) *f is continuous and Y is locally compact;*

(4) *$X = Y = \mathbf{R}$ and f is a Darboux Baire 1 function;*

(5) *each $f^{-1}(x)$ is closed and f maps open sets onto F_σ sets;*

(6) *$Y = \mathbf{R}^n$, each $f^{-1}(x)$ is convex and f maps open sets onto F_σ sets;*

(7) *f is closed and each $f^{-1}(x)$ is closed.*

If Y is a σ-compact metric space, the same conclusion holds with:

(8) *$\mathrm{gr} f$ is an F_σ set;*

(9) *f is continuous;*

(10) *$Y = \bigcup_{n=1}^{\infty} A_n$ where each A_n is closed and $A_n \cap A_m = \emptyset$ when $n \neq m$ and each $f \urcorner A_n$ is continuous.*

Of these results only (4) and possibly (10) are new. The original results from which these results follow are in order: Cor. 3, Cor. 3, Cor. 3, Cor. 3, Th. 3, Th. 5, Th. 7, Th. 4, Th. 4, and Th. 4, respectively.

We will not state the f^{-1} analogues for the theorems in Sections 1 and 3.

Again it is possible to formulate Questions 2, 5, 6, 7, 9, 10 in terms of multifunctions of the form f^{-1}.

We conclude this section with two examples the first of which shows that each hypothesis of part (1) of Theorem 13 is necessary. The first example is a strengthening of Example 10.

Example 11 (Christensen[2]). *There exists a continuous function from* $\mathbf{N}$ *onto* $[0, 1]$ *such that* f^{-1} *has no Borel selection. Moreover, there exists a Borel* 1 *function* $g: \mathbf{R} \to [0, 1]$ *such that* g^{-1} *has no Borel* 1 *selection.*

Proof. The second statement follows from the first (found in [2]) since a continuous function on a dense G_δ set can be extended to a Borel 1 function on all of $\mathbf{R}$.

The next example shows that there are l. s. c. multifunctions of the form $f^{-1}: \mathbf{R} \to 2^{\mathbf{R}}$ having no Borel selections.

Example 12. *There exists a l. s. c.* $f^{-1}: \mathbf{R} \to 2^{\mathbf{R}}$ *which has no Borel selection.*

Proof. Let $\{B_\alpha\}_{\alpha<c}$ be a well-ordering of all uncountable Borel subsets B of $\mathbf{R}^2$ such that each horizontal line hits B at most once and $\Pi_x B$ has cardinality c. It suffices to construct an open function $f: \mathbf{R} \overset{\text{onto}}{\to} \mathbf{R}$ which does not contain any B_α.

Let $\mathcal{O}$ consist of all open intervals in $\mathbf{R}$. Let $\{(\mathcal{O}_\alpha, r_\alpha)\}_{\alpha<c}$ be a well-ordering of $\mathcal{O} \times \mathbf{R}$. By transfinite induction suppose we have chosen for each $\alpha < \beta$ points z_α and w_α in the plane with the following properties:

$$z_\alpha \in B_\alpha \qquad w_\alpha \in \mathcal{O}_\alpha \times \{r_\alpha\} ;$$

$$w_\alpha \neq z_\xi \quad \text{for} \quad \alpha < \xi < \beta ;$$

$$1^{\text{st}} \text{ coord } z_\alpha \neq 1^{\text{st}} \text{ coord } z_\xi \quad \text{for} \quad \alpha < \xi < \beta ;$$

$$1^{\text{st}} \text{ coord } \omega_\alpha \neq 1^{\text{st}} \text{ coord } w_\xi \quad \text{for} \quad \alpha < \xi < \beta .$$

Now we proceed to choose z_β and w_β. Pick

$$z_\beta \in B_\beta - \bigcup\{V(w_\alpha) \cup V(z_\alpha): \alpha < \beta\}$$

where $V(x)$ denotes the vertical line through x. The choice of z_β is possible since $\Pi_x B_\beta$ has cardinality c. Next, pick

$$w_\beta \in (\mathcal{O}_\beta \times \{r_\beta\}) - \bigcup\{V(w_\alpha) \cup V(z_\alpha): \alpha < \beta\} - V(z_\beta) .$$

Clearly the inductive hypotheses are satisfied for $\beta + 1$, completing the induction.

Let $Z = \{z_\alpha: \alpha < c\}$ and $W = \{w_\alpha: \alpha < c\}$. Then $W \cap Z = \emptyset$ and no two points of either W or Z lie on the same vertical line. Define f as follows:

$f(x) = y$ if $(x, y) = w_\alpha$ for some α,

$f(x) \in V(x) - Z$ otherwise.

Since $f \cap W$ takes on each real value in each interval we have $f(I) = \mathbf{R}$ for each open interval I. Hence f is open. Since $f \cap Z = \emptyset$ and $Z \cap B_\alpha \neq \emptyset$ for each α, f contains no B_α.

Example 12 can be considered a stronger version of Example 8 in which the values of $\Phi(x)$ are all disjoint (however, not open as in Example 8). In view of Example 12 and part (1) of Theorem 13 we can pose the following

Question 11. *If f is an open Borel 2 function from $\mathbf{R}$ onto $\mathbf{R}$, does f^{-1} have a Borel 1 selection?*

This can be considered to be a simplifed version of Question 3.

5. MISCELLANEOUS

Another condition to impose upon a multifunction for the admission of a "nice" selection is that the multifunction restricted to each set of a family of "small" sets has a "nice" selection. A result of this kind is the following

Theorem 14 (Ceder [1]). *Let $\Phi: \mathbf{R} \to \mathscr{F}\mathscr{K}(\mathbf{R})$. Then Φ has a Borel 1 selection if and only if $\Phi \mid P$ has a Borel 1 selection for each perfect, no-where dense subset P, of $\mathbf{R}$.*

In a subsequent paper we will investigate the possibility of extending this result in various ways.

Addendum: After this paper was completed several of the questions have been answered. Question 3 has an affirmative answer [13]. Question 4 has an affirmative answer when the continuum hypothesis is assumed [14]. Questions 6 and 7 have negative answers even when Φ is l.s.c. [14]. Question 2 has a negative answer [15]. Questions 5 and 10 have negative answers even when grF is closed [15].

References

[1] *J. Ceder:* On Baire 1 selections. Ric. Mat. *30,* 1981, pp. 305 — 315.

[2] *J. P. R. Christensen:* Topology and Borel structure. North Holland Mathematics Studies (10), Amsterdam 1974.

[3] *M. M. Čoban:* Many-valued mappings and Borel sets, I. Trans. Mosc. Math. Soc. *22,* 1970. Transl. Am. Math. Soc. pp. 258—280).

[4] *M. M. Čoban:* Many-valued mappings and Borel sets, II. Trans. Mosc. Math. Soc. *23,* 1970. (Transl. Am. Math. Soc. pp. 286—310).

[5] *K. Debs:* Sélections d'une multi-application à valeurs G_δ. Bull. Cl. Sci. Math. Acad. R. Belg., 1979, pp. 211—216.

[6] *R. Engelking:* Selectors of the first Baire class for semicontinuous set-valued functions. Bull. Acad. Pol. Sci. XVI, 1968 pp. 277—282.

[7] *J. Jayne, C. A. Rogers:* Upper semicontinuous set-valued functions. Acta Math. *149*, 1982, pp. 87 — 125.

[8] *K. Kuratowski:* Topology, vol. I. Academic Press, New-York, 1966.

[9] *K. Kuratowski, C. Ryll-Nardzewski:* A general theorem on selectors. Bull. Acad. Pol. Sci. XIII, 1965, pp. 397—403.

[10] *E. Michael:* Continuous selections, I. Ann. Math., *63*, 1956, pp. 361—382.

[11] *D. H. Wagner:* Survey of measurable selection theorems. SIAM J. Control Optimization, *15*, 1977, pp. 859—903.

[12] *D. H. Wagner:* Survey of measurable selection theorems: an update. Proc. Oberwolfach 1979 conference in measure theory. Lecture notes in Mathematics, n$^\delta$ 794, Springer Verlag.

[13] *Z. Grande:* Sur une fonction de classe 2 de Baire dont le graphe coupe les graphes de toutes fonctions de classe 1. Real Analysis Exchange *8*, 1982—1983, pp. 509—510.

[14] *W. Ślezak:* Some counterexamples in multifunction theory. Real Analysis Exchange *8*, 1987—1983, pp. 494—501.

[15] *W. Ślezak:* Sharpness of some graph conditioned theorems on Borel 1 selectors (to appers).

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