Časopis pro pěstování matematiky

Jaroslav Hylán

Asymptotical properties of the Wronski determinant of a certain class of linear differential equations of the 2nd order

Časopis pro pěstování matematiky, Vol. 110 (1985), No. 1, 13--18

Persistent URL: http://dml.cz/dmlcz/118218

Časopis pro pěstování matematiky, roč. 110 (1985), Praha

ASYMPTOTICAL PROPERTIES OF THE WRONSKI DETERMINANT OF A CERTAIN CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF THE 2ND ORDER

JAROSLAV HYLÁN, Žilina
(Received May 25, 1982)

1. INTRODUCTION

In [1] asymptotical properties of the solution of the following linear differential equation of the 2nd order have been investigated:

$$y'' + [l^r - q(x, l)]\, y = 0\,, \tag{1.1}$$

where $q(x, l) = \sum_{\nu=0}^{r'} a_\nu(x)\, l^\nu$ with real functions $a_\nu(x)$ continuous on the interval $[0, a]$, r is a natural number, $r' \leqq r$ an integer and l a complex variable.

The functions $\varphi(x, l)$ and $\psi(x, l)$ which are the solution of equation (1, 1) satisfying the initial conditions

$$\begin{aligned} &\varphi(0, l) = \alpha_1\,, \quad \varphi'(0, l) = \alpha_2\,, \quad \alpha_1^2 + \alpha_2^2 > 0 \\ &\psi(a, l) = \beta_1\,, \quad \psi'(a, l) = \beta_2\,, \quad \beta_1^2 + \beta_2^2 > 0 \end{aligned} \tag{1.2}$$

have been proved to be entire functions of the complex variable l for every $x \in [0, a]$.

By the substitution $s = l^{r/2} = |l|^{r/2}\, e^{i\alpha}$, $\alpha \in [0, \pi r)$, where $l = |l|\, e^{i\beta}$, $\beta \in [0, 2\pi]$, the differential equation (1, 1) transforms into the differential equation

$$y'' + [s^2 - q(x, l)]\, y = 0\,. \tag{1.3}$$

Denoting $\gamma = 2r'/r < 1$, asymptotic estimates for the functions $\varphi(x, l)$ and $\psi(x, l)$, their derivatives with respect to x and s and their integrals for $|s| = \varrho \to +\infty$ have been established.

In this work, the methods of [1] are extended in order to obtain asymptotical properties of the Wronski determinant of the functions $\varphi(x, l)$ and $\psi(x, l)$.

In this part we quote briefly those results from [1] that will be needed in the sequel:

$$\varphi(x, l) = \alpha_1 \cos(sx) + \alpha_1 s^{-1} \sin(sx) + [\alpha_1 + \alpha_2 \varrho^{-1}]\, \varrho^{\gamma-1}\, e_1(x)\, c_1(x)\,; \tag{1.4}$$

$$\begin{aligned} \frac{\partial \varphi(x, l)}{\partial x} = & -\alpha_1 x \sin(sx) + \alpha_2 x s^{-1} \cos(sx) - \\ & - \alpha_2 s^{-2} \sin(sx) + [\alpha_1 + \alpha_2 \varrho^{-1}]\, \varrho^{\gamma-1}\, e_1(x)\, c_2(x)\,, \end{aligned} \tag{1.5}$$

where $e_1(x) = \exp [x\varrho |\sin \alpha|]$ and $c_i(x)$ (for $i = 1, 2, \ldots$) are functions of the variable x on the interval $[0, a]$, with positive constants c_i independent of $\varrho > 1$, $\alpha \in (-\pi, \pi]$ and $x \in [0, a]$.

2. NOTATION AND PRELIMINARY INFORMATION

The following definitions and notation are used: $k_i(s)$ ($i = 1, 2, \ldots$) denotes a function of the variables $\varrho = |s|$ and $\alpha = \arg s$, i.e. a function of the complex variable s, such that

$$|k_i(s)| < k_i$$

where k_i is a positive constant independent of s. The numbering of these functions is independent in every section and the index i of the function $k_i(s)$ in a relation which is to be proved need not coincide with the index j of the function $k_j(s)$, which is used in the proof of this relation.

3. Theorem. *Let $w(l)$ be the Wronski determinant of the functions $\varphi(x, l)$ and $\psi(x, l)$ which were defined above in Section 1.*

Then

$$w(l) = \alpha_1\beta_1 s \sin(as) + [\alpha_1\beta_2 - \alpha_2\beta_1] \cos(as) + \\ + \alpha_2\beta_2 s^{-1} \sin(as) + [-\alpha_1\beta_1 + (\beta_2\alpha_1 - \beta_1\alpha_2)\varrho^{-1} + \alpha_2\beta_2\varrho^{-2}]\varrho^{\gamma} e_1(a) k_1(s), \tag{3.1}$$

where $e_1(a)$ is defined in Section 2.

Proof. Liouville's formula implies that the Wronski determinant $w(l)$ is indepedent of x (the coefficient at y in equation (1.1) is equal to zero).

Further, an arbitrary number x can be substituted into the functions $\varphi(x, l)$ and $\psi(x, l)$ and their derivatives in the Wronski determinant $w(l)$.

Substituting $x = a$ into $w(l)$ we obtain by virtue of (1.2), (1.4) and (1.5)

$$w(l) = \begin{vmatrix} \varphi(a, l) & \varphi'(a, l) \\ \psi(a, l) & \psi'(a, l) \end{vmatrix} = \begin{vmatrix} \varphi(a, l) & \varphi'(a, l) \\ \beta_1 & \beta_2 \end{vmatrix} = \tag{3.2}$$

$$= \beta_2\, \varphi(a, l) - \beta_1\, \varphi'(a, l) = \beta_2\{\alpha_1 \cos(as) + \alpha_2 s^{-1} \sin(as) + \\ + [\alpha_1 + \alpha_2\varrho^{-1}]\varrho^{\gamma-1} e_1(a) k_2(s)\} - \beta_1\{-\alpha_1 s \sin(as) + \\ + \alpha_2 \cos(as) + [\alpha_1 + \alpha_2\varrho^{-1}]\varrho^{\gamma} e_1(a) k_3(s).$$

Formula (3,1) follows by easy calculation.

4. Theorem. *Let $u(s) = w(l)$, where s is defined in Section 1. Then*

a) $\alpha_1\beta_1 \neq 0 \Rightarrow$

(4.1) $u(s) = \alpha_1\beta_1 s \sin(as)[1 + \varrho^{\gamma-1} k_1(s)] = e_1(a) k_2(s);$

b) $\alpha_1 = \beta_1 = 0,\ \alpha_2\beta_2 \neq 0 \Rightarrow$

(4.2) $u(s) = \alpha_2\beta_2 s^{-1} \sin(as)[1 + \varrho^{\gamma-1} k_3(s)] = \varrho^{-1} e_1(a) k_4(s);$

c) $\alpha_1 = 0,\ \beta_1 \neq 0 \Rightarrow$

(4.3) $u(s) = -\alpha_2\beta_1 \cos(as)[1 + \varrho^{\gamma-1} k_5(s)] = e_1(a) k_6(s);$

d) $\alpha_1 \neq 0,\ \beta_1 = 0 \Rightarrow$

(4.4) $u(s) = \alpha_1\beta_2 \cos(as)[1 + \varrho^{\gamma-1} k_7(s)] = e_1(a) k_8(s).$

Proof results from (3,1) by using the notation introduced in Section 2.

5. Theorem

(5.1)
$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \alpha_1\beta_1[as\cos(as) + \sin(as)] +$$
$$+ [\alpha_2\beta_1 - \alpha_1\beta_2]\, a \sin(as) - \alpha_2\beta_2 s^{-1}[s^{-1}\sin(as) - a\cos(as)] +$$
$$+ [\alpha_1\beta_1 + (\alpha_2\beta_1 + \alpha_1\beta_2)\varrho^{-1} + \alpha_2\beta_2\varrho^{-2}]\, \varrho^{\gamma} e_1(a) k_1(s).$$

Proof. We differentiate (3.2) with respect to s and make use of the expressions derived in [1]:

(a)
$$\frac{\mathrm{d}\varphi(a, l)}{\mathrm{d}s} = -\alpha_1 a \sin(as) + \alpha_2 a s^{-1}\cos(as) -$$
$$- \alpha_2 s^{-2}\sin(as) + [\alpha_1 + \alpha_2\varrho^{-1}]\, \varrho^{\gamma-1} e_1(a) c_1(a);$$

(b)
$$\left[\frac{\partial}{\partial s}\left(\frac{\partial\varphi(x, l)}{\partial x}\right)\right]_{x=a} = -\alpha_2 a \sin(as) - \alpha_1 \sin(as) -$$
$$-\alpha_1 as\cos(as) + [\alpha_1 + \alpha_2\varrho^{-1}]\, \varrho^{\gamma} e_1(a) c_2(a).$$

6. Theorem

a) $\alpha_1\beta_1 \neq 0 \Rightarrow$

(6.1)
$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \alpha_1\beta_1 as\cos(as)[1 + \varrho^{\gamma-1} k_1(s)] = \varrho\, e_1(a) k_2(s);$$

b) $\alpha_1 = \beta_1 = 0,\ \alpha_2\beta_2 \neq 0 \Rightarrow$

(6.2)
$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \alpha_2\beta_2 a s^{-1}\cos(as)[1 + \varrho^{\gamma-1} k_3(s)] = e_1(a) k_4(s);$$

c) $\alpha_1 = 0,\ \beta_1 \neq 0 \Rightarrow$

(6.3)
$$\frac{\mathrm{d}\omega(s)}{\mathrm{d}s} = \alpha_2\beta_1 a \sin(as)[1 + \varrho^{\gamma-1} k_5(s)] = e_1(a) k_6(s);$$

d) $\alpha_1 \neq 0,\ \beta_1 = 0 \Rightarrow$

(6.4) $$\frac{d\omega(s)}{ds} = \alpha_1\beta_2 a \sin(as)\left[1 + \varrho^{\gamma-1} k_7(s)\right] = e_1(a)\, k_8(s).$$

Proof follows from the modified equation (5.1).

7. Note. In what follows, n represents a natural number and c_i $(i = 1, 2, \ldots)$ are positive constants independent of n, $\varrho > 1$, $\tau \in [0, \pi r]$.

8. Theorem. *Let* $\{\varrho_n\}$ *be an increasing sequence of real numbers such that*

(8.1) $$\lim_{n\to+\infty} \varrho_n = +\infty$$

and let there exist $c_1 \in (0, 1)$ *independent of* n *and such that*

(8.2) $$|\sin(a\varrho_n)| > c_1 .$$

Then for $\tau \in [0, \pi r]$ *the inequality*

(8.3) $$|\operatorname{cosec}(a\varrho_n e^{i\tau})| < c_2\, e_1(-a)$$

holds for almost all n.

Similarly,

(8.4) $$|\cos(a\varrho_n)| > c_3 \Rightarrow |\sec(a\varrho_n e^{i\tau})| < c_4\, e_1(-a)$$

for almost all n *and for* $c_3 \in (0, 1)$ *independent of* n; $e_1(-a) = \exp\left[-a\varrho_n|\sin\tau|\right]$.

Proof. Let $s = \varrho_n e^{i\tau}$. Then an elementary calculation yields

(8.5) $$|\operatorname{cosec}(as)| = |\operatorname{cosec}(a\varrho_n e^{i\tau})| =$$
$$= 2\, e_1(-a)\left[(e_1^2(-a) - 1)^2 + 4\, e_1^2(-a)\sin^2(a\varrho_n\cos\tau)\right]^{-1/2} .$$

(a) Let $\tau \in [0, \pi/a\varrho_n]$, where obviously $\pi/a\varrho_n < \pi/2$ for almost all n.

It may be easily proved that

(8.6) $$\cos\tau > 1 - \frac{\pi}{a\varrho_n} \quad \text{for almost all } n .$$

Considering (8.2) and (8.6) we conclude

(8.7) $$\sin^2(a\varrho_n\cos\tau) > \sin^2\left[a\varrho_n\left(1 - \frac{\pi}{a\varrho_n}\right)\right] = \sin^2(a\varrho_n) > c_1^2 .$$

Since

(8.8) $$e_1(-a) < e^{-\pi} \quad \text{for} \quad \tau \in \left[0, \frac{\pi}{a\varrho_n}\right],$$

(8.5) together with (8.7) and (8.8) yields

$$|\operatorname{cosec}(a\varrho_n e^{i\tau})| < c_2 \, e_1(-a) .$$

Thus the relationship (8.3) is proved for $\tau \in [0, \pi/a\varrho_n]$.

Further, let

(b) $$\tau \in \left[\frac{\pi}{a\varrho_n}, \frac{\pi}{2}\right].$$

We apply the inequality

(8.9) $$\sin \tau > \frac{2\tau}{\pi} \quad \text{for} \quad \tau \in \left(0, \frac{\pi}{2}\right),$$

obtaining

(8.10) $$e_1(-a) < e^{-2} < 1 .$$

It can be seen that even in this case, (8.5) together with (8.2), (8.9) and (8.10) yields

$$|\operatorname{cosec}(a\varrho_n e^{i\tau})| < c_3 \, e_1(-a) .$$

This proves the relation (8.3) for $\tau \in [\pi/a\varrho_n, \pi/2]$.

The equation (8.3) may be proved similarly.

For the other values of τ the proof is based on the fact that cosec (as) is an odd function assuming real values for s real, while sec (as) is an even function with real values for s real.

9. Theorem.

Let

(9.1) $$s = \varrho_n e^{i\tau}, \quad \tau \in [0, \pi r],$$

where ϱ_n *satisfies conditions* (8.1) *to* (8.4).

Then

a) $\alpha_1 \beta_1 \neq 0 \Rightarrow$

(9.2) $$\omega^{-1}(s) = (\alpha_1 \beta_1 s)^{-1} \operatorname{cosec}(as) [1 + \varrho^{\gamma-1} k_1(s)] = \varrho^{-1} e_1(-a) \, k_2(s) ;$$

b) $\alpha_1 = \beta_1 = 0, \ \alpha_2 \beta_2 \neq 0 \Rightarrow$

(9.3) $$\omega^{-1}(s) = (\alpha_2 \beta_2)^{-1} \, s \operatorname{cosec}(as) [1 + \varrho^{\gamma-1} k_3(s)] = \varrho \, e_1(-a) \, k_4(s) ;$$

c) $\alpha_1 = 0, \ \beta_1 \neq 0 \Rightarrow$

(9.4) $$\omega^{-1}(s) = -(\alpha_2 \beta_1)^{-1} \sec(as) [1 + \varrho^{\gamma-1} k_5(s)] = e_1(-a) \, k_6(s) ;$$

d) $\alpha_1 \neq 0, \ \beta_1 = 0 \Rightarrow$

(9.5) $$\omega^{-1}(s) = (\alpha_1 \beta_2)^{-1} \sec(as) [1 + \varrho^{\gamma-1} k_7(s)] = e_1(-a) \, k_8(s) .$$

Proof. For s from (9.1) it follows from (4.1) and (8.2) that

$$\omega(s) = \alpha_1 \beta_1 s \sin(as) [1 + \varrho^{\gamma-1} k_1(s)] ,$$

hence

$$u^{-1}(s) = (\alpha_1 \beta_1 s)^{-1} \operatorname{cosec}(as) [1 + \varrho^{\gamma-1} k_2(s)] = \varrho^{-1} e_1(-a) k_3(s) ,$$

which proves (9.2).

Relations (9.3), (9.4) and (9.5) are derived analogously.

References

[1] *J. Hylán:* Asymptotical properties of solution of a certain class of linear differential equations of the 2nd order (Czech). Proceedings 6th Scientific Conference in Žilina, 1979 at Technical University for Transport Engineering.

[2] *E. C. Titchmarsh:* Eigenfunction expansions associated with second-order differential equations. Clarendon Press, Oxford 1946 (Russian translation, Moskva 1960).

Author's address: 010 00 Žilina, Marxa-Engelsa 25 (katedra matematiky Vysoké školy dopravy a spojů).