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REMARKS ABOUT MEASURES ON ORTHOMODULAR POSETS

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INTRODUCTION

The set of measures on an orthomodular poset may be fairly different from the set of measures on a σ-algebra. There exists a (finite) orthomodular poset without any measure at all or with exactly one measure (see [2], [9]). In this paper we take up the problem of how (and when) we can distinguish noncompatible elements of an orthomodular poset by a measure and in particular, when we can separate noncompatible elements "up to a given ε". We construct two examples illustrating that the situation may be considerably complex. As the counterexamples are lattice, our results may find applications also in the lattice theory. Besides, the problems investigated here are related to the foundations of quantum theories as the paper [7] indicates.

1. PRELIMINARIES

Definition 1.1. An *orthomodular poset* (abbr. OMP) is a triple $(L, \geqq, ')$, where L is a nonvoid set endowed with a partial ordering $\geqq$, and where $'$ is a unary operation on L such that

(i) there is a least element 0 in L,

(ii) if $a, b \in L$ and $a \leqq b$ then $a' \geqq b'$,

(iii) if $a \in L$ then $(a')' = a$,

(iv) if $a, b \in L$ and $a \leqq b$ then $b = a \vee (b \wedge a')$, where, in all what follows, $\vee$, $\wedge$ mean the lattice – theoretic operations induced by $\geqq$,

(v) if $\{a_i, i \in N\}$ is a sequence of elements of L and if $a_i \leqq a_j'$ for any distinct $i, j \in N$ then $\bigvee_{i=1}^{\infty} a_i$ exists in L.

Throughout the paper, the letter L will be reserved for OMP's.

Definition 1.2. A mapping $m : L \to \langle 0, 1 \rangle$ is called a *(probability) measure* on L if

(i) $m(1) = 1$,

(ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \leqq a'_j$ for any distinct indices $i, j \in N$.

Proposition 1.1. *If m is a measure on L and if $a \leqq b$ then $m(a) \leqq m(b)$.*

Proof. Obvious.

Definition 1.3. Two elements $a, b \in L$ are called *compatible* (in symbols: $a \leftrightarrow b$) if there exist three elements $a_1, b_1, c \in L$ such that $a_1 \leqq b'_1$, $b_1 \leqq c'$, $a_1 \leqq c'$ and $a = a_1 \vee c$, $b = b_1 \vee c$.

Proposition 1.2. (i) *If $a \leqq b$ then $a \leftrightarrow b$.*

(ii) *If $a \leftrightarrow b$ then $a \vee b$, $a \wedge b$ exist in L and we have $a \wedge b = 0$ if and only if $a \leqq b'$.*

(iii) *$a \leftrightarrow b$ if and only if $a \leftrightarrow b'$.*

Proof. See [5], [10].

Proposition 1.3. *Let $\{m_i\}$ be a sequence of measures on L and let $\{\alpha_i\}$ be a collection of nonnegative real numbers such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then $m = \sum_{i=1}^{\infty} \alpha_i m_i$ is a measure on L. We say that m is a convex combination of m_i, $i = 1, 2, 3, \ldots$.*

Proof. Obvious.

Definition 1.4. Let L be an OMP. We call L *reasonable* if for any $a \in L$ with $a \neq 0$ there exists a measure m on L such that $m(a) = 1$.

It is known that there exist OMP's with "small" sets of measures (see [2]). Since such OMP's seem fairly useless as regards the potential applications in quantum theories or elsewhere, we restrict ourselves to those OMP's which possess reasonable collections of measure.

Proposition 1.4. *Let L be reasonable. Then for any noncompatible $a, b \in L$ there is a measure m on L and a real number $r \geqq \frac{1}{2}$ such that $m(a) = r = m(b)$.*

Proof. If there is a measure m on L with $m(a) = 1 = m(b)$ then we have nothing to prove. In the opposite case, there are measures m_1, m_2 on L such that $m_1(a) = 1$, $m_2(b) = 1$, $m_1(b) = r_1 < 1$, $m_2(a) = r_2 < 1$ and we put

$$m = \frac{(1 - r_2) m_1 + (1 - r_1) m_2}{(1 - r_1) + (1 - r_2)}.$$

Corollary 1.5. *Let L be reasonable. Then for any $a, b \in L$ with $a \not\leftrightarrow b$ there exists a measure m on L such that $m(a) = \frac{1}{2} = m(b)$.*

Proof. According to Proposition 1.4, there are measures m_1, m_2 on L with $m_1(a) = m_1(b) = r_1 \geqq \frac{1}{2}$, $m_2(a') = m_2(b') = 1 - r_2 \geqq \frac{1}{2}$. If $r_1 = r_2$ then $r_1 = r_2 = \frac{1}{2}$. If this is not the case, we put

$$m = \frac{(\frac{1}{2} - r_2)\, m_1 + (r_1 - \frac{1}{2})\, m_2}{r_1 - r_2},$$

and we obtain the desired measure.

There is a natural question whether any r, $0 < r < 1$, can be placed instead of the number $\frac{1}{2}$ in the corollary. The answer will be given in Theorem 2.4.

2. MEASURES ON ORTHOMODULAR POSETS

In the following definition we introduce certain classes of OMP's which we shall deal with in the sequel.

Definition 2.1. Let L be a reasonable OMP. Let us define *conditions* (C1)–(C5) as follows:

(C1) if $a, b \in L$ and $a \nleftrightarrow b$ then there exists a measure m on L such that $m(a) \neq m(b)$,

(C2) if $a, b \in L$ and $a \nleftrightarrow b$ then there exists a measure m on L such that $m(a) = 1$, $m(b) \neq 1$,

(C3) if $a, b \in L$ and $a \nleftrightarrow b$ then there exists a measure m on L such that $m(a) = 1 = m(b)$,

(C4) if $a, b \in L$ and $a \nleftrightarrow b$ and if we are given a real number r, $0 < r < 1$, then there exists a measure m on L such that $m(a) = 1$, $m(b) \geqq r$,

(C5) if $a, b \in L$ and $a \nleftrightarrow b$ and if we are given a real number r, $0 < r < 1$, then there exists a measure m on L such that $m(a) \geqq r$, $m(b) \geqq r$.

The conditions (C1), (C2), (C3), (C4) have appeared in [4], [5], [7], [8]. In this paper we shall continue the investigation of the respective classes of OMP's and we also take up the question of how the condition (C5) is related to the previous ones. It is worthwhile to note at the moment that e.g. the Hilbert logic $L(H)$, dim $H \geqq 3$ fulfils (C2) but does not fulfil (C5).

Theorem 2.1. *We have the following implications*:

a) (C3) ⇒ (C4) ⇒ (C2) ⇒ (C1),

b) (C4) ⇒ (C5) ⇒ (C1).

Proof. a) For (C3) ⇒ (C4) and for (C4) ⇒ (C2) see [8], the third implication is obvious.

b) The implication (C4) ⇒ (C5) holds trivially. If $a, b \in L$ and $a \nleftrightarrow b$ then (C5) yields the existence of a measure m on L such that $m(a) \geqq \frac{3}{4}$, $m(b') \geqq \frac{3}{4}$. Hence $m(b) \leqq \frac{1}{4}$ which proves the implication (C5) ⇒ (C1).

We shall show that none of the implications in Theorem 2.1 is an equivalence. Prior to that, let us consider the case of finite OMP's.

Theorem 2.2. *Let L be a finite* OMP. *Then the conditions* (C3), (C4) *and* (C5) *are equivalent.*

Proof. It remains to prove that (C5) implies (C3). Let L satisfy (C5) and let $a, b \in L$ and $a \nleftrightarrow b$. For any $n \in N$ let us take a measure m_n on L such that $m_n(a) \geqq 1 - 1/n$ and $m_n(b) \geqq 1 - 1/n$. For any $c \in L$ the sequence $\{m_n(c)\}$ contains a convergent subsequence. Since L is finite, we can successively find a subsequence $\{m_{n_k}\}$ of $\{m_n\}$ such that $\{m_{n_k}(c)\}$ converges for any $c \in L$. Let us define a mapping $m : L \to \langle 0, 1 \rangle$ by putting $m(c) = \lim_{k \to \infty} m_{n_k}(c)$. We shall show that m is a measure on L. Trivially, $m(1) = 1$. Let us suppose that $c_1 \leqq c_2'$. Then $m(c_1 \vee c_2) = \lim m_{n_k}(c_1 \vee c_2) = \lim (m_{n_k}(c_1) + m_{n_k}(c_2)) = \lim m_{n_k}(c_1) + \lim m_{n_k}(c_2) = m(c_1) + m(c_2)$. We have thus obtained that m is a measure on L. Obviously, $m(a) = 1 = m(b)$. The proof is complete.

Theorem 2.3. *None of the implications in the formulae* a), b) *of Theorem* 2.1 *is an equivalence.*

Proof. It is known that the implications (C2) $\Rightarrow$ (C4) and (C4) $\Rightarrow$ (C3) fail in general (see [8]). The implication (C1) $\Rightarrow$ (C2) has been disproved by R. Godowski (see [1]). (Also our Example 2.1 presented later on shows that (C1) $\Rightarrow$ (C2) fails.) Since the counterexample (C1) $\not\Rightarrow$ (C3) is finite, we have (C1) $\not\Rightarrow$ (C5) (Theorem 2.2). It remains to show that (C5) does not imply (C4). We shall construct the example by making use of a Greechie diagram.

We assume that the reader is more or less familiar with the interpretation of Greechie diagrams. Let us only recall that the "points" of a Greechie diagram are interpreted as the atoms of the corresponding OMP and the straight line segments

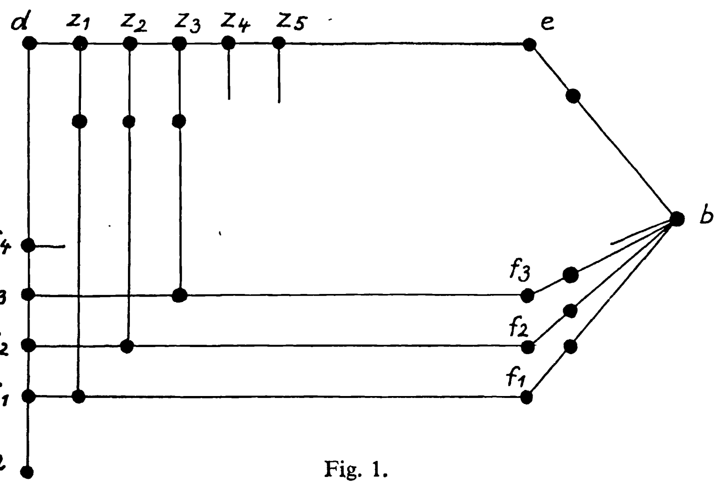

Fig. 1.

(or possibly the curve segments) group together those atoms which belong to a maximal orthogonal set (Boolean block). There is a one-to-one relation between the measures on the corresponding OMP and the so called weights on the Greechie diagram. (A weight is such a (nonnegative) evaluation of the points of the diagram that the sums over Boolean blocks equal 1.) We shall not make any distinction between the weights on a Greechie diagram and the measures on the corresponding OMP. The precise description of Greechie diagrams may be found in [2] or [9].

Let us return to the example disproving (C5) ⇒ (C4). We start with a preliminary construction.

Example 2.1. The orthomodular poset L given by Fig. 1 has the following property: If m is a measure on L and if $m(b) = 1$ then $m(a) = 0$.

Proof. Suppose that $m(b) = 1$ for a measure m on L. Then $m(f_i) = 0$ for any $i = 1, 2, 3, \ldots$. Therefore $m(d) + \sum m(z_i) \leqq m(d) + \sum m(c_i)$. Since $m(e) = 0$, the left hand side of the last inequality equals one. Hence $m(d) + \sum m(c_i) = 1$ and therefore $m(a) = 0$. This was to prove. Observe that we have also proved that the identity $m(b) = 1$ for a measure m implies $m(a') = 1$ and thus L does not satisfy (C2). On the other hand, if $\varepsilon > 0$ and $r \in (0, 1\rangle$ are given, we may find a measure m such that $m(b) = 1 - \varepsilon$ and $m(a) = r$. Obviously, if we take two noncompatible elements $p, q \in L$ such that $(p, q) \neq (a, b)$, there exists a measure m on L with $m(p) = 1 = m(q)$. Hence L satisfies (C1) and, as noted in the proof of Theorem 2.1, we

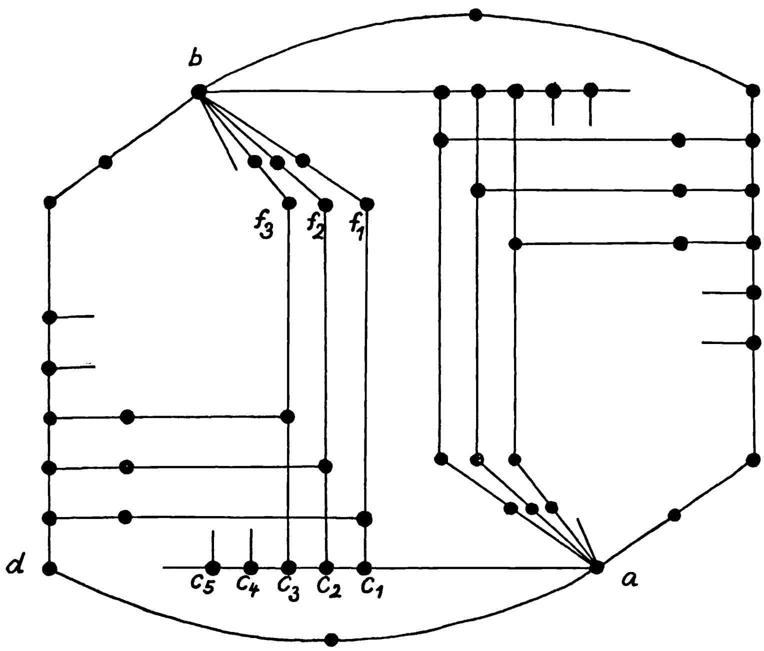

Fig. 2.

have another example of (C1) $\nRightarrow$ (C2). If $d = 0$ (it means there does not exist such an atom in L), then L becomes an orthomodular lattice.

The desired example establishing (C5) $\nRightarrow$ (C4) is now constructed as follows.

Example 2.2. An orthomodular poset (lattice) L given by Fig. 2 satisfies (C2), (C5) but does not satisfy (C4).

Proof. If we take noncompatible atoms $(p, q) \neq (a, b)$, then there obviously exists a measure m on L such that $m(p) = 1 = m(q)$. Let us consider the pair (a, b). If $m(b) = 1$, then $m(f_i) = 0$ for $i = 1, 2, 3, \ldots$ and therefore $m(d) + \sum m(c_i) = 1$ (see Example 2.1). Therefore $m(a) \leqq \frac{1}{2}$ and L does not satisfy (C4). On the other hand, L satisfies (C5). Indeed, if we are given an $\varepsilon > 0$, we may simply construct a measure m with $m(b) = 1 - \varepsilon$ and $m(f_i) \in \langle 0, \varepsilon \rangle$ (in the manner similar to the construction in Example 2.1). The fact that L satisfies (C2) is obvious. Theorem 2.3 is thus completely proved.

The following theorem says that (C5) is in fact equivalent to a condition apparently stronger.

Theorem 2.4. *A reasonable orthomodular poset L fulfils (C5) if and only if L fulfils the following condition (C6): If a, b are two noncompatible elements of L and if we are given a real number $r \in (0, 1)$ then there exists a measure m on L such that $m(a) = r = m(b)$.*

Proof. The condition (C6) is obviously sufficient for (C5). To prove necessity, let a real number $r < 1$ be given. We may and shall suppose that $r \geqq \frac{1}{2}$ – otherwise we take up the equivalent assertion with a', b' and $r' = 1 - r$. We shall construct two measures m_1, m_2 on L such that $m_1(a) = m_1(b) = r_1 \in \langle 0, \frac{1}{2} \rangle$ and $m_2(a) = m_2(b) = r_2 \in \langle r, 1 \rangle$. The required measure m on L can be then constructed by putting

$$m = \frac{(r_2 - r)\, m_1 + (r - r_1)\, m_2}{r_2 - r_1}.$$

First we shall obtain m_1. If there is a measure m on L such that $m(a') = 1 = m(b')$, we put $m_1 = m$. If this is not the case, the assumption of Theorem 2.4 guarantees the existence of measures m_3, m_4 on L with $m_3(a') = 1$, $m_3(b') = r_3 < 1$, $m_4(a') = r_4 < 1$, $m_4(b') = 1$. We put

$$m_1 = \frac{(1 - r_4)\, m_3 + (1 - r_3)\, m_4}{(1 - r_3) + (1 - r_4)}.$$

Let us consider the construction of m_2. If there is a measure m on L with $m(a) = 1 = m(b)$, we put $m_2 = m$ and the proof is complete. If there is no such measure, we consider measures m_5, m_6 and m_7 such that $m_5(a) = 1$, $m_5(b) = r_5 < 1$, $m_6(a) = r_6 < 1$, $m_6(b) = 1$ and further, if we set $s = (\max r, r_5, r_6)$, we require $m_7(a) = r_7 > s$, $m_7(b) = r_8 > s$. The assumptions of Theorem 2.4 guarantee the existence of such measures. If $r_7 = r_8$, we put $m_2 = m_7$. If $r_7 < r_8$, then

$$m_2 = \frac{(r_8 - r_7)\, m_5 + (1 - r_5)\, m_7}{(1 - r_5) + (r_8 - r_7)}\,.$$

If $r_8 < r_7$, the construction of m_2 proceeds dually. The proof of Theorem 2.4 is complete.

References

[1] *R. Godowski:* Varieties of orthomodular lattices with a strongly full set of states. Demonstratio Math. *14* (1981), 725–733.

[2] *R. Greechie:* Orthomodular lattices admitting no states. J. Comb. Theory *10* (1971), 119 to 132.

[3] *S. Gudder:* Uniqueness and existence properties of bounded observables, Pacific Journal Math. *19* (1966), 81–93, 578–589.

[4] *S. Gudder:* Axiomatic quantum mechanics and generalized probability theory. Probabilistic Methods in Applied Mathematics, Vol. 2, (A. Bharucha - Reid, ed.), Academic Press, New York, 1970.

[5] *S. Gudder:* Stochastic Methods in Quantum Mechanics, North Holland, New York, 1979.

[6] *S. Gudder, K. Rüttimann, R. Greechie:* Measurements, Hilbert space and quantum logics, J. Math. Phys. *23* (1982), 2381—86.

[7] *P. Pták, V. Rogalewicz:* Regularly full logics and the uniqueness problem for observables. Ann. Inst. H. Poncaré *38* (1983), 69–74.

[8] *P. Pták, V. Rogalewicz:* Measures on orthomodular partially ordered sets. Journal Pure Applied Algebra *28* (1983), 75–80.

[9] *F. W. Schultz:* A characterization of state space of orthomodular lattices, J. Comb. Theory A *17* (1974), 317–325.

[10] *V. S. Varadarajan:* Geometry of Quantum Theory I, Van Nostrand Reinhold, Princeton, New Jersey, 1968.

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