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ON PARTITION OF TOPOLOGICAL SPACES

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0. INTRODUCTION

The problem considered in our paper goes back to a classical work of F. Bernštein [1] in which he proved that in every complete separable metric space Y there is a subset A such that neither A nor $Y \setminus A$ contain a homeomorphic copy of the Cantor discontinuum D^{ω_0}. In its modern interpretation this problem was apparently set for the first time by Z. Frolík about 1972. He asked whether there exists a Hausdorff space Y such that for every its subspace A the Cantor set D^{ω_0} is embeddable either in A or in $Y \setminus A$ (or, less generally, either A or $Y \setminus A$ contains the segment $I = [0, 1]$). This problem is of importance only in the case when Y is asked to be Hausdorff: rather a simple construction of J. Pelant, J. Nešetřil and V. Rödl allows, for every T_1-space X, to find a T_1-space Y such that if $A \subset Y$ then either A or $Y \setminus A$ contains a subset (but essentially not closed!) homeomorphic to X (see e.g. [2]).

During the recent years a number of papers devoted to this problem has appeared (see e.g. [3], [4], [5], [6], [7], [8], [32]).

In our paper all these results are improved and generalized in various directions. First, the only topological properties of D^{ω_0} and I essential in our considerations are the countable compactness and unscatteredness. (Recall that a space I is called unscattered if it contains a subspace which is dense in itself.)

Thus in our paper the following result is proved under some set-theoretical assumptions.

Theorem. *In every topological space there are subspace X_0 and X_1 such that $X = X_0 \cup X_1$ and neither X_0 nor X_1 contain a closed (in X) countably compact regular unscattered subspace F. In particular, if X is Hausdorff, then every compactum contained in X_0 or X_1 is scattered. (See theorems (8.6) and (9.7).)*

The set-theoretic assumption ACP$^\#$ under which we prove this theorem is essentially weaker than the assumption $\mathrm{ACP}_1^\# \equiv (\aleph_1 < \mathfrak{c}) \& (\mu^{\aleph_0} < \mu^+)$ for every $\mu \geqq \mathfrak{c}$. An other assumption which is also sufficient for us is the axiom of constructibility $\mathrm{V} = \mathrm{L}$. Hence this theorem improves W. Weiss's result which is the strongest one among the results of this kind obtained before, and which asserts that under $\mathrm{V} = \mathrm{L}$, in every Hausdorff space X there exist subsets X_0 and X_1 neither of which contain D^{ω_0} and $X = X_1 \cup X_2$.

However, we believe that the most natural is the following general version of our theorem (see Sections 8, 9).

Theorem [ACP#] $\vee$ [V = L]. *For every topological space X there exists a partition (see 2.1) $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ such that for every closed regular countably compact subset F without isolated points the intersections $R_\alpha \cap F$ are everywhere dense in F for all R_α (thus $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ is a decomposition (see 2.2) for all such F simultaneously). In particular, for every Hausdorff compact space X without isolated points there exists a (universal) decomposition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ such that if F is a closed subset without isolated points then $[F \cap R_\alpha] = F$ for every $\alpha < \mathfrak{c}$.*

Let us give the outline of the exposition adopted in the paper. We begin with *preliminaries* where some basic notations, definitions and concepts used in the paper are introduced and discussed. The *second* section also has an auxiliary character; we prove there some elementary facts on partitions and decompositions. In the *third* section the fundamental for our theory concept of a *relatively countable closedness*, or rc-closedness, is introduced and studied. Although the majority of the results about rc-closedness obtained here are used in the basic construction (Section 6), some are adduced only for the sake of completeness. The *fourth* section is devoted to the continuous mappings with values in I. The results of this section are also essentially used in the basic construction, but we consider them to be of interest by themselves and therefore they are given here sometimes in a more general form than it is necessary for our applications. In the short *fifth* section the notion of a τ-universal partition of a set A in a space X is introduced (it is a specification of the usual partition). This notion is basic for the induction in the proof of main results (see Section 6). We draw attention to the fact that the deepest results on the τ-universal partitions are obtained in case when the partitioned set A is rc-closed in X.

The next *sixth* section contains the principal amout of the technique in the whole work. We introduce here the notion "the statement $S_\tau(X \mid \mu)$ holds for a space X and cardinals τ and μ" (Definition 6.1) and prove by induction that this statement really holds "very often" (see e.g. (6.18)). As a corollary we obtain that "usually" topological spaces have $\aleph_1$-iniversal partitions (6.19), (6.20)).

As was mentioned above, to apply the results of Section 6 for the proof of our principal results we need some additional set-theoretic assumptions. The majority of these assumptions are formulated and discussed in the next section – the *seventh* section.

In the *eighth* section the set-theoretic assumptions discussed in Section 7 are applied to the results of Section 6 to get the main theorems of the work (see (8.1)–(8.10)); some of these theorems were already stated above.

The last four sections, 9–12, have the nature of appendices. In Section 9 the results of W. Weiss [3] on partition of topological spaces under the assumption of the axiom of constructibility are improved and generalized. The proof of this generalization is again based on the technique of Section 6.

In Sections 10 and 11 the main results are applied to obtain some corollaries of rather a varied nature.

The concept of an rc-closed set which is the key notion for the whole work (see Section 4) leads us also to a new class of topological spaces called quasi-sequential spaces. This class enters rather naturally the hierarchy of classes of spaces, the topology of which is defined by countable subsets (spaces of countable tightness, sequential spaces etc.). Quasi-sequential spaces are studied in the last Section 12.

1. PRELIMINARIES

The notions and concepts which are used but not defined in the paper can be found in usual handbooks on general topology, for example, in the well-known Engelking's book [10]. There one can find also facts and theorems used in the paper if no other reference is given.

No separation axiom is assumed unless explicitly stated. All mappings of topological spaces are continuous. A mapping $f : X \to Y$ is called an injection if it is one-to-one, i.e., if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. A mapping $f : X \to Y$ is called a surjection if $f(X) = Y$. A mapping which is both a surjection and an injection is called a bijection.

Script letters always signify a family of sets, while capital letters are usually used to denote topological spaces and sets. Specifically, the letters X, Y and Z will always signify topological spaces. Small Greek letters are used only for cardinals and ordinals. We do not distinguish in notations between a cardinal τ and the minimal ordinal of the cardinality τ. Moreover, sometimes we identify in notations an ordinal τ and the set $T(\tau)$, consisting of all ordinals σ less than τ. Thus the equality $\tau = T(\tau)$ means exactly that τ is a cardinal. Nevertheless, following the tradition we use for a countable cardinal and the first uncountable cardinal either the notations ω_0 and ω_1 or the notations $\aleph_0$ and $\aleph_1$, respectively. If τ is a cardinal then τ^+ denotes the least cardinal larger than τ; $\mathrm{cf}(\tau)$ means, as usual, the cofinal character of the cardinal τ (see e.g. [11]). If A is a set, then $|A|$ denotes the dardinality of A. As usual, we write $\mathfrak{c}$ for the cardinality of continuum.

Let X be a topological space and A its subset. Then $[A]_X$ or simply $[A]$ denotes the closure of A in X, A' denotes the set of limit points of A. The notation $d(X)$ stands for the density of the space X. As usual, we use R and I, respectively, as designations for the real line and the segment $[0, 1]$ endowed with the usual topology, N is the set of all natural numbers.

Let L be a linearly ordered set with a linear order $\leqq$ and let $\alpha, \beta \in L$. Then we write shortly $[\alpha, \beta]$, $[\alpha, \beta[$, $]\alpha, \beta]$ and $]\alpha, \beta[$ instead of $\{x : x \in L,\ \alpha \leqq x \leqq \beta\}$, $\{x : x \in L,\ \alpha \leqq x < \beta\}$, $\{x : x \in L,\ \alpha < x \leqq \beta\}$ and $\{x : x \in L,\ \alpha < x < \beta\}$, respectively.

The notation $X \approx Y$ will mean that spaces X and Y are homeomorphic, $X \hookrightarrow Y$ will mean that X is homeomorphic to a subset $\tilde{X}$ of Y while $X \underset{\mathrm{cl}}{\hookrightarrow} Y$ will mean that X is homeomorphic to a closed subset $\tilde{X}$ of Y.

2. SOME ELEMENTARY FACTS ON PARTITIONS

Although the next two definitions are well-known and have been used by various authors we reproduce them here both for the sake of completeness and to specify these fundamental for our paper notions.

(2.1) **Definition.** A disjoint family $\mathscr{R}$ of subsets of a set X is called *a partition* of X (or a partition in X) if $\bigcup\{R : R \in \mathscr{R}\} = X$.

(2.2) **Definition.** If $\mathscr{R}$ is a partition of a topological space X and $[R] = X$ for every $R \in \mathscr{R}$ then $\mathscr{R}$ is called *a decomposition* of X.

The next result can be found e.g. in [12], p. 434:

(2.3) **Proposition.** *Let $\mathscr{F}$ be a family of subsets of a set X, $|\mathscr{F}| \leqq \tau$ and $|F| \geqq \tau$ for every $F \in \mathscr{F}$. Then there exists a partition $\mathscr{R} = \{R_\sigma : \sigma < \tau\}$ of the set X such that $|R_\sigma \cap F| \geqq \tau$ for every $\sigma < \tau$ and every $F \in \mathscr{F}$.*

This proposition leads one easily to the following statement which was already used in A. V. Archangelskij's paper [9].

(2.4) **Proposition.** *Let X be a topological space such that $|X|^\lambda = |X| = \tau$ and $\mathscr{F} = \{F : F \underset{\mathrm{cl}}{\subset} X,\ |F| = \tau,\ d(F) \leqq \lambda\}$. Then there exists a partition $\mathscr{R} = \{R_\sigma : : \sigma < \tau\}$ of the set X such that $|R_\sigma \cap F| = \tau$ for all σ and all $F \in \mathscr{F}$.*

Proof. Every closed subset F of X with $d(F) \leqq \lambda$ is uniquely determined by a set $G \subset F$ of cardinality λ. Hence $|\mathscr{F}| \leqq |X|^\lambda = \tau$ and therefore the assertion of (2.4) follows immediately from (2.3).

The last proposition implies the following well-known statement (see e.g. [12], p. 524).

(2.5) **Corollary.** *In every topological space X of cardinality $\mathfrak{c}$ there exists a partition $\mathscr{R} = \{R_\sigma : \sigma < \mathfrak{c}\}$ such that $|R_\sigma \cap F| = \mathfrak{c}$ for every $\sigma < \mathfrak{c}$ and every closed separable subset F of cardinality $\mathfrak{c}$.*

3. RELATIVELY COUNTABLY CLOSED SUBSETS

The notion of a relatively countably closed set, which is defined below and plays a fundamental role in our theory, belongs to the sort of "folklore" notions. Many authors have used it without giving it a special name in various constructions and proofs. For example, practically all constructions of countably compact extensions of topological spaces as well as completions of countably compact spaces are based on this notion; it was also essentially used by A. Gryzlov [13], W. Comfort [14], [15] and others.

(3.1) **Definition.** A subset $A \subset X$ is called *relatively countable closed* (or relatively countably compact, if one prefers it) in X if every countable set $N \subset A$ which is not closed in X, is not closed in A either (i.e., $N' \neq \emptyset$ implies $A \cup N' \neq \emptyset$). We shall usually abbreviate "relatively countably closed" as "*rc-closed*".

(3.2) Remark. One can easily notice that a subset A of a T_1-*space* X is rc-closed iff every sequence (x_n) $n \in N$ which has a cluster point x_0 in X also has a cluster point $a_0 \in A$.

The proofs of the following propositions about rc-closed subsets are obvious and therefore omitted.

(3.3) **Proposition.** *Every countably compact subset A of a T_1-space X is rc-closed in X.*

(3.4) **Proposition.** *If X is a countably compact T_1-space and A is rc-closed in X, then A is countably compact.*

(3.5) **Proposition.** *Every $\aleph_0$-closed*[1]) *(and therefore also every closed) subset A of a space X is rc-closed in X.*

(3.6) **Proposition.** *If $A \subset B \subset X$ and A is rc-closed in B while B is rc-closed in X, then A is rc-closed in X as well.*

(3.7) **Proposition.** *If $A \subset B \subset X$ and A is rc-closed in X, then A is rc-closed in B as well.*

(3.8) **Proposition.** *If A is rc-closed in X and F is $\aleph_0$-closed in X, then $A \cap F$ is rc-closed in F.*

Proof. The intersection $A \cap F$ is obviously $\aleph_0$-closed in A and hence by (3.5) $A \cap F$ is rc-closed in A. Therefore by (3.6) $A \cap F$ is rc-closed in X and hence also in F.

(3.9) **Proposition.** *Every rc-closed subset A of a T_1-space X is sequentially closed*[2])

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[1]) A subset A of X is called $\aleph_0$-closed if $[A]_{\aleph_0} = A$, where $[A]_{\aleph_0} = \bigcup\{[N]_X : N \subset A, |N| \leqq \aleph_0\}$. Obviously every closed set is sure to be $\aleph_0$-closed. It is known (and easy to be checked that a space X has a countable tightness (i.e. $t(X) \leqq \aleph_0$) iff every its $\aleph_0$-closed subset is closed.

[2]) A subspace A of X is called *sequentially closed* in X if A contains the limits of all its converging sequences. A space X is called *sequential* if every its sequentially closed subset is closed. For example, all first countable spaces are sequential.

in X. Hence in a sequential T_1-space every rc-closed subspace is closed.

The proof immediately follows from (3.2).

(3.10) **Remark.** From (3.5) and (3.9) we conclude that if X is a T_1-space, then every its $\aleph_0$-closed set is rc-closed and every rc-closed set is sequentially closed. The converses do not hold – for examples see (12.4) and (12.5).

(3.11) **Proposition.** *Let X, Y be topological spaces. where Y is moreover a T_1-space, and $f : X \to Y$ a closed mapping. If A is an rc-closed subset in X, then $f(A)$ is an rc-closed subset in Y.*

Proof. Let $M = \{y_n : n \in N\} \subset f(A)$ and $M' \cap (Y \setminus f(A)) \neq \emptyset$. We have to show that in this case $M' \cap f(A) \neq \emptyset$, too. Since Y is a T_1-space, wihout loss of generality we may assume that $y_n \neq y_m$ if $n \neq m$. For every $n \in N$ take $x_n \in f^{-1}(y_n) \cap A$ and consider a set $P = \{x_n : n \in N\} \subset A$. This set is not closed in X (otherwise $M = f(A)$ would be closed!) and since A is an rc-closed set, there exists $x_0 \in P' \cap A$. It is easy to notice that $y_0 = f(x_0)$ is an accumulation point for M and hence $M' \cap f(A) \neq \emptyset$.

(3.12) **Proposition.** *Let $f : X \to Y$ be a closed mapping, X a T_1-space and B an rc-closed subset of Y. Then the preimage $f^{-1}(B)$ is an rc-closed subset in X.*

Proof. Notice first that without loss of generality one can assume that f is a bijection and hence Y is a T_1-space, too. Really, since f is closed, the image $Y_0 = f(X)$ is a closed T_1-subspace of Y, and we may speak about the preimage of the set $B_0 = B \cap \cap Y_0$ instead of the preimage of the set B.

Let $x_0 \in X$ be a cluster point of a sequence $(x_n)_{n \in N} \subset f^{-1}(B)$ where $x_n \neq x_m$ if $n \neq m$ and consider the sequence $(y_n = f(x_n))_{n \in N}$. Obviously $y_0 = f(x_0)$ is its cluster point and hence there exists a cluster point $b_0 \in B$. Moreover, without loss of generality one can assume that $y_n \neq b_0$ for every $n \in N$, and therefore $x_n \notin f^{-1}(b_0)$ for every $n \in N$. Now to complete the proof it is sufficient to show that there exists $a_0 \in f^{-1}(b_0)$ which is a cluster point of $(x_n)_{n \in N}$.

Really, otherwise since X is a T_1-space there would be a neighbourhood U of $f^{-1}(b)$ such that $(x_n)_{n \in N} \subset X \setminus U$ and therefore $f(X \setminus U)$ would be a closed set in Y which contains the whole sequence $(y_n)_{n \in N}$ but not b_0.

(3.13) **Proposition.** *Let $f : X \to Y$ be a closed mapping, where Y is a sequential T_1-space and A an rc-closed subset of X. Then $f(A)$ is closed in Y. Moreover, the mapping $f' = f|_A : A \to Y$ is closed*[1]).

Proof. The first statement follows directly from (3.11) and (3.5). Now the second can be easily obtained from (3.8) and (3.9).

(3.14) **Remark.** In case of a countably compact set A the propositions (3.9), (3.11) and (3.13) are well-known and easily proved.

(3.15) **Proposition.** *For every set $A \subset X$ there exists an rc-closed set $\tilde{A}$ in X such that $A \subset \tilde{A}$ and $|\tilde{A}| \leqq |A|^{\aleph_0}$* [2]).

Proof. Let $A_0 = A$ and suppose that for every $\alpha < \beta$ where $\beta < \omega_1$ we have defined sets $A_\alpha \subset X$ in such a way that the following three conditions hold:

(a) $|A_\alpha| \leqq |A|^{\aleph_0}$,

(b) if $\alpha' < \alpha$ then $A_{\alpha'} \subset A_\alpha$,

(c) if $\alpha' < \alpha$, $(\alpha' \neq \alpha)$ $M \subset A_{\alpha'}$, $|M| \leqq \aleph_0$ and $M' \neq \emptyset$, then $M' \cap A_\alpha \neq \emptyset$.

Let $A^*_\beta = \bigcup\{A_\alpha : \alpha < \beta\}$, and for every countable set $M \subset A^*_\beta$ satisfying $M' \neq \emptyset$ take a point $x(M) \in M'$ and define $A_\beta = A^*_\beta \cup \{x(M) : M \subset A^*_\beta, |M| \leqq \aleph_0, M' \neq \emptyset\}$. Obviously A satisfies the conditions (b) and (c). To check the condition (a) notice that $|A_\beta| \leqq |A^*_\beta|^{\aleph_0} \leqq (\sum_{\alpha<\beta} |A_\alpha|)^{\aleph_0} \leqq (\aleph_0 |A|^{\aleph_0})^{\aleph_0} = |A|^{\aleph_0}$.

Let now $\tilde{A} = \bigcup\{A_\beta : \beta < \omega_1\}$. Since $\mathrm{cf}(\omega_1) = \omega_1 > \omega_0$, the conditions (b) and (c) allow us to conclude that $\tilde{A}$ is rc-closed in X. On the other hand, by (a) we have $|\tilde{A}| \leqq \omega_1 |A_\beta| = \omega_1 |A|^{\aleph_0} = |A|^{\aleph_0}$ [2]).

(3.16) **Corollary.** *If $A \subset X$ and $|A| \leqq \mathfrak{c}$ then there exists an rc-closed set $\tilde{A}$ in X which contains A and $|\tilde{A}| \leqq \mathfrak{c}$.*

Appyling Proposition (3.4) to (3.15) one obtains (3.16)

(3.17) **Corollary.** *If X is a countably compact T_1-space then for every $A \subset X$ there exists a countably compact set $\tilde{A} \subset X$ which contains A and $|\tilde{A}| \leqq |A|^{\aleph_0}$; in particular, if $|A| \leqq \mathfrak{c}$, then $|\tilde{A}| \leqq \mathfrak{c}$.*

4. SOME REMARKS ON MAPPINGS INTO I

The first type of problems considered here is the problem of discovering such subsets in topological spaces which can be (continuously) mapped onto the segment $I = [0, 1]$. The method employed here is the one which is essentially developed in [16]. It is based on the notion of a dyadic system defined by B. Efimov in 1970 [17] and as a matter of fact goes back to the paper pf P. S. Aleksandroff and V. I. Ponomarev [18].

(4.1) **Definition.** A system $\mathfrak{A} = \{\mathscr{A}_\alpha : \alpha \in L\}$ where $\mathscr{A}_\alpha = \{F^0_\alpha, F^1_\alpha\}$ for every $\alpha \in L$ and all F^0_α, F^1_α are closed nonempty subsets of a topological space X is called dyadic if

(a) $\emptyset \notin \mathscr{A}_{\alpha_1} \wedge \dots \wedge \mathscr{A}_{\alpha_n}$ $(\mathscr{A}_{\alpha_1}, \dots, \mathscr{A}_{\alpha_n} \in \mathfrak{A})$,

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[1]) $f' = f|_A$ denotes the restriction of a mapping f to a set A.

[2]) To be more precise, $|\tilde{A}| \leqq |A|^{\aleph_0} \aleph$, if we do not want to exclude the case $|A| = 1$. This remark is essential, however, only if X is not a T_1-space.

(b) $F_\alpha^0 \cap F_\alpha^1 = \emptyset$ for every $\alpha \in L$.

Here $\mathscr{A}_{\alpha_1} \bigwedge \dots \bigwedge \mathscr{A}_{\alpha_n}$ denotes the so called structural intersection of the families $\mathscr{A}_{\alpha_1}, \dots, \mathscr{A}_{\alpha_n}$, i.e. $\mathscr{A}_{\alpha_1} \bigwedge \dots \bigwedge \mathscr{A}_{\alpha_n} = \{F_{\alpha_1}^{\varepsilon_1} \cap \dots \cap F_{\alpha_n}^{\varepsilon_n} : F_{\alpha_i}^{\varepsilon_i} \in \mathscr{A}_{\alpha_i},\ i = 1, \dots, n, \varepsilon_i = 0$ or $\varepsilon_i = 1\}$. Note that this definition differs quite unessentially from the one given in [17]; see also [16].

(4.2) **Theorem.** *In every Urysohn*[1]) *countably compact unscattered space X there exists a closed separable subspace H which can be mapped onto I by a closed mapping.*

Proof. Without loss of generality we assume that X is dense in itself. Take $\mathscr{U}_0 = \{U_0^0, U_0^1\}$, where U_0^0 and U_0^1 are arbitrary nonempty sets satisfying $[U_0^0] \cap [U_0^1] = \emptyset$. Assume that for all $k \leqq n - 1$ we have defined pairs of nonempty open sets $\mathscr{U}_k = \{U_k^0, U_k^1\}$ in such a way that the system $\{\mathscr{A}_k = \{[U_k^0], [U_k^1]\} : k = 0, \dots, n - 1\}$ is dyadic and consider the family $\mathscr{G}_n = \bigwedge\{\mathscr{U}_k : k \leqq n - 1\}$. It is disjoint and consists of 2^n nonempty open set: $\mathscr{G}_n = \{G_i : i = 0, \dots, 2^n - 1\}$. Since for every $= 0, \dots, 2^n - 1$ the set G_i is open and hence contains more than one point, there exist nonempty open sets V_i^0, V_i^1 such that $[V_i^0] \cap [V_i^1] = \emptyset$ and $V_i^0 \cup V_i^1 \subset G_i$. Let now $U_n^\varepsilon = \bigcup\{V_i^\varepsilon : i = 0, \dots, 2^n - 1\}$, $\varepsilon = 0, 1$ and $\mathscr{A}_n = \{[U_n^0], [U_n^1]\}$. Thus we obtain a dyadic system $\{\mathscr{A}_k : k = 0, \dots, n\}$. Obviously, the system $\mathscr{A} = \{\mathscr{A}_n : n < \omega_0\}$ is dyadic as well.

Since the space X is countably compact, the family $\mathscr{G} = \bigwedge\{\mathscr{A}_n : n < \omega_0\}$ does not contain empty sets. Define $F = \bigcap\{[U_n^0] \cup [U_n^1] : n < \omega_0\}$; $F_n^\varepsilon = F \cap [U_n^\varepsilon]$, where $\varepsilon = 0, 1$. Note that $F_n^0 \cup F_n^1 = F$ and $F_n^0 \cap F_n^1 = \emptyset$ for every $n < \omega_0$, while all F_n are nonempty.

For every $n < \omega_0$ consider a mapping $f_n : F \to D_n = \{0, 1\}$ such that $f_n(F_n^\varepsilon) = \{\varepsilon\}$, $\varepsilon = 0, 1$. It is easy to notice that the diagonal mapping $f = \Delta\{f_n : n < \omega_0\} : F \to D^{\omega_0} = \prod\{D_n : n < \omega_0\}$ maps the space F onto the space D^{ω_0}. Really, for every $x \in D^{\omega_0}$ according to the definition of f we have

$$f^{-1}(x) = \bigcap\{f_n^{-1}(x) : n < \omega_0\} = \bigcap\{F_n^{\varepsilon(x,n)} : n < \omega_0\} = \\ = F \cap (\bigcap\{[U_n^{\varepsilon(x,n)}] : n < \omega_0\} = \bigcap\{[U_n^{\varepsilon(x,n)}] : n < \omega_0\} .$$

(Here $\varepsilon(x, n) = 0$ if $x \in U_n^0$ and $\varepsilon(x, n) = 1$ if $x \in U_n^1$.) From the definition of the structural intersection it follows immediately that

$$f^{-1}(x) = \bigcap\{[U_n^{\varepsilon(x,n)}] : n < \omega_0\} \in \mathscr{G} = \bigwedge\{\mathscr{A}_n : n < \omega_0\}$$

and hence (as $\mathscr{A}$ is dyadic) $f^{-1}(x) \neq \emptyset$.

Take an arbitrary countable set N in F such that $[f(N)] = D^{\omega_0}$. Since $[N]$ is countably compact, $f[N]$ is countably compact in D^{ω_0} and hence $f[N] = [f(N)] = D^{\omega_0}$.

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[1]) Recall that a space X is called Urysohn if for every two points $x_1, x_2 \in X$ there exist neighbourhoods U_{x_1} and U_{x_2} such that $[U_{x_1}] \bigcap [U_{x_2}] = \emptyset$.

Let $H = [N], f' = f|_H : H \to D^{\omega_0}$ and consider a surjection $r : D^{\omega_0} \to I$ (see [16]). To complete the proof one has to notice only that the composition $h = r \circ f'$ is a mapping of a separable countably compact closed subspace H of X onto I.

(4.3) **Remark.** This theorem was proved for compacta and Čech complete spaces by one of the authors in the course of writing the paper [19] (see also [16], [20], [21] etc.). Furthermore, for these classes of spaces the converse is true as well. Namely, if there exists a compactum $H \subset X$ which can be mapped onto I, then X is unscattered. Really, in this case the corresponding mapping $f : H \to I$ may be chosen *irreducible*, therefore there exists a dense in itself subset $S \subset H$ and hence X is unscattered. On the other hand, for a countably compact space X this is not true. The corresponding example constructed under certain additional set theoretic assumptions is contained in [21].

(4.4) **Remark.** One can easily generalize Theorem (4.2) to the class of $G_{I\delta}$subsets of countably compact Urysohn spaces. We shall now formulate the most general result of this type (see (4.5)). Since it will not be further used in the paper the proof is omitted.

(4.5) **Proposition.** *Suppose X is an Urysohn unscattered space and there exists a sequence of π-bases $\mathscr{B} = \{\mathscr{B}_n : n < \omega_0\}$ (see e.g. [16]) with the following property:*

(p) *if $\mathscr{G} = \{G_n : n < \omega_0\}$ is a sequence of sets where $G_n \in \mathscr{B}_n$ and $[G_n] \subset G_{n+1}$, then $\bigcap \mathscr{G} \neq \emptyset$.*

In this case there exist a closed surjection $f : H \to I$ where H is a closed subset of X. If, moreover,

(p′) *$\mathscr{G}$ is a base for the set $\bigcap \mathscr{G}$,*

then H can be chosen separable.

It is easy to see that every G_δ subset of a countably compact Urysohn space has a countable system of π-bases with the properties (p) and (p′). Therefore Theorem (4.2) is also a corollary of this proposition.

The second group of problems considered here concerns the problem which is in a known sense opposite to the first one. Namely, we ascertain here some cases when the image $f(X)$ of a mapping $f : X \to I$ is so "small" in I that $I \setminus f(X)$ contains D^{ω_0}. The central result in this group is the following theorem.

(4.6) **Theorem.** *If there is no mapping of a space X onto I, then $D^{\omega_0} \subsetneq I \setminus f(X)$ for every mapping $f : X \to I$.*

Proof. Let $f : X \to I$ be a mapping such that $I \neq f(X)$ and consider a surjection $r : I \to I^{\omega_0} = \prod\{I_n : n < \omega_0\}$ (here $I_n = I$ for every $n < \omega_0$; the existence of such a mapping is ensured e.g. by [16]). Now let $\tilde{f} = r \circ f : X \to I^{\omega_0}$ and $\tilde{f}_n = \pi_n \circ f : X \to I_n$, where $\pi_n : I^{\omega_0} \to I_n$ is the corresponding projection.

For every $n < \omega_0$ take a point $b_n \in I_n \setminus \tilde{f}_n(X)$ and let $a = (a_n)_{n \in N}$ be an arbitrary point in I^{ω_0}. Denote $C(a) = \prod\{\{a_n, b_n\} : n < \omega_0\}$. We shall first show that $C(a) \setminus \setminus \{a\} \subset I^{\omega_0} \setminus \tilde{f}(X)$.

Really, if $x \in C(a)$, $x \neq a$, then there exists $n^* < \omega_0$ such that $\pi_{n^*}(x) = b_n$ and hence $\pi_{n^*}(x) \in I_n \setminus \pi_{n^*}(\tilde{f}(X))$. Therefore, obviously, $x \in I^{\omega_0} \setminus f(X)$.

Now, since $C(a) \approx D^{\omega_0}$, we may find $C^0 \subset C(a) \setminus \{a\}$ such that $C^0 \approx D^{\omega_0}$. Take a compactum $H \subset I$ for which $r(H) = C^0$ and $r|_H$ is irreducible (see e.g. [10], p. 179). It is easy to notice that r is a homeomorphism and therefore $H \approx D^{\omega_0}$.

Thus, to complete the proof we have to notice only that $H \subset I \setminus f(X)$, but this is obvious, since $H \subset r^{-1}(C^0) \subset r^{-1}(I^{\omega_0} \setminus \tilde{f}(X)) \subset I \setminus f(X)$.

An easy consequence of Theorem (4.6) is the following

(4.7) **Proposition.** *Let A be a subset of I containing D^{ω_0} and $M \subset I$, $|M| < \mathfrak{c}$. Then $A \setminus M$ also contains D^{ω_0}.*

Proof. In case $A = I$ this is an immediate corollary of the previous theorem. Suppose now that $A \approx D^{w_0}$ and consider a mapping r of A onto I (see [16] or Theorem (4.2)). If $M \subset A$, $|M| < \mathfrak{c}$, then $r(M) = M^0 \subset I$, $|M^0| < \mathfrak{c}$ and hence by (4.6), $D^{\omega_0} \approx B \subset I \setminus M^0$. Take now a compactum $H \subset A$ for which $r(H) = B$ and $r' = r|_H$ is irreducible. Then r' is a homeomorphism and hence, obviously, $D^{\omega_0} \approx \approx H \subsetneq A \setminus M$.

The following statement is well-known (see e.g. [5]).

(4.8) **Proposition.** *If F is a closed subset of I and $|F| > \aleph_0$, then $D^{\omega_0} \subsetneq F$ and hence $|F| = \mathfrak{c}$.*

Finally, we can get the following statement which is fundamental for us.

(4.9) **Proposition.** *If the mapping $f : X \to I$ is closed, $M \subset X$ and $|f(X)| > \aleph_0$ but $|f(M)| < \mathfrak{c}$ then there exists a closed separable subset H of X, such that $H \subset \subset X \setminus M$ and $D^{\omega_0} \approx f(H) \subset I \setminus f(M)$* [1].

Proof. According to (4.8), $D^{\omega_0} \subsetneq f(X)$ and hence by (4.6), $D^{\omega_0} \approx K \subset f(X) \setminus \setminus f(M)$. Take any countable $N \subset f^{-1}(K)$ such that $f(N) = K$ and let $H = N$. Obviously $f[N] = [f(N)]$ and therefore $[N] \subset X \setminus M$ and $f(H) = K$.

5. UNIVERSAL PARTITIONS

The following two definitions present a concretization and a specification of the notion of a partition; they will play the basic role for the induction in the next section.

[1]) In particular, $|H| \geqq \mathfrak{c}$ in this case.

Let τ be a cardinal, $\tau \geqq \aleph_1$, and let A be a subset of a topological space X.

(5.1) **Definition.** A *partition* $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ of a set A is called *τ-universal* (shortly, a u_τ-partition) of A in X if for every $\aleph_0$-closed subset F in X and every closed mapping $f : F \to I$ the inequality $|f(F \cap A)| \geqq \tau$ implies $|f(F \cap R_\alpha)| \geqq \mathfrak{c}$ for all $\alpha < \mathfrak{c}$. In case $X = A$ we call $\mathscr{R}$ a *u_τ-partition of the space X.*

(5.2) **Definition.** A subset A of X is called a *u_τ-set* if there exists a u_τ partition of A in X. In particular, if $A = X$, then X is called a *u_τ-space.*

(5.3) **Remark.** We shall usually write just "u-partition" and "u-set" instead of "$u_{\mathfrak{c}}$-partition" and "$u_{\mathfrak{c}}$-set", respectively.

(5.4) **Remark.** We extend *formally* the notions introduced in (5.1) and (5.2) to the case $\tau = \aleph_0$, defining a *$u_{\aleph_0}$-partition* and a *$u_{\aleph_0}$-set* as a $u_{\aleph_1}$-partition and a $u_{\aleph_1}$-set, respectively. Besides, we usually abbreviate them as "u_0-partition" and "u_0-set", respectively.

(5.5) **Remark.** Since $|I| = \mathfrak{c}$, it is obvious that the definitions (5.1) and (5.2) are substantial only if $\tau \leqq \mathfrak{c}$. Therefore in what follows when speaking about u_τ-partitions and u_τ-sets we shall always assume that $\aleph_0 \leqq \tau \leqq \mathfrak{c}$.

(5.6) **Remark.** If $\tau \leqq \sigma \leqq \mathfrak{c}$, then every u_τ-partition is obviously a u_σ-partition, too. In particular, every u_τ-partition (for $\tau \leqq \mathfrak{c}$) is also a u-partition. The following proposition offers the case when the converse is true.

(5.7) **Proposition.** *If A is an rc-closed subset in a topological space X and $\mathscr{R}$ is its u_τ-partition, then $\mathscr{R}$ is a u_0-partition of A, too. In particular, every u-partition of an rc-closed set is a u_0-partition.*

Proof. Let F be an $\aleph_0$-closed set in X. Then $A \cap F$ is an rc-closed set in F (3.8) and therefore for every closed mapping $f : F \to I$ the set $f(A \cap F)$ is rc-closed and hence also closed in I (see (3.11), (3.9)). If, moreover, $|f(A \cap F)| \geqq \tau > \aleph_0$ then by (4.8), $|f(A \cap F)| = \mathfrak{c}$.

(5.8) **Corollary.** *If A is an rc-closed u_τ-subset of a space X then A is also a u_0-subset of X.*

(5.9) **Definition.** A partition $\mathscr{R}$ of a space X is called *hereditary with respect to* a family $\mathscr{F} \subset 2^X$ if for every $F \in \mathscr{F}$ the system $\mathscr{R}_F = \{F \cap R : R \in \mathscr{R}\}$ is a partition of F.

(5.10) **Denotation.** Let $\mathscr{F}^*(X)$ denote the family of all $\aleph_0$-closed subsets F of X for which there exist closed mappings $f : F \to I$ such that $|f(F)| > \aleph_0$. The family of all oversets of elements F from $\mathscr{F}^*(X)$ will be denoted by $\mathscr{F}(X)$, i.e.

$$\mathscr{F}(X) = \{F \subset X : \exists F^* \subset F,\ F^* \in \mathscr{F}^*(X)\}\ .$$

Directly from the definition we get the following

(5.11) **Proposition.** *A u-partition $\mathscr{R}$ of a space X is hereditary with respect to $\mathscr{F}(X)$.*

6. INDUCTION

(6.1) **Denotation.** By $S_\tau(X \mid \mu)$, where X is a topological space, τ and μ are cardi-nals, we denote the following statement $S_\tau(X \mid \mu)$: "For every pair of subsets $A \subset B$ where B is rc-closed in X and $|A| \leqq \mu$ there exists a u_τ-set $\tilde{A}$ in X such that $A \subset \tilde{A} \subset B$ and $|\tilde{A}| \leqq \mu$.

(6.2) **Remark.** In conformity with (5.3) we write $S(X \mid \mu)$ instead of $S_{\mathfrak{c}}(X \mid \mu)$. Furthermore, in conformity with (5.4) we define additionally the statement $S_{\aleph_0}(X \mid \mu)$ as the statement $S_{\aleph_1}(X \mid \mu)$ and write simply $S_0(X \mid \mu)$ in this case.

(6.3) **Remark.** According to (5.5) the statement $S_\tau(X \mid \mu)$ is substantial only for $\tau \leqq \mathfrak{c}$. Therefore in what follows when writing $S_\tau(X \mid \mu)$ we shall always assume $\tau \leqq \mathfrak{c}$. Besides, it is easy to notice that the minimal μ for which $S_\tau(X \mid \mu)$ has an appropriate sense is $\mathfrak{c}$, so further we assume that $\mu \geqq \mathfrak{c}$.

(6.4) **Remark.** The remark (5.6) implies that if $\tau \leqq \sigma \leqq \mathfrak{c}$ and the statement $S_\tau(X \mid \mu)$ holds, then the statement $S_\sigma(X \mid \mu)$ holds as well.

(6.5) **Proposition.** *Let $S_\tau(X \mid \mu)$ hold for some cardinals τ and μ. If A is an rc-closed set in a space X and $|A| \leqq \mu$ then A is a u_0-set in X. In particular, $S_\tau(X \mid |X|)$ implies that X is a u_0-space.*

Proof. Let $B = A$, then by (6.1) A is a u_τ-set in X and therefore (see (5.7)) A is also a u_0-set in X.

If $\mu = \mu^{\aleph_0}$ then the previous proposition can be improved in the following way:

(6.6) **Proposition.** *If $\mu = \mu^{\aleph_0}$ then the statement $S_\tau(X \mid \mu)$ is equivalent to the statement $S_0(X \mid \mu)$. In particular, the statement $S(X \mid \mu)$ is equivalent to $S_0(X \mid \mu)$ in this case.*

Proof. Suppose $A \subset B$, $|A| \leqq \mu$ and B is an rc-closed set in X. Then by (3.15) there exists a set $\tilde{A}$ which is rc-closed in B, and therefore by (3.6) also in X, such that $A \subset \tilde{A} \subset B$ and $|\tilde{A}| \leqq \mu$. By (6.5), $\tilde{A}$ is a u_0-set and thus the statement $S_0(X \mid \mu)$ holds.

The aim of this section is, roughly speaking, to show that the statement $S_0(X \mid \mu)$ holds "very often". This will be proved by induction on the cardinal μ. We begin with proving the basis for this induction, i.e., the statement $S_0(X \mid \mathfrak{c})$ (recall that according to our assumption $\mu \geqq \mathfrak{c}$).

(6.7) **Lemma.** *The statement $S_0(X \mid \mathfrak{c})$ holds for every space X with $|X| \geqq \mathfrak{c}$.*

Proof. Let $A \subset B \subset X$, $|A| \leqq \mathfrak{c}$ and let B be rc-closed in X. By (3.16) and (3.6) there exists an rc-closed set $\tilde{A}$ in X such that $|\tilde{A}| = \mathfrak{c}$ and $A \subset \tilde{A} \subset B$. Now by (2.5) there exists a partition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ of $\tilde{A}$ in X such that

(*) $|R_\alpha \cap H| \geqq \mathfrak{c}$ for every $\alpha < \mathfrak{c}$ and every closed separable subset $H \subset \tilde{A}$, $|H| = \mathfrak{c}$.

Let now F be an $\aleph_0$-closed subset of X and $f : F \to I$ – a closed mapping such that $|f(F \cap \tilde{A})| > \aleph_0$. To complete the proof we have to show the inequality $|f(F \cap R_\alpha)| \geqq \mathfrak{c}$ for all α.

By (3.8) the set $\tilde{F} = F \cap \tilde{A}$ is rc-closed in F and hence the mapping $f' = f|_{\tilde{F}} : \tilde{F} \to I$ is also closed (see (3.13)). Suppose that $|f(R_{\alpha_0} \cap F)| < \mathfrak{c}$ for some α_0. Since obviously $R_\alpha \cap F = R_\alpha \cap \tilde{F}$ for every α, this inequality means that $|f'(R_{\alpha_0} \cap \tilde{F})| < \mathfrak{c}$ and therefore according to (4.9) there exists a separable closed subset H of $\tilde{F}$ such that $|H| \geqq \mathfrak{c}$ and $H \cap R_{\alpha_0} = \emptyset$. Since $\tilde{F}$ is $\aleph_0$-closed in $\tilde{A}$ the set H is also closed in $\tilde{A}$. The existence of such H contradicts (*). Hence $|f(F \cap R_\alpha)| \geqq \mathfrak{c}$ for all α and therefore R is a u_0-partition of $\tilde{A}$. Thus the statement $S_0(X \mid \mathfrak{c})$ holds.

To continue the induction we need first some simple facts about chains. We begin with the following well-known definition.

(6.8) **Definition.** *A chain* in a set X is a well-ordered by inclusion non-decreasing family $\mathscr{A}$ of subsets of X, i.e. $\mathscr{A} = \{A^\alpha : \alpha < \nu\}$, where ν is an ordinal and $A^\beta \subset A^\alpha$ iff $\beta < \alpha < \nu$.

(6.9) **Proposition.** *Let $\mathscr{A} = \{A^\alpha : \alpha < \nu\}$ be a chain in a space X and $|A^\alpha| < \tau$ for all $\alpha < \nu$. Then $|\bigcup\mathscr{A}| \leqq \tau$. Moreover,*

(6.9.a) *if* $\mathrm{cf}(\tau) \neq \mathrm{cf}(\nu)$, *then* $|\bigcup\mathscr{A}| < \tau$;

(6.9.b) *if all $A^\alpha \in \mathscr{A}$ are closed and $\bigcup\mathscr{A} \neq A^\alpha$ for $\alpha < \nu$, then* $\mathrm{cf}(\nu) \leqq d(\bigcup\mathscr{A})$;

(6.9.c) *if all $A^\alpha \in \mathscr{A}$ are closed, $\bigcup\mathscr{A} \neq A^\alpha$ for $\alpha < \nu$ and $d(\bigcup\mathscr{A}) < \mathrm{cf}(\tau)$, then* $|\bigcup\mathscr{A}| < \tau$.

Proof. Consider an arbitrary subset $M \subset \bigcup\mathscr{A}$ such that $|M| = \tau$ and for every $x \in M$ fix an ordinal $\alpha(x) < \nu$ satisfying $x \in A^{\alpha(x)}$. Let $\beta = \sup\{\alpha(x) : x \in M\}$; it is easy to notice that $\beta = \nu$ (otherwise $|A^\beta| \geqq |M| = \tau$!). Hence $\mathrm{cf}(\nu) \leqq |M| = \tau$ and therefore $|\bigcup\mathscr{A}| \leqq \tau\,\mathrm{cf}(\nu) = \tau$.

Assume now that $|\bigcup\mathscr{A}| = \tau$ and for every $\lambda < \tau$ fix an ordinal $\alpha(\lambda) < \nu$ such that $|A^{\alpha(\lambda)}| > \lambda$ (otherwise $|\bigcup\mathscr{A}| < \lambda\,\mathrm{cf}(\nu) \leqq \lambda\tau = \tau$!), therefore $\mathrm{cf}(\tau) \leqq \mathrm{cf}(\nu)$. Since the converse inequality $\mathrm{cf}(\nu) \leqq \mathrm{cf}(\tau)$ is obvious we get $\mathrm{cf}(\tau) = \mathrm{cf}(\nu)$ in this case and thus (6.9.a) holds.

To prove (6.9.b) choose $M \subset \bigcup\mathscr{A}$ such that $[M] \supset \bigcup\mathscr{A}$ and $|M| = d(\bigcup\mathscr{A})$ and let $\beta = \sup\{\alpha(x) : x \in M\}$. Notice that in this case $\beta = \nu$, too (otherwise $A^\beta \supset M$ and hence $A^\beta = [A^\beta] \supset \bigcup\mathscr{A}$), and therefore $\mathrm{cf}(\nu) \leqq |M| = d(\bigcup\mathscr{A})$.

Now (6.9.c) follows immediately from (6.9.a) and (6.9.b).

(6.10) **Denotation.** Let $\mathscr{A} = \{A^\xi : \xi < \nu\}$ be a chain in X and $\mathscr{R}^\xi = \{R^\xi_\alpha : \alpha < c\}$ a partition of A^ξ. For every $\xi < \nu$ and $\alpha < \mathfrak{c}$ let $\tilde{A}^\xi = \bigcup\{A^{\xi'} : \xi' < \xi\}$, $\tilde{R}^\xi_\alpha = R^\xi_\alpha \setminus \tilde{A}^\xi$ and $\tilde{R}_\alpha = \bigcup\{\tilde{R}^\xi_\alpha : \xi < \nu\}$. It is easy to notice that $\mathscr{R}^* = \{\tilde{R}_\alpha : \alpha < \mathfrak{c}\}$ is a partition of the set $\tilde{A} = \bigcup\mathscr{A}$. The following lemma specifies some properties of this partition.

(6.11) **Lemma.** *For every $\xi < \nu$ let $\mathscr{R}^\xi$ be a u_τ-partition of the set A^ξ in X.*

(6.11.a) *If $\tau < \mathfrak{c}$, then $\mathscr{R}^*$ is a $u_{\tau+}$-partition of A^ξ in X.*

(6.11.b) *If $\tau < \mathfrak{c}$ and* $\mathrm{cf}(\tau) \neq \mathrm{cf}(\nu)$, *then $\mathscr{R}^*$ is a u_τ-partition of the set $\tilde{A}$ in X.*

(6.11.c) *If $\tau < \mathfrak{c}$ and $\tilde{A}$ is rc-closed in X, then $\mathscr{R}^*$ is a u_0-partition of $\tilde{A}$ in X.*

(6.11.d) *If every A^ξ is rc-closed in X, then $\mathscr{R}^*$ is a u_0-partition of $\tilde{A}$ in X.*

Proof. We first prove the statements (6.11.a) and (6.11.b). Take an $\aleph_0$-closed subset F of X and a closed mapping $f : F \to I$ satisfying $|f(F \cap \tilde{A})| \geqq \tau^+$ in case (a) and $|f(F \cap \tilde{A})| \geqq \tau$ in case (b). It is obvious that $\{f(F \cap A^\xi) : \xi < \nu\}$ is a chain in I. Now by (6.9) in case (a) and by (6.9.a) in case (b) there exists a cardinal $\xi_0 < \nu$, such that

$(*)$ $|f(F \cap A^{\xi_0})| \geqq \tau$ but

$(**)$ $|f(F \cap \tilde{A}^{\xi_0})| \leqq \tau < \mathfrak{c}$.

(Notice that ξ_0 can be defined as $\xi_0 = \min\{\xi < \nu : |f(F \cap A^\xi)| \geqq \tau.\}$ Since $\mathscr{R}^{\xi_0}$ is a u_τ-partition of the set A^{ξ_0} in X, the inequality $(*)$ implies that $|f(F \cap R^{\xi_0}_\alpha)| \geqq \mathfrak{c}$ for every $\alpha < \xi_0$. Furthermore, $f(F \cap \tilde{R}_\alpha) \supset f(F \cap \tilde{R}^{\xi_0}_\alpha) \supset (f(F \cap R^{\xi_0}_\alpha) \setminus f(F \cap \tilde{A}^{\xi_0}_\alpha))$ and hence by $(**)$ $|f(F \cap \tilde{R}_\alpha)| \geqq \mathfrak{c}$ for every α. But this means exactly that $\mathscr{R}^*$ is a $u_{\tau+}$-partition in case (a) and a u_τ-partition in case (b) of the set $\tilde{A} = \cup\mathscr{A}$ in X.

If moreover, $\tilde{A}$ is rc-closed in X then by (5.7) we conclude that $\mathscr{R}^*$ is a u_0-partition of $\tilde{A}$ and thus (6.11.c) holds, too. We outline now the proof of (6.11.d). By (3.8), (3.10), (3.6) and (3.13) in this case every $f(F \cap A^\xi)$ is closed in I and therefore, applying (6.9.c) to the chain $\{f(F \cap A^\xi) : \xi < \nu\}$ and taking into consideration Theorem (4.8) one can conclude that there exists an ordinal $\xi_0 < \nu$, satisfying

$(*')$ $|f(F \cap A^{\xi_0})| \geqq \aleph_1$,

$(*'*')$ $|f(F \cap \tilde{A}^{\xi_0})| \leqq \aleph_0 < \aleph_1 \leqq \tau < \mathfrak{c}$.

$\mathscr{R}^{\xi_0}$ is a u_τ-partition of A^{ξ_0} in X and hence also a u_0-partition. Therefore the inequality $(*')$ implies that $|f(F \cap R^{\xi_0}_\alpha)| \geqq \mathfrak{c}$ for every $\alpha < \xi_0$. Now in the same way as in the proof of (6.11.a) one can show that $|f(F \cap \tilde{R}_\alpha)| \geqq \mathfrak{c}$ for every $\alpha < \mathfrak{c}$. But this means exactly that $\mathscr{R}^*$ is a u_0-partition of the set $\tilde{A}$ in X and thus (6.11.d) is proved.

(6.12) Recall that a cardinal ν is called $\aleph_0$-unaccessible (see e.g. [11]) if $\lambda^{\aleph_0} < \nu$ for every $\lambda < \nu$. Obviously a cardinal $\nu = (\mu^{\aleph_0})^+$ is $\aleph_0$-unaccessible.

(6.13) **Lemma.** *Let the statement $S_\tau(X \mid \lambda)$ hold for every $\lambda \in [\mu, \nu[$.*

(6.13.a) *If $\tau < \mathfrak{c}$, then $S_{\tau^+}(X \mid \nu)$ holds.*

(6.13.b) *If $\tau < \mathfrak{c}$ and $\mathrm{cf}(\tau) \neq \mathrm{cf}(\nu)$, then $S_\tau(X \mid \nu)$ holds.*

(6.13.c) *If $\tau < \mathfrak{c}$ and $\nu = \nu^{\aleph_0}$, then $S_0(X \mid \nu)$ holds.*

(6.13.d) *If ν is $\aleph_0$-unaccessible, then $S_0(X \mid \nu)$ holds.*

Proof. Consider a pair of subsets $A \subset B$ where $|A| \leqq \nu$ and B is rc-closed in X. Express A as $A = \{a_\alpha : \alpha < \nu\}$. To prove (6.13.a) and (6.13.b) assume that for every α, $\alpha < \beta < \nu$ we have defined sets $A^\alpha \subset X$ in such a way that

(*) $a_\alpha \in A^\alpha \subset B$ for every α,

(**) $|A^\alpha| \leqq \max(|\alpha|, \mu)$,

(***) if $\alpha' < \alpha$, then $A^{\alpha'} \subset A^\alpha$,

(i*) A^α is a u_τ-set in X.

Let $\bar{A}^\beta = \{a_\beta\} \cup (\bigcup\{A^\alpha : \alpha < \beta\})$. Obviously $|\bar{A}^\beta| \leqq |\beta|\, \mu < \nu$. Therefore, applying $S_\tau(X \mid \lambda)$, where $\lambda = |\beta|\, \mu$, to a pair $\bar{A}^\beta \subset B$ we conclude that there exists a u_τ-set $\tilde{A}^\beta$ such that $|\tilde{A}^\beta| \leqq \lambda$ and $\bar{A}^\beta \subset \tilde{A}^\beta \subset B$. Take $A^\beta = \tilde{A}^\beta$. It is evident that A^β satisfies the conditions (*)–(i*) and hence one can continue the induction and get a chain of sets $\mathscr{A} = \{A^\alpha : \alpha < \nu\}$ satisfying the conditions (*)–(i*). Let $\tilde{A} = \bigcup\mathscr{A}$. Then $A \subset \tilde{A} \subset B$ (by (*)) and $|\tilde{A}| \leqq \nu$ (by (**)). Applying Proposition (6.11.a) in the case (a) and (6.11.b) in the case (b) we obtain that $\tilde{A}$ is a u_{τ^+}-set in the case (a) and a u_τ-set in the case (b). Thus (6.13.a) and (6.13.b) are proved. To get (6.13.c) we must only apply (6.6) to (6.13.a).

To prove (6.13.d) we construct by induction a chain of sets $\mathscr{A} = \{A^\alpha : \alpha < \nu\}$ satisfying the conditions (*), (*'*'), (***), (i'*'), where

(*'*') $|A^\alpha| \leqq (\max(|\alpha|, \mu))^{\aleph_0}$,

(i'*') A^α is an rc-closed u_0-set in X.

If such sets A are already constructed for all $\alpha < \beta$ let $\bar{A}^\beta = \{a_\beta\} \cup (\bigcup\{A^\alpha : : \alpha < \beta\})$. By (3.15) there exists an rc-closed set $\tilde{A}^\beta$ such that $\bar{A}^\beta \subset \tilde{A}^\beta \subset B$ and $|\tilde{A}^\beta| \leqq |\bar{A}^\beta|^{\aleph_0} = (|\beta|\, \mu)^{\aleph_0} = \lambda < \nu$. According to (6.5), $\tilde{A}^\beta$ is a u_0-set in X in this case, therefore we may define $A^\beta = \tilde{A}^\beta$ and continue the induction. We complete the proof of (6.13.d) in the same manner as in the previous cases but employ (6.11.d) instead of (6.11.a) and thus conclude that A is a u_0-set in this case.

Applying the previous lemma for $\nu = \mu^+$ we obtain immediately the following

(6.14) **Corollary.** *Let the statement $S_\tau(X \mid \mu)$ hold.*

(6.14.b) *If* $\tau < \mathfrak{c}$, *then* $S_\tau(X \mid \mu^+)$ *holds.*

(6.14.c) *If* $\tau < \mathfrak{c}$ *and* $(\mu^+)^{\aleph_0} = \mu^+$, *then* $S_0(X \mid \mu^+)$ *holds.*

(6.14.d) *If* $\mu^{\aleph_0} = \mu$, *then* $S_0(X \mid \mu^+)$ *holds.*

(6.15) **Denotation.** For every ordinal α and a fixed cardinal μ define cardinals μ_α by induction. Let $\mu_0 = \mu$, for $\alpha = \beta + 1$ define $\mu_\alpha = \mu_\beta^+$ and if α is a limit ordinal let $\mu_\alpha = \sup\{\mu_\beta : \beta < \alpha\}$.

Thus μ_α is the α-th cardinal in the class $K_\mu = \{\nu : \nu \geqq \mu\}$ of all cardinals which are not less than μ and ordered by the natural order.

(6.16) **Lemma.** *Let the statement* $S_\tau(X \mid \mu)$ *hold for some cardinals* τ *and* μ.

(6.16.a) *If* $\tau < \mathfrak{c}$ *and* τ *is regular, then for every* $\lambda \in [\mu, \mu_\tau[$ *the statement* $S_\tau(X \mid \lambda)$ *holds. Moreover, the statement* $S_{\tau^+}(X \mid \mu_\tau)$ *holds, too.*

(6.16.b) *If* $\aleph_1 < \mathfrak{c}$, *then for every* $\lambda \in [\mu, \mu_{\aleph_1}[$ *the statement* $S_0(X \mid \lambda)$ *holds. Moreover, the statement* $S_{\aleph_2}(X \mid \mu_{\aleph_1})$ *holds, too.*

(6.16.c) *If* $\tau = \aleph_1 < \mathfrak{c}$ *and* $\nu < \mathfrak{c}$, *then for every* $\lambda \in [\mu, \mu_\nu[$ *the statement* $S_\nu(X \mid \lambda)$ *holds. Moreover, the statement* $S_{\nu^+}(X \mid \mu_\nu)$ *holds, too.*

(6.16.d) *If* $\mu^{\aleph_0} = \mu$, *then for every* $\lambda \in [\mu, \mu_{\omega_0}]$ *the statement* $S_0(X \mid \lambda)$ *holds.*

Proof. Let $\lambda = \mu_\alpha$ where $0 < \alpha < \tau$ and assume that for every $\beta < \alpha$ the corresponding statement of the lemma is proved. To prove (6.16.a) notice first that for a regular cardinal the inequality $\mathrm{cf}(\mu_\alpha) \neq \mathrm{cf}(\tau)$ holds (if α is a limit cardinal, then $\mathrm{cf}(\mu_\alpha) = \mathrm{cf}(\alpha) \leqq |\alpha| < \tau = \mathrm{cf}(\tau)$, if $\alpha = \beta + 1$, then $\mu_\alpha = \mu_\beta^+$ and hence $\mathrm{cf}(\mu_\alpha) = \mu_\alpha \geqq \mathfrak{c} > \tau = \mathrm{cf}(\tau)$). Therefore, applying (6.13.b) we get the statement $S_\tau(X \mid \lambda)$ and therefore also the statement $S_\tau(X \mid \lambda)$ for every $\lambda \in [\mu, \mu_\tau]$. Moreover, applying (6.13.a) we get $S_{\tau^+}(X \mid \mu_\tau)$ and thus complete the proof of (6.16.a).

Proposition (6.16.b) is a trivial corollary of (6.16.a).

To prove (6.16.c) we may assume that $\nu \geqq \aleph_1$ (since the case $\nu < \aleph_1$ is contained in (6.16.b.)!). Take $\alpha < \nu$ and let $\sigma = |\alpha|^+ \aleph_1$. Obviously, $\aleph_1 \leqq \sigma \leqq \nu < \mathfrak{c}$ and $\mathrm{cf}(\sigma) = \sigma$. As the statement $S_{\aleph_1}(X \mid \mu)$ holds by assumption and $\sigma \geqq \aleph_1$, therefore the statement $S_\sigma(X \mid \mu)$ is true, too. Now applying (6.16.a) to $S_\sigma(X \mid \mu)$ we get the statement $S_\sigma(X \mid \lambda)$ for every $\lambda \in [\mu, \mu_\sigma]$ and therefore, obviously, the statement $S_\nu(X \mid \lambda)$ holds for every $\lambda \in [\mu, \mu_\nu]$. To complete the proof of (6.16.c) we only have to apply (6.13.a) once again and get $S_{\nu^+}(X \mid \mu_\nu)$.

To prove (6.16.d) notice first that if $\alpha \leqq \omega_0$ and $\mu^{\aleph_0} = \mu$, then μ_α is $\aleph_0$-unaccessible. Therefore (6.16.d) is an immediate corollary of (6.13.d).

(6.17) **Lemma.** *Let the statement* $S(X \mid \mu)$ *hold for a space* X *and a cardinal* $\mu = \mu^{\aleph_0}$. *Then for every* $\lambda \in [\mu, \mu_\nu[$ *where* $\nu < \mathfrak{c}$ *the statements* $S_\nu(X \mid \lambda)$ *and* $S_{\nu^+}(X \mid \mu_\nu)$ *hold. Therefore* $S(X \mid \lambda)$ *holds for every* $\lambda \in [\mu, \mu_\nu[$.

Proof. If $\nu \leqq \aleph_0$ then the statement of the lemma follows directly from (6.16.d). Consider now $\nu \in [\aleph_1, \mathfrak{c}[$. By (6.6) the statement $S(X \mid \mu)$ is equivalent in this case to the statement $S_0(X \mid \mu)$ and hence we can apply Proposition (6.16.c) and get the statements $S_\nu(X \mid \lambda)$ and $S_{\nu^+}(X \mid \mu_\nu)$ for all $\lambda \in [\mu, \mu_\nu[$.

Since $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$, from Lemmas (6.7) and (6.17) we obtain the following

(6.18) **Proposition.** *Let X be an arbitrary topological space and $\tau < \mathfrak{c}$. Then*

(6.18.a) *$S_\tau(X \mid \lambda)$ holds for every $\lambda \in [\mathfrak{c}, \mathfrak{c}_\tau[$,*

(6.18.b) *$S_{\tau^+}(X \mid \mathfrak{c}_\tau)$ holds,*

and therefore

(6.18.c) *the statement $S(X \mid \lambda)$ holds for every $\lambda \in [\mathfrak{c}, \mathfrak{c}_\tau[$.*

(6.19) **Proposition.** *If there exists $\tau < \mathfrak{c}$ such that the inequality $\mathfrak{c} \leqq |X| \leqq \mathfrak{c}_\tau$ holds, then X is a u_0-space.*

Proof. By Proposition (6.18.c) the statement $S(X \mid \lambda)$ holds for every $\lambda \in [\mathfrak{c}, \mathfrak{c}_\tau]$ and in particular, for $\lambda = |X|$. Hence by (6.5), X is a u_0-space.

(6.20) **Corollary.** *Let $|X| \geqq \mathfrak{c}$. Then X is a u_0-space in each of the following cases:*

(6.20.a) *if $|X| \leqq \mathfrak{c}_{\omega_0}$;*

(6.20.b) *if $|X| \leqq \mathfrak{c}_{\omega_1}$ and $\aleph_1 < \mathfrak{c}$;*

(6.20.c) *if $|X| < \mathfrak{c}_\mathfrak{c}$ and $\mathfrak{c}$ is a limit cardinal.*

7. SOME SET-THEORETIC AXIOMS

To apply the results of Section 6 for deriving our main theorems in the next section we need some additional set-theoretic axioms which are discussed below.

It is well-known that the statement "$\mu^{\aleph_0} \leqq \mu^+$ for every infinite cardinal μ" does not depend on the ZFC system of axioms [11]. Moreover, it was already used by some authors to obtain topological results (see e.g. [22], [23]).

(7.1) **Denotation.** By ACP_1 (The First Countable Power Axiom) we denote the following statement

$\mathrm{ACP}_1 \equiv$ "$\mu^{\aleph_0} \leqq \mu^+$ for every cardinal $\mu \geqq \mathfrak{c}$".

In general for an arbitrary ordinal $\alpha > 0$, $0 < \alpha < \mathfrak{c}$, let

$\mathrm{ACP}_\alpha \equiv$ "$\mu^{\aleph_0} \leqq \mu_\alpha$ for every cardinal $\mu \geqq \mathfrak{c}$" (see (6.15)).

Besides, if $\alpha > 1$, let

$\mathrm{ACP}(\alpha) \equiv$ "$\mu^{\aleph_0} < \mu_\alpha$ for every cardinal $\mu \geqq \mathfrak{c}$".

(7.2) **Remark.** It is obvious that for every ordinal α, $0 < \alpha < \mathfrak{c}$, the stament ACP_α is equivalent to the statement $\mathrm{ACP}(\alpha + 1)$. Moreover, if $\beta \geqq \alpha$, then the statement $\mathrm{ACP}(\alpha)$ implies the statement $\mathrm{ACP}(\beta)$.

(7.3) **Denotation.** By ACP (The General Countable Power Axiom) we denote the following statement

ACP $\equiv$ "For every cardinal $\mu \geqq \mathfrak{c}$ there exists a cardinal $\tau < \mathfrak{c}$ such that $\mu^{\aleph_0} \leqq \mu_\tau$".

(7.4) **Denotation.** By $\mathrm{ACP}^{\#}$, $\mathrm{ACP}(\alpha)^{\#}$ and $\mathrm{ACP}_\alpha^{\#}$ we denote the statements $\mathrm{ACP} \,\&\, (\aleph_1 < \mathfrak{c})$, $\mathrm{ACP}(\alpha) \,\&\, (\aleph_1 < \mathfrak{c})$ and $\mathrm{ACP}_\alpha \,\&\, (\aleph_1 < \mathfrak{c})$, respectively.

(7.5) **Remark.** It is known (see e.g. [11]) that $\mathrm{ACP}_1^{\#}$ is independent of the ZFC system of axioms, hence the rest of the statements considered in (7.4) are also independent of the ZFC system of axioms.

(7.6) **Remark.** From (7.2) it follows that if $\alpha > 1$ then

$\mathrm{ACP}_1^{\#} \Rightarrow \mathrm{ACP}(\alpha)^{\#} \Rightarrow \mathrm{ACP}(\alpha + 1)^{\#} \Leftrightarrow \mathrm{ACP}_\alpha^{\#}$.

If, moreover, $|\alpha|^+ < \mathfrak{c}$, then

$\mathrm{ACP}_\alpha^{\#} \Rightarrow \mathrm{ACP}^{\#}$.

In particular,

$\mathrm{ACP}_1^{\#} \Rightarrow \mathrm{ACP}(\omega_0)^{\#} \Rightarrow \mathrm{ACP}(\omega_1)^{\#} \Leftrightarrow \mathrm{ACP}_{\omega_0}^{\#} \Rightarrow \mathrm{ACP}^{\#}$.

(7.7 **Remark.** It is easy to notice that $\mathrm{ACP}^{\#}$ is equivalent to the following statement:

$[\mathrm{ACP}(\mathfrak{c}) \,\&\, (\mathfrak{c}$ is a limit cardinal$)] \vee \mathrm{ACP}_\tau \,\&\, (\mathfrak{c} = \tau^+ > \aleph_1)$.

It is just the statement $\mathrm{ACP}^{\#}$, the weakest of the assumptions considered above, which will be employed in the next section to derive the main results.

8. THE MAIN RESULTS

(8.1) **Theorem** $[\mathrm{ACP}^{\#}]$. *For every topological space X with $|X| \geqq \mathfrak{c}$ and every cardinal $\lambda \geqq \mathfrak{c}$ there exists a cardinal $\tau < \mathfrak{c}$, for which the statement $S_\tau(X \mid \lambda)$ holds. Hence the statement $S(X \mid \lambda)$ holds for every space X and every cardinal $\lambda \geqq \mathfrak{c}$.*

Proof. According to Lemma (6.7) the statement $S_0(X \mid \mathfrak{c})$ holds for a space X and hence the statement $S(X \mid \mathfrak{c})$ holds, too. Assume that the statement $S(X \mid \nu)$ is already proved for every $\nu \in [\mathfrak{c}, \lambda[$.

Let $\mu = \lambda^{1/\aleph_0}$ (i.e. $\mu = \min\{\alpha : \alpha^{\aleph_0} \geqq \lambda\}$).

Obviously μ is $\aleph_0$-unaccessible and $\mu \in [\mathfrak{c}, \lambda[$, hence by (6.13.d) the statement $S_0(X \mid \mu)$ holds.

According to $\mathrm{ACP}^{\#}$ there exists a cardinal τ such that $\mu^{\aleph_0} \leqq \mu_\tau$ and therefore

$$\mu \leqq \lambda \leqq \mu^{\aleph_0} \leqq \mu_\tau .$$

Consider now the three cases:

(1) If $\mu < \lambda < \mu^{\aleph_0}$, then obviously $\lambda \in [\mu, \mu_\tau[$ and therefore by (6.16.c) the statement $S_\tau(X \mid \lambda)$ holds.

(2) If $\lambda = \mu^{\aleph_0}$, then $\lambda \in [\mu, \mu_\tau]$ and Proposition (6.16.c) allows to conclude that at least the statement $S_{\tau^+}(X \mid \lambda)$ holds. But applying (6.6) we get that in this case the statement $S_0(X \mid \lambda)$ is also true.

(3) In the last case, i.e. $\lambda = \mu$, the statement $S_0(X \mid \lambda)$ has been already obtained before.

Thus in each of the cases the statement $S_\tau(X \mid \lambda)$ holds for some $\tau < \mathfrak{c}$.

The previous theorem together with Proposition (6.5) immediately imply the following

(8.2) **Corollary.** [ACP$^\#$] *Every topological space X with $|X| \geqq \mathfrak{c}$ is a u_0-space.*

With the help of Proposition (5.11) we can combine Proposition (6.19) and the above lemma in the following way:

(8.3) **Theorem.** *For a topological space X with $|X| \geqq \mathfrak{c}$ there exists a partition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ which is hereditary with respect to the family $\mathscr{F}(X)$* [1]) *in each of the following cases*:

(8.3.a) *if $|X| \leqq \mathfrak{c}_\tau$ for some cardinal $\tau < \mathfrak{c}$,*

(8.3.b) *if* ACP$^\#$ *holds.*

The following theorem is just another version of the previous one.

(8.4) **Theorem.** *For a topological space X with $|X| \geqq \mathfrak{c}$ there exists a partition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ which is a simultaneous decomposition of all closed regular countably compact subsets F without isolated points (i.e. $[F \cap R_\alpha] = F$ for every such F and all $\alpha < \mathfrak{c}$), if either* (8.3.a) *or* (8.3.b) *holds.*

Proof. Notice that according to (4.2) every closed countably compact subset F belongs to $\mathscr{F}^*(X)$. If, moreover, F is regular and has no isolated points then every its open subset U belongs to $\mathscr{F}(X)$. Therefore Theorem (8.3) implies that $[F \cap R_\alpha] = F$ for every $R_\alpha \in \mathscr{R}$.

(8.5) **Corollary.** *If X is a Hausdorff compactum without isolated points then under the assumption of* (8.3.a) *or* (8.3.b) *there exists a decomposition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ such that for every closed subset F without isolated points $[F \cap R_\alpha] = F$ holds for every $\alpha < \mathfrak{c}$.*

The next two theorems (8.6) and (8.8) which we assume to be the main results of the work are obvious corollaries of Theorem (8.3) or (8.4).

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[1]) See (5.9), (5.10).

(8.6) **Main theorem I.** *For every topological space X satisfying $\mathfrak{c} \leqq |X| \leqq \mathfrak{c}_\tau$ for some $\tau < \mathfrak{c}$ (in particular, $\tau \leqq \omega_0$) there exists a partition $\mathscr{R} = \{R_0, R_1\}$ of X such that every its countably compact $\aleph_0$-closed regular subspace F contained in R_ε $(\varepsilon = 0, 1)$ is scattered.*

(8.7) **Corollary.** *If X is a Hausdorff space and $\mathfrak{c} \leqq |X| \leqq \mathfrak{c}_\tau$ for some $\tau < \mathfrak{c}$ (in particular, $\tau \leqq \omega_0$) then there exists a partition $\mathscr{R} = \{R_0, R_1\}$ of X such that every compactum F contained in R_ε $(\varepsilon = 0, 1)$ is scattered.*

(8.8) **Main theorem II** [ACP$^\#$]. *For every topological space X there exists a partition $\{R_0, R_1\}$ such that every its countably compact $\aleph_0$-closed regular subspace F contained in R_ε $(\varepsilon = 0, 1)$ is scattered.*

(8.9) **Corollary.** [ACP$^\#$]. *For every Hausdorff space X there exists a partition $\{R_0, R_1\}$ such that every compactum F contained in R_ε $(\varepsilon = 0, 1)$ is scattered.*

(8.10) **Remark.** The previous results can be improved in the following way. Theorems (8.6) and (8.8) remain true if F is only a G_δ-subset of a countably compact $\aleph_0$-closed regular subspace G of a space X. Corollaries (8.7) and (8.9) are true also provided F is a Čech complete subspace of X. We shall not present here these modifications of our results in detail.

9. APPENDIX I. PARTITIONING OF TOPOLOGICAL SPACES UNDER THE AXIOM OF CONSTRUCTIBILITY

Assuming the Hödel axiom of constructibility [V = L] W. Weiss ([3], see also [4][1]) has proved that
every Hausdorff space X can be partitioned into two subsets R_0 and R_1, neither of which contain the Cantor set D^{ω_0}.

Here, employing the results of Section 6 we improve this theorem of W. Weiss (see Theorem (9.3)).

(9.1) **Denotation.** Following K. Devlin [25] we denote by $\square_\mu$ the following combinatorial principle.
There is a sequence of sets $\{C_\xi : \xi < \mu^+$ and ξ is a limit ordinal$\}$ such that

(9.1.a) C_ξ is closed and unbounded in ξ,

(9.1.b) $\mathrm{cf}(\xi) < \mu$ implies $|C_\xi| < \mu$,

1) Notice that the proof of the same result in [4] (see Corollary 7) is incomplete unless the axiom of constructibility is assumed.

(9.1.c) if γ is a limit point in C_ξ then $C_\gamma = \gamma \cap C_\xi$.

R. Jensen has shown that Gödel's axiom of constructibility implies that $\Box_\mu$ holds for every infinite cardinal μ.

Recall also that the axiom of constructibility implies the GCH.

The following lemma is just a modification of W. Weiss's Lemma 4 in [3] (see also Lemma 4 in [4]).

(9.2) **Lemma.** [V = L]. *If $S_0(X \mid \nu)$ holds for every $\nu \leqq \mu_{\omega_0}$ then $S_0(X \mid \mu_{\omega_0+1})$ holds as well.*

Proof. Let $A \subset B$ where B is an rc-closed subset of X and $|A| \leqq \mu_{\omega_0+1}$ (see (6.15)). Enumerate A as $A = \{a_\xi : \xi < \mu_{\omega_0+1}\}$. Inductively we shall construct a system $\mathscr{A} = \{A_n^\xi : n \in N,\ \xi < \mu_{\omega_0+1}\}$ with the following properties:

(1) $a_\xi \in A_n^{\xi+1} \subset B$,

(2) $A_n^\xi \subset A_n^{\xi+1}$,

(3) $|A_n^\xi| \leqq \mu_n$,

(4) A_n^ξ is an rc-closed u_0-set in X,

(5) if $\xi' < \xi$ then $\bigcup\{A_n^{\xi'} : n \in N\} \subset \bigcup\{A_n^\xi : n \in N\}$,

(6) if ξ is a limit ordinal and $|C_\xi| \leqq \mu_n$ then $\bigcup\{A_n^{\xi'} : \xi' \in C_\xi\} \subset A_n^\xi$.

Assume that for all $\xi < \lambda$ and for all $n \in N$ the sets A_n^ξ are already constructed. If $\lambda = \xi + 1$, for every $n \in N$ let $A_n^\lambda = \{a_\xi\} \cup A_n^\xi$. If λ is a limit ordinal, consider $m = \inf\{k : |C_\lambda| \leqq \mu_k\}$ (see (9.1)) and let $A_n^\lambda = \emptyset$ for $n < m$ and $\bar{A}_n^\lambda = \bigcup\{A_n^\xi : \xi \in C_\lambda\}$ for $n \geqq m$. Since, obviously, $|\bar{A}_n^\lambda| \leqq \mu_n$ and $\mu_n^{\aleph_0} = \mu_n$ according to Proposition (3.15), there exists an rc-closed set $\hat{A}^\lambda$ such that $\bar{A}_n^\lambda \subset \hat{A}_n^\lambda \subset B$ and $|\hat{A}_n^\lambda| \leqq \mu_n$. Moreover, since $S_0(X \mid \mu_n)$ holds, by (6.5) $\hat{A}_n$ is a u_0-set and we let $A_n^\lambda = \hat{A}_n^\lambda$. It is obvious that A_n^λ satisfies the conditions (1)–(6) and thus we get the system $\mathscr{A} = \{A_n^\xi : n \in N,\ \xi < \mu_{\omega_0+1}\}$ with properties (1)–(6). Moreover, $A \subset \bigcup\mathscr{A} \subset B$.

Let $A^\xi = \bigcup\{A_n^\xi : n \in N\}$. Since all A_n^ξ are u_0-sets, therefore A^ξ are u_0-sets, too; moreover, $\tilde{\mathscr{A}} = \{A^\xi : \xi < \mu_{\omega_0+1}\}$ is a chain. To complete the proof we shall show that $\tilde{A} = \bigcup\tilde{\mathscr{A}}$ is also a u_0-set.

Really, for each $\xi < \mu_{\omega_0+1}$ consider a u_0-partition $\mathscr{R}^\xi = \{R_\alpha^\xi : \alpha < \xi\}$ of the set A^ξ and let $\tilde{A}^\xi$, $\mathscr{R}^*$, $\tilde{R}_\alpha^\xi$, $\tilde{R}_\alpha$ and be defined in the same manner as in (6.10). Let F be an $\aleph_0$-closed subset in X and $I : F \to I$ a closed mapping such that $|f(F \cap \tilde{A})| \geqq \mathfrak{c} = \aleph_1$. Obviously $\{f(F \cap A^\xi) : \xi < \mu_{\omega_0+1}\}$ is a chain in I. Let $\xi_0 = \inf\{\xi : |f(F \cap A^\xi)| \geqq \mathfrak{c}\}$, it is easy to notice that $|f(F \cap A^{\xi_0})| \geqq \mathfrak{c}$. We shall show that $|f(F \cap \tilde{A}^{\xi_0})| \leqq \aleph_0$. In case $\xi_0 = \xi + 1$ this is obvious. Let now ξ_0 be a limit ordinal. Then for every $n \geqq m = \min\{k : |C_{\xi_0}| \leqq \mu_k\}$ the system $\{f(F \cap A_n^\xi) : \xi \in C_{\xi_0}\}$ is an increasing chain of countable subsets of I. Therefore also the sets $B_n = \bigcup\{f(F \cap A_n^\xi) : \xi \in C_{\xi_0}\}$ are countable, and hence the set $f(F \cap \tilde{A}^{\xi_0}) = \bigcup\{B_n : n \geqq m\}$ is countable, too.

Noticing that $f(F \cap \tilde{R}_\alpha) \supset (f(F \cap R_\alpha^{\xi_0}) \setminus f(F \cap \tilde{A}^{\xi_0})$ we can conclude that $|f(F \cap \cap \tilde{R}_\alpha)| \geqq \mathfrak{c}$ for every $\alpha < \nu$. Hence $\mathscr{R}^*$ is a u_0-partition of the set $\tilde{A}$ and therefore $\tilde{A}$ is a u_0-set. Since the inequality $|\tilde{A}| \leqq \mu_{\omega_0+1}$ is obvious we can conclude now that the statement $S_0(X \mid \mu_{\omega_0+1})$ is true.

The following theorem is a natural analog of Theorem (8.1).

(9.3) **Theorem** [V = L]. *For every topological space X and every cardinal $\lambda \geqq \mathfrak{c}$ the statement $S_0(X \mid \lambda)$ holds.*

Proof. Notice first that since the axiom of constructibility implies the GCH the statements $S_0(X \mid \lambda)$ and $S(X \mid \lambda)$ are equivalent.

According to (6.18.c) the statement $S(X \mid \mathfrak{c})$ is true. Assume the correctness of the statement $S(X \mid \nu)$ for all $\nu \in [c, \lambda[$ and let $\mu = \lambda^{1/\aleph_0}$ (see (8.1)). It is easy to notice that $\mu \leqq \lambda \leqq \mu^{\aleph_0}$ and therefore either $\mu^{\aleph_0} = \mu$ (provided $\mathrm{cf}(\mu) > \omega_0$) or $\mu^{\aleph_0} = \mu^+$ (provided $\mathrm{cf}(\mu) = \omega_0$) [11]. Since, obviously, μ is $\aleph_0$-unaccessible, the statement $S_0(X \mid \mu)$ holds according to (6.13.d). Hence to complete the proof we have to consider the case when $\lambda = \mu^+$, i.e. $\mathrm{cf}(\mu) \leqq \omega_0$. But in this case there exists an ordinal ν ($\nu \geqq \mathfrak{c}$) satisfying $\mu = \nu_{\omega_0}$ and the statement $S_0(X \mid \xi)$ holds for every $\xi < \nu_{\omega_0}$. Employing the previous lemma we get from here the statement $S_0(X \mid \nu_{\omega_0+1})$ as well. To complete the proof one only has to notice that $\nu_{\omega_0+1} = \mu^+ = \lambda$ in this case.

Applying (6.5) to the previous theorem we obtain the following

(9.4) **Corollary** [V = L]. *Every topological space X is a u_0-space.*

Quite in the same manner as Theorems (8.4)–(8.9) were derived from (8.2) one can easily get the following statements just from (9.4).

(9.5) **Corollary** [V = L]. *A topological space X with $|X| \geqq \mathfrak{c}$ has a partition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ which is hereditary with respect to the family $\mathscr{F}(X)$ (see (5.10)).*

(9.6) **Corollary** [V = L]. *For every topological space X there exists a partition $\{R_0, R_1\}$ such that every countably compact $\aleph_0$-closed regular subset F contained in R_ε ($\varepsilon = 0, 1$) is scattered.*

(9.7) **Corollary** [V = L]. *For every Hausdorff space X there is a partition $\{R_0, R_1\}$ such that every compactum F contained in R_ε ($\varepsilon = 0, 1$) is scattered.*

(9.8) **Corollary** [V = L]. *For a topological space X with $|X| \geqq \mathfrak{c}$ there exists a partition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ which is a simultaneous decomposition of all closed regular countably compact subsets F without isolated points.*

(9.9) **Corollary** [V = L]. *If X is a Hausdorff compactum without isolated points then there exists a decomposition $\mathscr{R} = \{R_\alpha : \alpha < \mathfrak{c}\}$ such that for every closed subset F without isolated points $[F \cap R_\alpha] = F$ holds for every $\alpha < \mathfrak{c}$.*

10. APPENDIX II. SOME TOPOLOGICAL APPLICATIONS OF THE MAIN RESULTS

Here we derive some simple corollaries of the results obtained in the previous sections.

(10.1) **Theorem** [ACP#] $\vee$ [V = L]. *A Hausdorff compactum X without non-trivial convergent sequences can be partitioned into two subsets $X = R_0 \cup R_1$ in such a way that every compactum contained in R_0 or in R_1 is finite.*

Proof follows immediately from (8.9), (9.7) and the known fact that every scattered Hausdorff compactum contains a non-trivial convergent sequence [26].

(10.2) **Remark.** For the Čech-Stone compactification βN of a countable discrete space N the statement of the previous theorem holds without any additional set-theoretic assumption because one can apply Proposition (2.4) instead of Theorem (8.7) in this case.

The next theorem allows to discover some differences in the properties of the connectedness and linear connectedness.

Recall that a topological space is called hereditarily disconnected if it does not contain connected subsets of cardinality larger than 1 [27]. Analogously we shall call a space hereditarily linearly disconnected if it does not contain any linearly connected subset of cardinality larger than 1.

It is not difficult to show that already the plane $\mathbb{R}^2$ cannot be partitioned into two hereditarily disconnected subsets R_0 and R_1. The situation with the hereditarily linear disconnectedness is quite different: it is established by the following theorem (10.3).

(10.3) **Theorem** [ACP#] $\vee$ [V = L]. *Every Hausdorff space can be partitioned into two hereditarily linearly disconnected subsets.*

Proof follows immediately from (8.9) and (9.7).

11. APPENDIX III. APPLICATION TO COMBINATORICAL SET THEORY

A well-known theorem of P. Erdös and R. Rado [28] states that for every infinite set X its exponent (i.e. the set of all its subsets) $\exp X$ can be represented as a union of two sets $\exp X = P_0 \cup P_1$ in such a way that for every infinite $A \in P_\varepsilon$, $\varepsilon = 0, 1$, there exists $B \subset A$ which belongs to $P_{1-\varepsilon}$. The aim of this section is to obtain a certain generalization of this fact (Theorem 11.2)). Its proof is based on the results of Sections 8 and 9. We begin with the following definition.

(11.1) **Definition.** Let $A \subset X$ and $B \subset A$. By the *exponent of A modulo B* we call the set

$$\exp(A/B) = \{C : B \subset C \subset A\}.$$

(11.2) **Theorem** [ACP*] $\vee$ [V = L]. *The exponent of an infinite set X can be represented as a union of two subsets* $\exp X = P_0 \cup P_1$ *in such a way that for every pair of subsets $B \subset A$ satisfying $|A \setminus B| \geqq \aleph_0$ neither P_0 nor P_1 contain* $\exp(A/B)$.

Proof. Define a mapping $\varphi : \exp X \to D^X$ (D is a discrete two-point space) by the equality $\varphi(A) = Z = (\delta x)_x$, where $A \in \exp X$ and $\delta_x = 1$ if $x \in A$ and $\delta_x = 0$, if $x \notin A$. It is obvious that φ is a bijection. According to Theorem (8.9) or (9.7) the space D^X can be partitioned into two subsets R_0 and R_1 in such a way that every compactum contained in R_ε, $\varepsilon = 0, 1$ is scattered. Let $P_\varepsilon = \varphi^{-1}(R_\varepsilon)$, $\varepsilon = 0, 1$. It is obvious that $\exp X = P_0 \cup P_1$. Moreover, neither P_0 nor P_1 contain some $\exp(A/B)$ with $|A \setminus B| \geqq \aleph_0$. Really, if there is a pair $B \subset A$ such that $|A \setminus B| \geqq \aleph_0$ and $\exp(A/B) \subset P_\varepsilon$, then

$$\begin{aligned}\varphi(\exp(A/B)) = \Pi\{D_x : x \in A \setminus B\} \times \Pi\{\{0_x\} : x \in X \setminus A\} \times \\ \times \Pi\{1_x\} : x \in B\} \approx D^{|A \setminus B|} \subset R_\varepsilon\end{aligned}$$

but this contradicts our assumptions.

(11.3) **Remark.** If $B = \emptyset$ then Theorem (11.2) turns into the above mentioned theorem of P. Erdös and R. Rado (up to the set theoretic assumptions, of course).

(11.4) **Remark.** The problem of representing the exponent of a set X as a union of two subsets neither of which contain an exponent $\exp(A/B)$ for some pair $B \subset A$ ($\subset X$) with infinite difference was considered by one of the authors in [7]. There, this problem was reduced to that of patritioning topological spaces. The technique employed here in the proof of Theorem (11.2) is approximately the same as in [7].

12. APPENDIX IV. ON QUASI-SEQUENTIAL SPACES

The notion of an rc-closed subset of a topological space, introduced in Section 3 and used essentially in Section 6 in the process of induction seems also interesting from another point of view. Namely, it allows naturally to distinguish a new class of spaces (the so called quasi-sequential spaces) which occupies a place between the class of sequential spaces and the class of spaces with countable tightness. Furthermore, the class of quasi-sequential spaces seems to be substantial in the theory of spaces, the topologies of which are characterized by means of countable sets. The aim of this section is to begin the investigation of these spaces.

(12.1) **Definition.** A space X is called *quasi-sequential* if every its rc-closed subset A is closed.

Proposition (3.9) implies immediately the following

(12.2) **Proposition.** *Every sequential T_1-space X is quasi-sequential.*

On the other hand, applying (3.15) we get

(12.3) **Proposition.** *A quasi-sequential space X has a countable tightness. Moreover, $|[A]| \leqq |A|^{\aleph_0}$ for every subset A of X*[1]*).*

(12.4) **Remark.** I. Juhász [23] has constructed a hereditarily separable space X, every countable subset of which has a closure of cardinality $2^{\mathfrak{c}}$. This space, obviously, represents an example of *a space with the countable tightness which is not quasi-sequential.* Recall also that a Hausdorff compactum with the properties of Juhász's example was constructed later by V. Fedorčuk [29].

(12.5) **Remark.** It is easy to notice that the topological space constructed in [30, p. 68] is an example of a countable Hausdorff *quasi-sequential space which is not sequential.* On the other hand, according to Proposition (12.9) there cannot be a Hausdorff compactum with such properties.

Applying Remark (3.2) one easily comes to the following characterization of quasi-sequential spaces.

(12.6) **Proposition.** *A T_1-space X is quasi-sequential iff for every its non-closed subset A there exists a sequence $(x_n)_{n \in N}$ which has a cluster point a in X but no cluster points in A.*

With the help of this characterization or just from the definitions one can easily come to the following fact:

(12.7) **Proposition.** *A quasi-sequential sequentially compact space is sequential.*

(12.8) **Proposition.** *A countably compact T_1-space (in particular, a compact T_1-space) X is quasi-sequential iff every its countably compact subset A is closed in X(i.e. iff it is cc-closed [31]).*

Proof follows easily from Propositions (3.3) and (3.4).

(12.9) **Theorem** [AM] $\vee$ [LH]. *A Hausdorff k-space X is sequential iff it is quasi-sequential. In particular, every quasi-sequential compactum is sequential.*

Proof. Let X be quasi-sequential and let A be its subset which is not closed. Then A is not rc-closed either, i.e., there exists a countable set $M \subset A$ such that $M' \cap \cap (X \setminus A) \neq \emptyset$ but $M' \cap A = \emptyset$. Since X is a k-space, without loss of generality we can choose M in such a way that $[M]$ is compact. According to Proposition (12.3), $|[M]| \leqq \mathfrak{c}$. Now, applying Theorem (1.3) from [22] which is proved under the

[1]) To be precise, this estimate is true if $|A| > 1$ or if X is a T_1-space.

assumption of $[AM] \vee [LH]$ we conclude that $[M]$ is sequentially compact and therefore according to Proposition (12.7) the space X is sequential.

(12.10) **Proposition.** *If X is a quasi-sequential T_1-space and $f: X \to Y$ is a closed surjection then Y is quasi-sequential as well.*

Proof. Take an rc-closed subset B in Y. According to (3.12) the preimage $f^{-1}(B)$ is rc-closed in X and hence $f^{-1}(B)$ is also closed in X. Therefore $B = ff^{-1}(B)$ is a closed subset in Y and hence Y is quasi-sequential.

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