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SOME REMARKS ABOUT DIGRAPHS WITH NON-ISOMORPHIC 1- OR 2-NEIGHBOURHOODS

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1. INTRODUCTION

Let $G = (V(G), U(G))$ be a digraph with a vertex set $V(G)$ and an arc set $U(G)$. Let t, $t \geqq 1$, be an integer. By $N_t(x, G)$ we denote the subdigraph of G induced by the set of vertices of G for which the length of the shortest directed path from x to them is equal to t. We call this subdigraph the *t-neighbourhood* of x in G. Moreover, let us assume that $\mathscr{DC}_t$ denotes the class of digraphs with non-isomorphic t-neighbourhoods, i.e. $G \in \mathscr{DC}_t$ iff it satisfies the following condition:

$$\forall_{x,y \in V(G)}\, x \neq y \Rightarrow N_t(x, G) \text{ non} \cong N_t(y, G)\,.$$

Other definitions not contained in this introduction can be found in [2] and [3].

J. Sedláček [4] considered the problem of existence of graphs with non-isomorphic 1-neighbourhoods. He obtained the following interesting theorem:

Theorem 1.1. [4]. *For every n, $n \geqq 6$, there exists a graph with non-isomorphic 1-neighbourhoods.*

The same problem, but for 2-neighbourhoods, and relations between classes of graphs with non-isomorphic 1- and 2-neighbourhoods were examined in [1].

In this paper we consider asymmetric digraphs with the properties:

(a) $V(N_1(x, G)) \neq \emptyset$ for all $x \in V(G)$ in Section 2, and
(b) $V(N_1(x, G)) \neq \emptyset$ and $V(N_2(x, G)) \neq \emptyset$ for all $x \in V(G)$ in Section 3,

where by an asymmetric digraph we mean a digraph G satisfying the following condition:

$$(x, y) \in U(G) \Rightarrow (y, x) \notin U(G)\,, \quad \text{for} \quad x, y \in V(G)\,.$$

The paper contains results concerning existence of digraphs in the class $\mathscr{DC}_1$ and in the class $\mathscr{DC}_2$, and relations between $\mathscr{DC}_1$ and $\mathscr{DC}_2$. In figures in this paper, double lines with arrows from the subdigraph G_1 of a digraph G to the subdigraph G_2 of G denote that $(x, y) \in U(G)$, for all $x \in V(G_1)$ and $y \in V(G_2)$.

2. ASYMMETRIC DIGRAPHS IN THE CLASS $\mathscr{DC}_1$

In this section we consider the problem of existence of asymmetric digraphs in the class $\mathscr{DC}_1$, assuming that all 1-neighbourhoods are non-isomorphic to the digraph $(\emptyset, \emptyset)$. We have

Proposition 2.1. *For an integer n, $1 < n \leqq 6$, every asymmetric digraph G with n vertices has the following property*:

$$\exists_{x,y \in V(G); x \neq y} N_1(x, G) \cong N_1(y, G) .$$

Proof. Note that the proof is immediate for $n < 6$. So we examine the case $n = 6$. Assume that there exists an asymmetric digraph G with 6 vertices satisfying the condition

(1) $$\forall_{x,y \in V(G)}\, x \neq y \Rightarrow N_1(x, G) \text{ non} \cong N_1(y, G) .$$

Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Since the digraph G may have at most 15 arcs and satisfies (1), so

(2) $$1, 2, 2, 3, 3, 3$$

is the only possible distribution of outdegrees of vertices in G, and then the digraph G has 14 arcs. Therefore there are two vertices in G which are not connected by an arc. Without loss of generality we assume that these are the vertices v_1 and v_2. Since necessarily exists a vertex v_i such that $N_1(v_i, G) \cong (\{x, y\}, \emptyset)$, so $(v_i, v_2) \in U(G)$, $(v_i, v_1) \in U(G)$ and $(v_l, v_i) \in U(G)$ for $l = 3, 4, 5, 6$ and $l \neq i$. We can assume that, for example, $i = 3$ (see Fig. a).

Among the vertices belonging to $\{v_4, v_5, v_6\}$ there must exist a vertex v_k with the outdegree equal to 3 such that (v_k, v_2), $(v_k, v_1) \in U(G)$. Note that $(v_k, v_3) \in U(G)$. Hence $(v_l, v_k) \in U(G)$ for $l = 4, 5, 6$ and $l \neq k$. We can assume that, for example, $k = 5$ (see Fig. b).

The above considerations imply that the outdegrees of v_1 and v_2 may be at most equal to two. By (2) we have that one of them has the outdegree equal to 2 and the other one must have the outdegree equal to 1.

Case 1. Assume that v_1 has the outdegree equal to 2 and v_2 has the outdegree equal to 1. So (v_1, v_4), $(v_1, v_6) \in U(G)$. Then we have two subcases.

Case 1a. Let $(v_2, v_6) \in U(G)$. Then $(v_6, v_4) \in U(G)$ and $(v_4, v_2) \in U(G)$. Let us consider $N_1(v_4, G)$ and $N_1(v_6, G)$. They are isomorphic (see Fig. c), a contradiction with (1).

Case 1*b*. Let $(v_2, v_4) \in U(G)$. As above we have a contradiction with (1).

Case 2. Similar considerations lead to a contradiction with (1) by the assumption that the outdegree of v_2 equals 2 and the outdegree of v_1 is equal to 1 (see Fig. d).

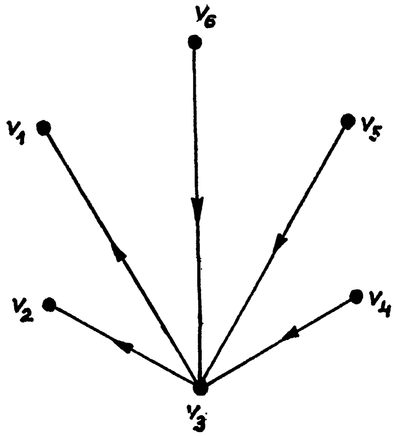


Fig. a.

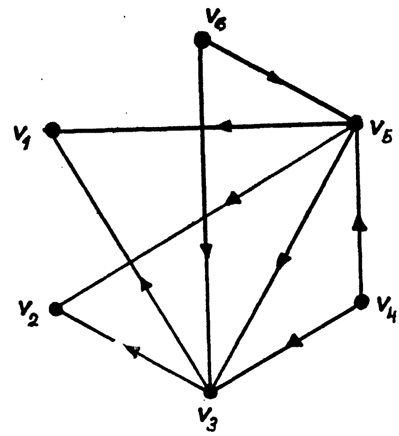


Fig. b.

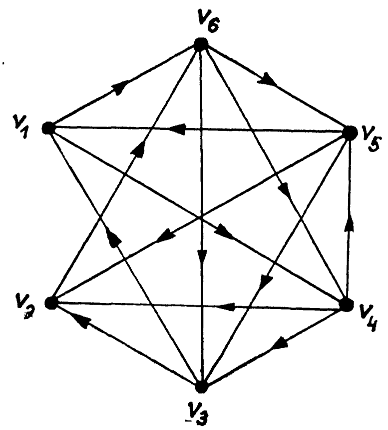


Fig. c.

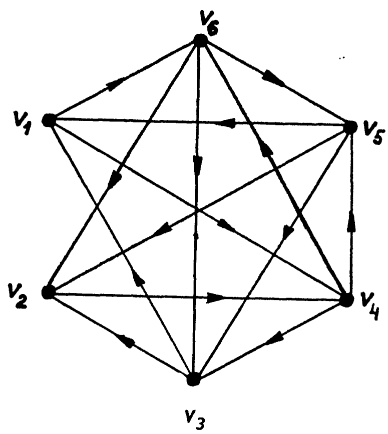


Fig. d.

So we restrict our considerations to digraphs with at least seven vertices. We have the following result:

Theorem 2.1. *For every* n, $n \geqq 7$, *there exists an asymmetric digraph with* n *vertices belonging to* $\mathscr{D}\mathscr{C}_1$.

Proof. We prove this theorem by induction on the number of vertices of the digraph. The digraph with 7 vertices belonging to $\mathscr{D}\mathscr{C}_1$ is presented in Fig. 3a.

Assume that there exists a digraph G with n vertices belonging to $\mathcal{DC}_1$. The construction shown in Fig. 1 gives a digraph with $(n + 1)$-vertices belonging to $\mathcal{DC}_1$ (note that $N_1(x_{n+1}, G') \cong G$ and $N_1(x_i, G') \cong N_1(x_i, G)$ for all $x_i \in V(G)$).

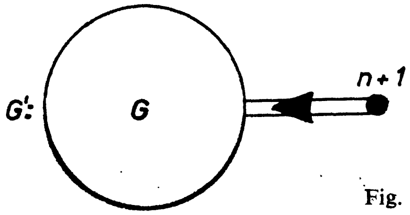


Fig. 1.

This completes the proof for all $n \geqq 7$.

3. ON RELATIONS BETWEEN $\mathcal{DC}_1$ AND $\mathcal{DC}_2$

First we deal with the class $\mathcal{DC}_2$. It is easy to see that no asymmetric digraph with n vertices, for $2 \leqq n \leqq 5$, belongs to $\mathcal{DC}_2$.

For asymmetric digraphs with the number of vertices greater than 5 we have

Theorem 3.1. *For every n, $n \geqq 6$, there exists an asymmetric digraph with n vertices belonging to $\mathcal{DC}_2$.*

Proof. Our proof consists of two parts.

Part 1. We prove this theorem for even n by induction on k, where k denotes $n/2$. For $k = 3$ the digraph presented in Fig. 5a belongs to $\mathcal{DC}_2$. Assume that the theorem holds for some k, i.e., there is an asymmetric digraph G with $2k$ vertices in $\mathcal{DC}_2$. In Fig. 2 we show an asymmetric digraph with $2k + 2$ vertices which is in $\mathcal{DC}_2$ (see Tab. 1), i.e., the theorem is true for $k + 1$. It completes the proof for all k, $k \geqq 3$.

Tab. 1.

vertex x	$N_1(x, G'')$	$N_2(x, G'')$
x_{n+1}	•	G
x_{n+2}	G	•
x_i $1 \leq i \leq n$	H_i	F_i

$H_i \cong N_1(x_i, G)$ $F_i \cong N_2(x_i, G)$

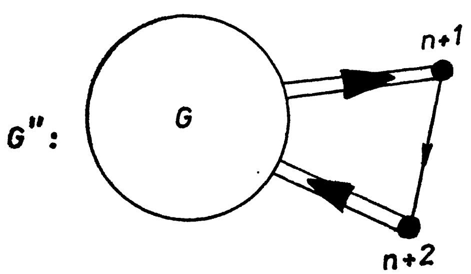
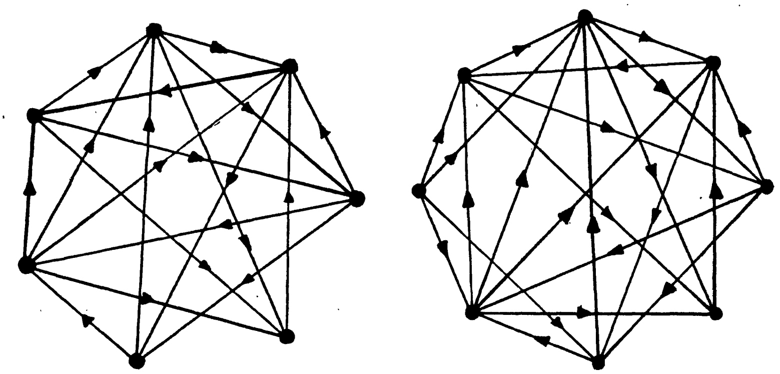


Fig. 2.

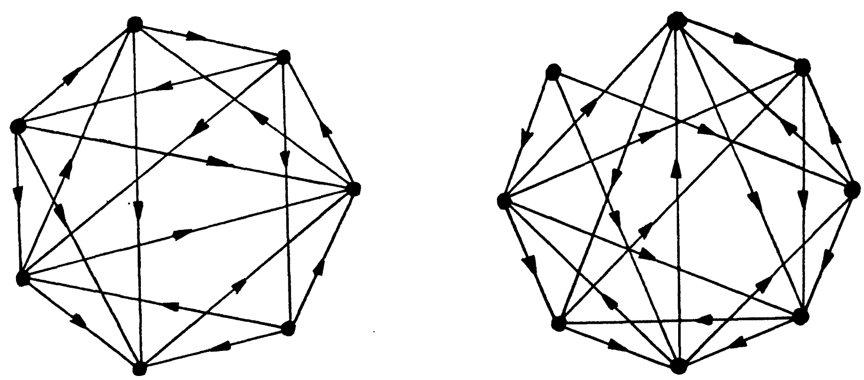
Part 2. The proof for odd n is done by induction on l, where l denotes $(n-1)/2$, and it is similar to the proof for even n. (Remark. For $l = 3$ the asymmetric digraph presented in Fig. 5b is a member of $\mathscr{DC}_2$.)

Now we proceed to the discussion of relations between the classes $\mathscr{DC}_1$ and $\mathscr{DC}_2$. It is sufficient to examine the digraphs in Figs. 3, 4 and 5, and the construction presented in Fig. 2 in order to obtain the following theorems.



(a) (b)

Fig. 3. Digraphs with 7 and 8 vertices belonging to $\mathscr{DC}_1 \cap \mathscr{DC}_2$.



(a) (b)

Fig. 4. Digraphs with 7 and 8 vertices belonging to $\mathscr{DC}_1 - \mathscr{DC}_2$.

Theorem 3.2. *For every $n, n \geqq 7$, there exists an asymmetric digraph with n vertices belonging to $\mathscr{DC}_1 \cap \mathscr{DC}_2$.*

Theorem 3.3. *For every $n, n \geqq 7$, there exists an asymmetric digraph with n vertices belonging to $\mathscr{DC}_1 - \mathscr{DC}_2$.*

Theorem 3.4. *For every n, $n \geqq 6$, there exists an asymmetric digraph with n vertices belonging to $\mathscr{D}\mathscr{C}_2 - \mathscr{D}\mathscr{C}_1$.*

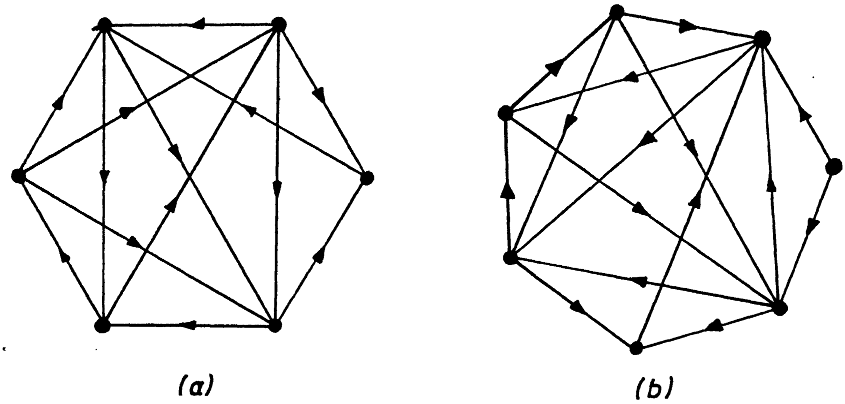


Fig. 5. Digraphs with 6 and 7 vertices belonging to $\mathscr{D}\mathscr{C}_2 - \mathscr{D}\mathscr{C}_1$.

Open problem: What can be said about the existence of asymmetric digraphs in classes $\mathscr{D}\mathscr{C}_t$ for $t \geqq 3$?

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