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A SHARPENING OF DISCRETE ANALOGUES OF WIRTINGER'S INEQUALITY

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Discrete analogues of Wirtinger's inequality have been already studied by various authors (see e.g. [1], [2], [3], [5], [6], [7], [8]). Z. Nádeník in [4] proved the following sharpening of Wirtinger's inequality:

Theorem 1. *Let $f(\varphi)$ denote a continuous function with the period 2π, which has the symmetrical derivative $f^{\bullet}(\varphi) = \lim_{\varepsilon \to 0} [f(\varphi + \varepsilon) - f(\varphi - \varepsilon)] : (2\varepsilon)$. Let $f^{\bullet}(\varphi)$ be of bounded variation in $\langle 0, 2\pi \rangle$. If*

$$\int_0^{2\pi} f(\varphi)\, \mathrm{d}\varphi = 0\,, \tag{0.1}$$

then

$$\int_0^{2\pi} f^{\bullet 2}(\varphi)\, \mathrm{d}\varphi - \int_0^{2\pi} f^2(\varphi)\, \mathrm{d}\varphi - \frac{\pi}{2}\,[f(0) + f(\pi)]^2 \geqq 0 \tag{0.2}$$

with the equality holding only for

$$f(\varphi) = a \cos \varphi + b \sin \varphi + c(2 - \pi|\sin \varphi|)\,, \quad a, b, c = \text{const.} \tag{0.3}$$

In this paper we prove sharpenings of two discrete analogues of Wirtinger's inequality, which are analogous to Theorem 1 (Theorems 2, 3), using real trigonometric polynomials. Then we show a geometrical application – a sharpening of the isoperimetric inequality for some polygons (Theorem 4).

1. LIST OF THEOREMS

Theorem 2. *Let $n = 2m$, let $x_1, \ldots, x_n$ be n real numbers such that*

$$\sum_{i=1}^{n} x_i = 0\,. \tag{1.1}$$

Let us define $x_{n+1} = x_1$. *Then*

$$(1.2)\qquad \sum_{i=1}^{n}(x_i - x_{i+1})^2 \geqq 4\sin^2\frac{\pi}{n}\sum_{i=1}^{n}x_i^2 + n\sin\frac{\pi}{n}\left(\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\right)(x_m + x_{2m})^2 .$$

The equality in (1.2) *holds if and only if*

$$(1.3)\qquad x_i = A\cos\frac{2\pi i}{n} + B\sin\frac{2\pi i}{n}, \quad i = 1, \ldots, n, \quad A, B = \text{const.}$$

Theorem 3. *Let* $x_1, \ldots, x_n$ *be n real numbers satisfying* (1.1), $n \geqq 2$. *Then*

$$(1.4)\qquad \sum_{i=1}^{n-1}(x_i - x_{i+1})^2 \geqq 4\sin^2\frac{\pi}{2n}\sum_{i=1}^{n}x_i^2 + $$

$$+ 2n\sin\frac{\pi}{2n}\left(\sin\frac{\pi}{n} - \sin\frac{\pi}{2n}\right)(x_1 + x_n)^2 .$$

The equality in (1.4) *holds if and only if*

$$(1.5)\qquad x_i = A\cos\frac{(2i-1)\pi}{2n}, \quad i = 1, \ldots, n, \quad A = \text{const.}$$

Theorem 4. *Let* $n = 2m$. *Let* $P = A_1 \ldots A_n$ *denote an equilateral closed n-gon in* E_2 *of area F and perimeter L. Let us denote by* d_i *the distance of the center of* A_iA_{i+m} *and the centroid of P,* $d = \max\{d_1, \ldots, d_m\}$. *Then*

$$(1.6)\qquad L^2 \geqq 4n\,\mathrm{tg}\frac{\pi}{n}F + 2n^2\,\mathrm{tg}^2\frac{\pi}{n}\left(2\cos\frac{\pi}{n} - 1\right)d^2$$

with the equality holding only for a regular n-gon.

2. NOTATIONS AND AUXILIARY THEOREMS

Let $n = 2m$. In [1] it is shown that there exist numbers c_k, c_l^*, $k = 0, \ldots, m$, $l = 1, \ldots, m-1$, such that

$$x_i = c_0 + \sum_{k=1}^{m-1}\left(c_k\cos ki\frac{2\pi}{n} + c_k^*\sin ki\frac{2\pi}{n}\right) + (-1)^i c_m, \quad i = 1, \ldots, n.$$

The assumption (1.1) implies that $c_0 = 0$. The following identities hold:

$$(2.1)\qquad \sum_{i=1}^{n}x_i^2 = \frac{n}{2}\sum_{k=1}^{m-1}(c_k^2 + c_k^{*2}) + nc_m^2;$$

$$(2.2)\qquad \sum_{i=1}^{n}(x_i - x_{i+1})^2 = 2n\sum_{k=1}^{m-1}(c_k^2 + c_k^{*2})\sin^2 k\frac{\pi}{n} + 4nc_m^2 ,$$

$$(2.3)\qquad (x_m + x_{2m})^2 = 4(\sum_{l=1}^{M} c_{2l})^2\,, \quad \text{where} \quad M = [n/4]\,.$$

Remark. Recall that $[a] = a$ for a being an integer, $[a] = b$, where b is the biggest integer smaller than a, otherwise.

Let us denote

$$(2.4)\qquad A(n) = n \sin\frac{\pi}{n}\left(\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\right).$$

By virtue of (2.1)–(2.4) the inequality (1.2) can be written as

$$\sum_{k=1}^{m-1}(c_k^2 + c_k^{*2})\left(\sin^2 k\frac{\pi}{n} - \sin^2\frac{\pi}{n}\right) + 2c_m^2\left(1 - \sin^2\frac{\pi}{n}\right) - \frac{2}{n}A(n)\,(\sum_{l=1}^{M} c_{2l})^2 \geqq 0\,.$$

The following inequalities hold:

$$(2.5)\qquad \sin^2 k\frac{\pi}{n} - \sin^2\frac{\pi}{n}\begin{cases} = 0\,, & k = 1\,,\\ > 0\,, & k = 2, \ldots, m-1\,,\end{cases}$$

$$(2.6)\qquad \sin^2\frac{\pi}{n} < 1\,.$$

Using (2.5), (2.6) we conclude that it is sufficient to prove that

$$(2.7)\qquad L = \sum_{k=1}^{M} c_{2k}^2\left(\sin^2\frac{2k\pi}{n} - \sin^2\frac{\pi}{n}\right) - \frac{2}{n}A(n)\,(\sum_{k=1}^{M} c_{2k})^2 \geqq 0\,.$$

Lemma 1. *Let us denote*

$$C_1(n) = 2\sin\frac{\pi}{n}\left(\sin\frac{2\pi}{n} + \sin\frac{\pi}{n}\right),$$

$$K_i(n) = \sin^2\frac{2(i+1)\pi}{n} - \sin^2\frac{\pi}{n}\,, \quad i = 1, \ldots, M-1\,.$$

Then

$$(2.8)\qquad C_1(n)\sum_{i=1}^{r}\frac{1}{K_i(n)} < 1\,, \quad r = 1, \ldots, M-1\,.$$

Proof. Clearly it is sufficient to prove (2.8) for $r = M - 1$. Denote

$$f(n) = C_1(n)\sum_{i=1}^{M-1}\frac{1}{K_i(n)}\,.$$

We have to show that $f(n) < 1$.

We can suppose $n \geqq 8$. Then

$$K_i(n) = \sin\frac{2i+3}{n}\pi\,\sin\frac{2i+1}{n}\pi\,.$$

We have

$$\operatorname{cotg} \alpha - \operatorname{cotg} (\alpha + \beta) = \frac{\sin \beta}{\sin \alpha \sin (\alpha + \beta)}$$

and therefore

$$\left(\text{for } \alpha = \frac{2i + 1}{n} \pi , \quad \beta = \frac{2}{n} \pi , \quad \text{i.e.} \quad \alpha + \beta = \frac{2i + 3}{n} \pi\right)$$

$$\frac{1}{K_i(n)} = \frac{1}{\sin \dfrac{2\pi}{n}} \left(\operatorname{cotg} \frac{2i + 1}{n} \pi - \operatorname{cotg} \frac{2i + 3}{n} \pi \right) .$$

After adding these expressions for $i = 1, \ldots, M - 1$ we get

$$\sum_{i=1}^{M-1} \frac{1}{K_i(n)} = \begin{cases} \dfrac{1}{\sin \dfrac{2\pi}{n}} \left(\operatorname{cotg} \dfrac{3\pi}{n} + \operatorname{tg} \dfrac{\pi}{n} \right) & \text{for} \quad n = 4M , \\ \dfrac{1}{\sin \dfrac{2\pi}{n}} \operatorname{cotg} \dfrac{3\pi}{n} & \text{for} \quad n = 4M + 2 . \end{cases}$$

For $n = 4M$ we conclude (by virtue of the identity $\sin 2\alpha = 2 \sin \alpha \cos \alpha$)

$$f(n) = \frac{\cos \dfrac{\pi}{2n} \cos \dfrac{2\pi}{n}}{\cos^2 \dfrac{\pi}{n} \cos \dfrac{3\pi}{2n}} .$$

Introducing the notation

$$x = \frac{\pi}{n} , \quad y = \cos^2 \frac{x}{2} , \quad \text{i.e.} \quad y = \cos^2 \frac{\pi}{2n} ,$$

we obtain

$$n \geqq 8 \Rightarrow x \in (0, \pi/8\rangle , \quad y \in I = \left\langle \cos^2 \frac{\pi}{16} , 1 \right) .$$

Substituting $x = \pi/n$ in $f(n)$ we get the function

$$f_1(x) = \frac{\cos \dfrac{x}{2} \cos 2x}{\cos^2 x \cos \dfrac{3x}{2}} ,$$

which is defined in I. Using the identities

$$\cos x = 2y - 1\,,$$
$$\cos 2x = 2(2y-1)^2 - 1\,,$$
$$\cos\frac{3x}{2} = \cos\frac{x}{2}(4y-3)\,,$$

we get the function

$$g(y) = \frac{8y^2 - 8y + 1}{(2y-1)^2(4y-3)}\,.$$

It is easy to show that $g(y) < 1$ in I and therefore $f(n) < 1$ for $n = 4M$, $n \geqq 8$.

Analogously, when considering $n = 4M + 2$, $n \geqq 10$, we can show that

$$f(n) = \frac{\cos\dfrac{\pi}{2n}\cos\dfrac{3\pi}{n}}{\cos\dfrac{\pi}{n}\cos\dfrac{3\pi}{2n}}$$

and with $x = \pi/n$, $y = \cos^2(x/2)$ we get the function

$$h(y) = \frac{16y^2 - 16y + 1}{4y - 3}\,,$$

$$y \in J = \left\langle \cos^2\frac{\pi}{20},\, 1\right)\,.$$

Using the inequality $h(y) < 1$, $y \in J$, we conclude that $f(n) < 1$ for $n = 4M + 2$, $n \geqq 10$.

So, Lemma 1 holds.

Lemma 2. *Using the notation from Lemma* 1, *define*

$$C_{r+1}(n) = \frac{C_1(n)}{1 - C_1(n)\sum_{i=1}^{r}\dfrac{1}{K_i(n)}}\,,\quad r = 1, \ldots, M-1\,.$$

Then

(2.9) $$K_r(n) - C_r(n) > 0\,,\quad r = 1, \ldots, M-1\,.$$

Proof. It is easy to show that (2.9) holds for $r = 1$. Let $r \geqq 2$. It can be shown that

$$K_r(n) - C_r(n) = \frac{K_r(n)\left[1 - C_1(n)\sum_{i=1}^{r}\dfrac{1}{K_i(n)}\right]}{1 - C_1(n)\sum_{i=1}^{r-1}\dfrac{1}{K_i(n)}}\,.$$

Now, (2.8) implies (2.9) for arbitrary $r = 1, \ldots, M-1$.

Lemma 3. *For $r = 1, \ldots, M$, we have*

$$L = \sum_{k=1}^{r} [c_{2k} D(k, n) - 2B(k, n) \sum_{l=k+1}^{M} c_{2l}]^2 + \tag{2.10}$$

$$+ \sum_{k=r+1}^{M} \left(\sin^2 \frac{2k\pi}{n} - \sin^2 \frac{\pi}{n}\right) c_{2k}^2 - C(r, n) \left(\sum_{k=r+1}^{M} c_{2k}\right)^2 ,$$

where

$$C(1, n) = 2 \sin \frac{\pi}{n} \left(\sin \frac{2\pi}{n} + \sin \frac{\pi}{n}\right) \quad [= C_1(n)] , \tag{2.11}$$

$$B(1, n) = \sin \frac{\pi}{n} ,$$

$$D(1, n) = \sin \frac{2\pi}{n} - \sin \frac{\pi}{n} ,$$

$$D^2(k + 1, n) = \sin^2 \frac{2(k + 1)\pi}{n} - \sin^2 \frac{\pi}{n} - C(k, n) ,$$

$$B(k + 1, n) = \frac{C(k, n)}{2D(k + 1, n)} ,$$

$$C(k + 1, n) = C(k, n) + 4B^2(k + 1, n) ,$$

$$k = 1, \ldots, M - 1 .$$

Remark. We have to verify that the definition of the numbers $D(k, n)$, $B(k, n)$, $C(k, n)$ in (2.11) is correct, i.e. that

$$K_k(n) - C(k, n) > 0 , \quad k = 1, \ldots, M - 1 .$$

We shall show that $C(k, n) = C_k(n)$, $C_k(n)$ being the numbers defined in Lemma 2. It is true for $r = 1$ [see (2.11)]. Let $C(k, n) = C_k(n)$ for an integer k, $1 \leqq k < M - 1$. Then $D^2(k + 1, n) = K_k(n) - C_k(n) > 0$ and therefore $C(k + 1, n)$ is defined in (2.11). It is easy to show that

$$C(k + 1, n) = \frac{C_1(n)}{1 - C_1(n) \sum_{i=1}^{k} \frac{1}{K_i(n)}} = C_{k+1}(n) .$$

The inequality $K_k(n) - C(k, n) > 0$ is now a consequence of Lemma 2.

Proof. Let us denote by L_i the representation of L in (2.10) for $r = i$. We have to show

$$L_r = L , \quad r = 1, \ldots, M .$$

We use the induction over r. Let $r = 1$. To prove that $P = L_1 - L = 0$ we write [see (2.11)]

$$P = \left[c_2\left(\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\right) - 2\sin\frac{\pi}{n}\sum_{l=2}^{M} c_{2l}\right]^2 - C_1(n)\left(\sum_{k=2}^{M} c_{2k}\right)^2 -$$

$$- c_2^2\left(\sin^2\frac{2\pi}{n} - \sin^2\frac{\pi}{n}\right) + \frac{2}{n}A(n)\left(\sum_{k=1}^{M} c_{2k}\right)^2 .$$

Adding and subtracting $(2/n)\,A(n)\left(\sum_{k=2}^{M} c_{2k}\right)^2$ we get

$$P = 2c_2^2\left(\frac{A(n)}{n} + 2\sin^2\frac{\pi}{n} - \sin\frac{2\pi}{n}\sin\frac{\pi}{n}\right) +$$

$$+ 4c_2\left(\sum_{k=2}^{M} c_{2k}\right)\left[\frac{A(n)}{n} - \sin\frac{\pi}{n}\left(\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\right)\right] +$$

$$+ \left(\sum_{k=2}^{M} c_{2k}\right)^2\left[4\sin^2\frac{\pi}{n} - C_1(n) + \frac{2}{n}A(n)\right].$$

From (2.11) it follows that $P > 0$.

Let $L_r = L$ for an integer r, $1 = r < M$. We show that $L_{r+1} = L$, i.e. $R = L_{r+1} - L_r = 0$. Analogously to the case $r = 1$ we get

$$R = c_{2(r+1)}^2\left[D^2(r+1, n) - \sin^2\frac{2(r+1)\pi}{n} + \sin^2\frac{\pi}{n} + C(r, n)\right] +$$

$$+ \left(\sum_{k=r+2}^{M} c_{2k}\right)^2\left[C(r, n) - C(r+1, n) + 4B^2(r+1, n)\right] +$$

$$+ c_{2(r+1)}\left(\sum_{k=r+2}^{M} c_{2k}\right)\left[-4B(r+1, n)\,D(r+1, n) + 2C(r, n)\right].$$

Using (2.11) we conclude $R = 0$.

So, (2.10) holds.

3. PROOFS OF THEOREMS

Theorem 2. The inequality (2.7) [and so (1.2) as well] is a consequence of Lemma 3. Choose $r = M$. Then

$$L = \sum_{k=1}^{M-1}\left[c_{2k}\,D(k, n) - 2B(k, n)\sum_{l=k+1}^{M} c_{2l}\right]^2 + D^2(M, n)\,c_{2M}^2 . \tag{3.1}$$

According to (2.11) and (2.9) we conclude that

$$L \geqq 0 . \tag{3.2}$$

Conditions for the equality follow from (2.5), (2.6) and (3.1).

Theorem 3. If we use Theorem 2 for real numbers $y_1, \ldots, y_{2n}$ defined as follows

$$y_k = \begin{cases} x_k, & k = 1, \ldots, n, \\ x_{2n-k+1}, & k = n + 1, \ldots, 2n, \end{cases}$$

we get the required inequality. (See the proof of Theorem 4 in [6].)

Theorem 4. In [6], Section 4, the following is proved:

$$8 \operatorname{tg} \frac{\pi}{n} F = \sum_{i=1}^{n} \left(1 - \operatorname{tg}^2 \frac{\pi}{n}\right) [(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2] +$$

$$+ 4 \operatorname{tg}^2 \frac{\pi}{n} \sum_{i=1}^{n} (x_i^2 + y_i^2),$$

where $A_i = [x_i, y_i]$, $i = 1, \ldots, n$, in the coordinate system $S = \{O, x, y\}$ in E_2 with O being the centroid of P. In the system S the assumptions of Theorem 2 for $\{x_i\}$, $\{y_i\}$ are satisfied. Hence Theorem 4 follows from Theorem 2.

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