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ON THE MAXIMUM NUMBER OF ARCS IN SOME CLASSES OF GRAPHS

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INTRODUCTION AND NOTATION

Under an oriented graph $G(X, U)$ we always understand a directed graph without loops and 2-cycles, with the set of points X and set of arcs U. If $|X| = p$, $|U| = q$, we also write $G(p, q)$. In such a graph, $d_G(x)$, for $x \in X$, denotes the sum of the out-degree and in-degree of x and

$$\delta(G) = \min \{d_G(x); x \in X\} .$$

We shall also denote, for a real t, by $\lceil t \rceil$ the integer satisfying

$$t \leqq \lceil t \rceil < t + 1 ,$$

by $\lfloor t \rfloor$ the integer satisfying

$$t - 1 < \lfloor t \rfloor \leqq t .$$

1. CALCULATION OF $f_1(p)$

We say that an oriented graph $G(p, q)$ satisfies property (P_1) if for all pairs of points x and y, there exists at most one path from x to y, and we let $\mathcal{G}_1$ be the set of all oriented graphs $G(p, q)$ satisfying property (P_1). Let $f_1(p) = \max \{q; G(p, q) \in \in \mathcal{G}_1\}$.

Theorem 1. $f_1(p) = \lfloor \frac{1}{4}p^2 \rfloor$ *for* $p \geqq 4$.

Proof. We first note some properties of the graphs $G(p, q)$ belonging to $\mathcal{G}_1$.

(a) If $[x_1, x_2, \ldots, x_r]$ is a directed path of $G = (X, U)$ and if $u \in X \setminus \{x_1, x_2, \ldots \ldots, x_r\}$, then u is joined by an arc

(1) to at most one x_i, $i = 1, 2, \dots, r$ and
(2) from at most one x_i, $i = 1, 2, \dots, r$.

If $[x_1, x_2, \dots, x_r, x_1]$ is a cycle, then u is joined by an arc (of any orientation) with at most one x_i altogether.

(b) If C is a cycle in $G \in \mathscr{G}_1$ of length $n \geqq 3$, and if G' is the graph obtained from G by contracting the points of C to a point v, with v joined to (from) a point u if any point of C is joined to (from) u, then G' also belongs to $\mathscr{G}_1$.

Now, consider the only two possible cases:

(i) The graph G contains a triangle with points x, y and z. The subgraph induced by $\{x, y, z\}$ is necessarily a 3-cycle. Let $G'(p - 2, q - 3)$ be the graph obtained by contracting this cycle as above; this gives $q - 3 \leqq f_1(p - 2)$ and so $q \leqq f_1(p - 2) + 3$.

(ii) The graph G does not contain a triangle and since G is antisymmetric then $q \leqq \lfloor \frac{1}{4}p^2 \rfloor$ (Turán). We deduce that

$$f_1(p) = \max\left(\left\lfloor \frac{p^2}{4} \right\rfloor; f_1(p - 2) + 3\right).$$

Now, since $f_1(2) = 1$ and $f_1(3) = 3$, we get that

$$f_1(p) \leqq \left\lfloor \frac{p^2}{4} \right\rfloor \quad \text{for all} \quad p \geqq 4 .$$

The value $\lfloor \frac{1}{4}p^2 \rfloor$ is attained for the complete bipartite graph (A, B, U) where $|A| = \lfloor \frac{1}{2}p \rfloor$ and $|B| = \lceil \frac{1}{2}p \rceil$ with arcs oriented from A to B. So

$$f_1(p) = \left\lfloor \frac{p^2}{4} \right\rfloor \quad \text{for all} \quad p \geqq 4 .$$

Remark. We note that if $G(p, q)$ is a directed graph (possible with 2-cycles) then

$$f_1(p) = 2p - 2 \quad \text{for all} \quad p \leqq 7$$

and

$$f_1(p) = \lfloor \tfrac{1}{4}p^2 \rfloor \quad \text{for all} \quad p \geqq 7 .$$

2. CALCULATION OF $f_2(p)$

We say that an oriented graph $G(p, q)$ has property (P_2) if, for all pairs of points x and y of G, there are at most two distinct directed paths from x to y and we denote by $\mathscr{G}_2$ be the set of all oriented graphs $G(p, q)$ with property (P_2). We let

$$f_2(p) = \max\{q; G(p, q) \in \mathscr{G}_2\} .$$

Theorem 2. $f_2(p) = \lfloor \frac{1}{2}(p-1) \rfloor + \lfloor \frac{1}{4}p^2 \rfloor$ *for all* $p \geqq 4$.

Proof. We shall say that a graph $G(p, q)$ satisfies the relation (R) if $q \leqq \leqq \lfloor \frac{1}{2}(p-1) \rfloor + \lfloor \frac{1}{4}p^2 \rfloor$.

We shall first establish the following two lemmas.

Lemma 1. *Let* $G(p, q)$ *be a graph having a point* x *such that* $d_G(x) \leqq \lceil \frac{1}{2}p \rceil$, *then* $G(p, q)$ *satisfies the relation* (R) *if the graph* $G' = G \setminus \{x\}$ *– obtained by deleting the point* x *and all arcs adjacent with* x *– satisfies the relation* (R).

Proof. We have

$$q - d_G(x) \leqq \left\lfloor \frac{p-2}{2} \right\rfloor + \left\lfloor \frac{(p-1)^2}{4} \right\rfloor,$$

which yields

$$q \leqq \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{p-2}{2} \right\rfloor + \left\lfloor \frac{(p-1)^2}{4} \right\rfloor.$$

Consequently, it is enough to show by inspection that

$$\left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{p-2}{2} \right\rfloor + \left\lfloor \frac{(p-1)^2}{4} \right\rfloor \leqq \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor.$$

Lemma 2. *For every graph* $G(p, q) \in \mathscr{G}_2$ *we have* $\delta(G) \leqq \lceil \frac{1}{2}p \rceil$.

Proof. If not then there exists a graph $G_0 \in \mathscr{G}_2$ such that $\delta(G_0) \geqq \lceil \frac{1}{2}p \rceil + 1$ and so, for any two points x and y of G_0 joined by an arc,

$$d_{G_0}(x) + d_{G_0}(y) \geqq 2\left\lceil \frac{p}{2} \right\rceil + 2 \geqq p + 2.$$

This implies the existence of at least two points of $X \setminus \{x, y\}$ simultaneously joined by arcs with x and y. We shall use this fact to show that all triangles in G_0 are 3-cycles (and deduce a contradiction). If not, without loss of generality, let u, v, w be the points of a triangle formed by the arc $[u, v]$ and the directed path $[u, w, v]$. By the above observation there is a point z, different from w, joined by arcs of any orientation with u and v. The triangle induced by $\{u, v, z\}$ is necessarily a 3-cycle (Fig. 1) by (P_2). We note that w and z cannot be joined by an arc by (P_2) so that

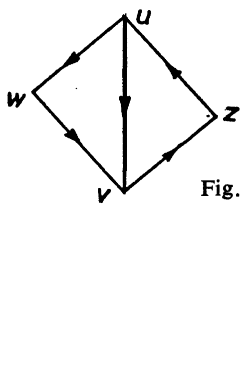

Fig. 1.

there is a point s different from w, other than u, which is joined by arcs with both points v and z. All the four possible orientations of the edges (v, s) and (s, z) lead to three or more distinct directed paths from a point in G_0 to another point (see Fig. 2 where □ marks the starting point of three or more distinct paths to the point marked ○). Now, to any arc $[u, v]$ of $G_0 \in \mathscr{G}_2$ correspond two distinct paths of length

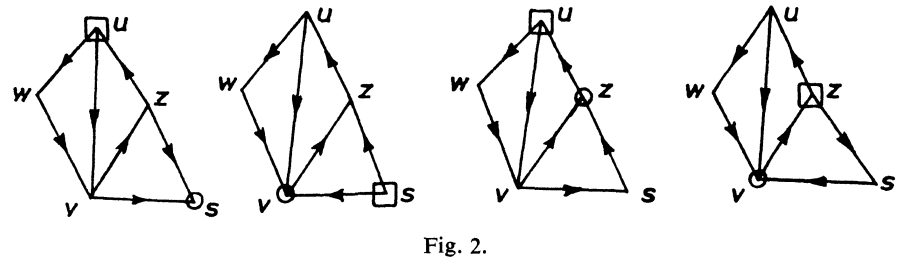

Fig. 2.

two from v to u in G_0, say $[v, w, u]$ and $[v, z, u]$ (see Fig. 3). But since w cannot be joined by an arc to (or from) z by (P_2), there exists another point different from u, say s, joined by arcs with both v and z, forming the 3-cycle $[v, z, s, v]$ (Fig. 4). Similarly, since u cannot be joined with s and w cannot be joined with z, there exists another point different from v, say r, joined by arcs with both z and s, forming the 3-cycle $[z, s, r, z]$ (Fig. 4). This leads, however, to three distinct directed paths from s to u contradicting (P_2). The proof is complete.

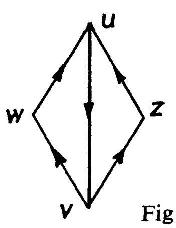

Fig. 3.

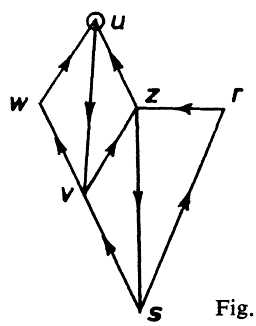

Fig. 4.

Now we are able to finish the proof of theorem 2. We shall use induction to show first that all graphs in $\mathscr{G}_2$ satisfy the relation (R). It is easy to see that all $G(p, q) \in \mathscr{G}_2$ of order $p \leqq 3$ satisfy (R). Assume that $n > 3$ and that the assertion is true for all graphs $G(p, q) \in \mathscr{G}_2$ such that $p \leqq n - 1$. Let $G(n, q) \in \mathscr{G}_2$. By lemma 2, there exists a point x in G such that $d_G(x) \leqq \lceil \frac{1}{2}p \rceil$. Since $G' = G \setminus \{x\}$ belongs to $\mathscr{G}_2$, it satisfies (R) by the induction hypothesis. Therefore, G satisfies (R) by lemma 1, i.e. for all $G(p, q) \in \mathscr{G}_2$, $q \leqq \lfloor \frac{1}{2}(p - 1) \rfloor + \lfloor \frac{1}{4}p^2 \rfloor$. Hence

$$f_2(p) \leqq \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor .$$

Finally, the complete tripartite graph (A, B, C, U) where $|A| = \lfloor\frac{1}{2}(p-1)\rfloor$, $B = \lceil\frac{1}{2}(p-1)\rceil$, $|C| = 1$, with orientation from A to B, from A to C and from C to B belongs to $\mathscr{G}_2$ and

$$|U| = q = \left\lfloor\frac{p-1}{2}\right\rfloor + \left\lfloor\frac{p^2}{4}\right\rfloor.$$

Hence

$$f_2(p) = \left\lfloor\frac{p-1}{2}\right\rfloor + \left\lfloor\frac{p^2}{4}\right\rfloor \text{for} \quad p \geqq 4\,.$$

References

[1] *C. Berge:* Théorie des graphes et hypergraphes. Dunod, Paris 1970.

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