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DE RHAM CURRENTS AND SPRAYS

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1. ALMOST — TANGENT STRUCTURE

1.1. Notations. Let M be a smooth $(= C^\infty)$ orientable differentiable manifold of dimension n. We shall denote by $\mathfrak{X}(M)$ the space of vector fields on M, by $A^p(M)$ and $\mathscr{D}'_p(M)$ the spaces of p-forms and p-currents on M, respectively. We recall that a p-current on M is an alternating p-linear and continuous map $T : \mathfrak{X}(M) \times \ldots \ldots \times \mathfrak{X}(M) \to \mathscr{D}'_0(M)$ ([2]), where $\mathscr{D}'_0(M)$ is the space of O-currents on M ([3]). In a local chart, $\mathscr{D}'_0(M)$ can be identified with the space of L. Schwartz distributions.

If $p_M : TM \to M$ and $p_{TM} : TTM \to TM$ is the tangent bundle and the second tangent bundle of M, respectively, and $p^T : TTM \to TM$ is the linear tangent map of p, then the diagram

$$\begin{array}{ccc} TTM & \xrightarrow{\;p^T\;} & TM \\ p_{TM}\downarrow & & \downarrow p \\ TM & \xrightarrow[\;p\;]{} & M \end{array}$$

is commutative.

If (U, φ) is a local chart on M, $TU[TTU]$ the corresponding chart on $TM[TTM]$, then a point $x \in M[z \in TU, Z \in TTU]$ can be represented as in [1] by $x \underset{U}{=} (x^1, \ldots, x^n)$ $z \underset{U}{=} (x^1, \ldots, x^n, y^1, \ldots, y^n)$ or briefly $z \underset{U}{=} (x, y)$, $Z \underset{U}{=} (x^1, \ldots, x^n, y^1, \ldots, y^n, X^1, \ldots \ldots, X^n, Y^1, \ldots, Y^n)$, or briefly $Z \underset{U}{=} (x, y, X, Y)$, respectively. We have $p'(Z) \underset{U}{=} (x, y)$, $p^T(Z) \underset{U}{=} (x, X)$ provided $p' = p_{TM}$. If we denote by $\bar{y}$ the vector fields on TM which satisfy $p'\bar{y} = p^T\bar{y}$, then $\bar{y} \underset{U}{=} (x, y, y, Y)$.

1.2. Fundamental sequences. The fundamental sequence on TM is an exact sequence

$$O \to TM \times_M TM \xrightarrow{\lambda} TTM \xrightarrow{\mu} TM \times_M TM \to O\,,$$

where $\mu = (p', p^T)$. If $y \underset{U}{=} (x, y)$; $z \underset{U}{=} (x, Z)$, then $\lambda(y, z) \overset{\text{def}}{=} (x, y, o, z)$.

1.3. Canonical fields. Let $\sigma : y \in TM \to \sigma(y) \stackrel{\text{def}}{=} (y, y) \in TM \times TM$, so that σ is a smooth section. Then we define:

1) the canonical vector-field $C \stackrel{\text{def}}{=} \lambda \circ \sigma \in \mathfrak{X}(TM)$;
2) the canonical (1.1) tensor-field $J \stackrel{\text{def}}{=} \lambda \circ \mu$.

1.4. Remarks.

1) From the exactness of the fundamental sequence it follows that $J^2 = 0$.
2) In a local chart one obtains:

$$C(x, y) \underset{U}{=} (x, y, o, y) \quad \text{and} \quad JZ \underset{U}{=} (x, y, o, X).$$

3) For the canonical tensor-field J we can introduce the operators of Frölicher-Nijenhuis d_J, i_J which are derivatives of degree zero:

a) $i_J : f \in C^\infty(M) \to i_J(f) = 0$

$i_J : \Omega \in A^p(TM) \to i_J(\Omega) \in A^p(TM)$

$$i_J(\Omega)(X_1, \ldots, X_p) \stackrel{\text{def}}{=} \sum_{i=1}^{p} \Omega(X_1, \ldots, JX_i, \ldots, X_p),$$

b) $d_J \stackrel{\text{def}}{=} i_J d - d i_J$.

For any $f \in C^\infty(TM)$, $d_J f = (\partial f/\partial y^i)\, dx^i$. These operators can be extended to the space of De Rham currents in the following manner:

a′) $i_J : T \in \mathscr{D}'_p(TM) \to i_J(T) \in \mathscr{D}'_p(TM)$

$$i_J(T)(X_1, \ldots, X_p) = \sum_{i=1}^{p} T(X_1, \ldots, JX_i, \ldots, X_p),$$

b′) $d_J T \stackrel{\text{def}}{=} i_J dT - d i_J T$.

As for the differential forms [1], for any $Z \in \mathfrak{X}(TM)$, $Y \in \mathfrak{X}(TM)$, $\bar{y} \in \mathfrak{X}(TM)$, $p'\bar{y} = p^T\bar{y}$, we have also for currents:

i₁) $[J, C] = -[C, J] = J$,
i₂) $[i_J, i_Z] = -i_{JZ}$; $[i_J, L_C] = i_J$,
i₃) $[i_C, d_J] = i_J$; $[d_J, L_C] = d_J$; $[i_J, d_J] = 0$, $[d, d_J] = 0$,
i₄) $[d_J, [i_Z, d_J]] = 0$,
i₅) $[i_{\bar{y}}, d_J] = L_C - i_{[\bar{y}, J]}$.

2. SEMI-BASIC AND HOMOGENEOUS CURRENTS

We shall denote by $T_0(M)$ the space of all nonzero vectors of TM.

2.1. Definition. *We shall say that $T \in \mathscr{D}'_p(T_0M)$ is homogeneous of degree r (we shall write $h(r)$) if*

$$L_C T = rT.$$

2.2. Proposition. *If* $T \in \mathscr{D}'_p(T_0M)$ *is* $h(r)$, *then* $i_J T$, $d_J T$ *are* $h(r-1)$.

2.3. Definition. *We shall say that* $T \in \mathscr{D}'_p(T_0M)$ *is semibasic if for any vertical vector-field v on* T_0M *we have*

$$i_v(T) = 0.$$

Remark. If T is a semi-basic current, then $d_J T$ is a semi-basic current and $i_J T = 0$. Now we can prove:

2.4. Proposition. *Let* $T \in \mathscr{D}'_p(T_0M)$ be $h(r)$ *and semibasic. Then for any* $\bar{y} \in \in \mathfrak{X}(T_0M)$ *for which* $J\bar{y} = C$, *we have* $i_{\bar{y}} d_J T + d_J i_{\bar{y}} T + (p + r) T$.

Proof. From 1.3 we obtain $i_{\bar{y}} d_J T + d_J i_{\bar{y}} T = L_C T - i_{[\bar{y},J]} T$. But $L_C T = rT$ and thus it is sufficient to verify the equality $i_{[\bar{y},J]} T = -pT$. For any $Z_1, \ldots, Z_p \in \mathfrak{X}(T_0M)$ we have

$$i_{[\bar{y},J]} T(Z_1, \ldots, Z_p) = \sum_{i=1}^{p} T(Z_1, \ldots, Z_{i-1}, [\bar{y}, J], Z_i, \ldots, Z_p).$$

But $[\bar{y}, J] Z = -JZ$. Indeed, $[\bar{y}, J] Z = [\bar{y}, JZ] - J[\bar{y}, Z]$. Because of $J^2 = 0$ we obtain $J[\bar{y}, J] Z = [C, JZ] - J[C, Z] = -JZ$. Q.E.D.

Corollary. *Let* $T \in \mathscr{D}'_p(T_0M)$ *be* $h(r)$, *semi-basic and* $d_J T = 0$. *Then T has the form*

$$T = \frac{1}{p + r} d_J i_{\bar{y}} T.$$

3. TWO — CURRENTS AND SPRAYS

First we shall recall the definition of a spray:

3.1. Definition. [1]. *We say that a vector field G on* T_0M *is a spray if* $[C, G] = 0$ *and* $J(G) = C$. *In a local chart,* $G \underset{U}{=} (x, y, y, X)$.

3.2. Definition. *We say that* $T \in \mathscr{D}'_2(T_0M)$ *is a generalized Finsler structure if*

f$_1$) $L_C T = T$,

f$_2$) $i_J T = 0$,

f$_3$) if for any $Y \in \mathfrak{X}(T_0M)$, $T(X, Y) = 0$, then $X = 0$.

3.3. Proposition. *Let* $\omega \in E_p(T_0M)$ *and let* $T_\omega \in \mathscr{D}'_p(T_0M)$ *be the current defined by* ω ([3]). *Then*

$$i_J T_\omega = T_{i_J \omega}.$$

Proof. For any $X_1, \ldots, X_p \in \mathfrak{X}(T_0M)$ we can write $i_J T_\omega(X_1, \ldots, X_p) = = \sum_{i=1}^{2n} T_\omega(X_1, \ldots, JX_i, \ldots, X_p) = \sum_{i=1}^{2n} \omega(X_1, \ldots, JX_i, \ldots, X_p) = T_{i_J\omega}(X_1, \ldots, X_p)$. Q.E.D.

Remark. From Proposition 3.3 and from the equality $L_X T_\omega = T_{L_X\omega}$ we deduce that the space of Finsler structures $\mathscr{F}(T_0M)$ is a subspace of the space of generalized Finsler structures $\mathscr{F}'(T_0M)$. We can also prove

3.4. Proposition. *Let M be a two-dimensional smooth differentiaable manifold. Then the space $\mathscr{F}(T_0M)$ is dense in the space $\mathscr{F}'(T_0M)$.*

Proof. The space $\mathscr{F}(T_0M)$ is a closed subspace of $\mathscr{F}'(T_0M)$ and consequently, it is a reflexive space. If $\omega \in \mathscr{F}(T_0M)$ is orthogonal to $\mathscr{F}(T_0M)$, then $\omega = 0$. Indeed, if ω is orthogonal to $\mathscr{F}(T_0M)$ then ω is orthogonal to ω ($\dim M = 2$) and then $\omega = 0$. This together with the Hahn-Banach theorem implies that $\mathscr{F}(T_0M)$ is dense in $\mathscr{F}'(T_0M)$. Q.E.D.

3.5. Conjecture. *Proposition 3.4 is true for M of any dimension.*

3.6. Proposition. *Let $T \in \mathscr{F}'(T_0M)$, $Z, Z' \in \mathfrak{X}(T_0M)$. If $t = i_Z T$, $t' = i_{Z'} T$ then $Z = JZ'$ if and only if $t = -Jt'$.*

Proof. $i_J i_{Z'} - i_Z i_J = -i_{JZ'}$, so that $i_J i_{Z'} T = -i_{JZ'} T$, or $i_J t' = -i_{JZ'} T$. If $Z = JZ'$, then $i_J t' = -i_Z T = -t$ and conversely, if $i_J t' = -t$ it follows that $i_Z T = t = i_{JZ'} T$. Then $T(JZ', Y) = T(Z, Y)$ for any $Y \in \mathfrak{X}(T_0M)$ and by Definition 3.2, the equality $JZ' = Z$ holds. Q.E.D.

Now it is easy to prove

3.7. Proposition. *Let $T \in \mathscr{F}'(T_0M)$ and $t_C \overset{\text{def}}{=} i_C T$. Then for any $t' \in \mathscr{D}'_2(T_0M)$ with the properties $h(2)$ and $Jt' = -t_C$, there exists $G \in \mathfrak{X}(T_0M)$ such that $J(G) = = C$, and $i_G T = t'$.*

Proof. We have $Jt' = -t_C$ and by Proposition 3.6 there exists $G \in \mathfrak{X}(T_0M)$ such that $J(G) = C$ and $i_G T = t'$. Q.E.D.

3.8. Definition. *Let $T \in \mathscr{F}'_2(T_0M)$. Then the Finsler energy of T is defined by*

$$E \overset{\text{def}}{=} \tfrac{1}{2} i_{\bar{y}} i_c T,$$

where $\bar{y} \in \mathfrak{X}(T_0M)$ and $J\bar{y} = C$.

3.9. Proposition. *Let $T \in \mathscr{F}'(T_0M)$ be such that $d_J i_C T = 0$. Then the vector field G defined by the equality $i_G T = -dE$ is a spray on M (the canonical spray).*

Proof. Since $d_J i_C T = 0$, for any $\bar{y} \in \mathfrak{X}(T_0M)$ such that $J\bar{y} = C$ we obtain

$$i_C T = \tfrac{1}{2} d_J (i_{\bar{y}} i_C T) \quad \text{or} \quad i_C T = d_J E .$$

But $d_J E = -i_J(-dE) = -J(-dE)$, so that there exists $G \in \mathfrak{X}(T_0 M)$ such that $JG = C$ and $i_G T = -dE$. Q.E.D.

We also have

3.10. Proposition. *The energy E is constant on the trajectories of the canonical spray.*

Proof. In fact $i_G T = -dT$, hence $i_G dT = 0$ or $L_G E = 0$. Q.E.D.

4. EXAMPLE

A very interesting case in mechanics is that for which T has the decomposition $T = dd_J E + S$, where S is a semi-basic two-current.

In this situation we can establish

4.1. Proposition. *Let $T \in \mathscr{F}'(T_0 M)$ and let E be the energy of T. Then $T - dd_J E$ is a semi-basic current if and only if $d_J E$ is a semi-basic current.*

Proof. If $S = T - dd_J E$ is semi-basic, then $d_J S = d_J T$ is semi-basic. Now we shall suppose that $d_J T$ is semi-basic, i.e. $i_J d_J T = 0$. For any $X, Y, Z \in X(T_0 M)$, $dT(JX, JY, Z) + dT(X, JY, JZ) + dT(JX, Y, JZ) = 0$. Let $X = \bar{y}$, where $J(\bar{y}) = C$. Then $dT(C, JY, Z) + dT(C, Y, JZ) + dT(\bar{y}, JY, Z) = 0$. But $i_J dT$ is semi-basic, therefore $i_J dT(\bar{y}, JY, Z) = 0$, so that we obtain

$$dT(C, JY, Z) + dT(y, JY, JZ) = 0 .$$

Then for any $Y, Z \in \mathfrak{X}(T_0 M)$, $dT(C, Y, JZ) = 0$ which implies that $i_J dT$ is semi-basic. Q.E.D.

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