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LOGICS OF BIPARTITE GRAPHS AND COMPLETE MULTIPARTITE GRAPHS

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In [1] the concept of the logic of a graph is introduced.

Let G be a finite undirected graph without loops and multiple edges, let $V(G)$ be its vertex set. If A is a non-empty subset of $V(G)$, then by $A^\perp$ we denote the set of all vertices of G which are adjacent to all vertices of A. For the empty set we put $\emptyset^\perp = V(G)$. Further, denote $A^{\perp\perp} = (A^\perp)^\perp$. For each $A \subseteq V(G)$ we have $A \cap A^\perp = \emptyset$, $A \subseteq A^{\perp\perp}$, $(A^{\perp\perp})^\perp = A^\perp$, $(A^{\perp\perp})^{\perp\perp} = A^{\perp\perp}$. For any two subsets A, B of $V(G)$ the inclusion $A \subseteq B$ implies $B^\perp \subseteq A^\perp$ and $A^{\perp\perp} \subseteq B^{\perp\perp}$. Hence the mapping $A \mapsto A^{\perp\perp}$ is a certain closure operation on the set of all subsets of $V(G)$. The sets A for which $A^{\perp\perp} = A$ will be called the $\perp\perp$-closed sets in G. The empty set and the whole set $V(G)$ are among them.

The intersection of two $\perp\perp$-closed sets is again a $\perp\perp$-closed set. For the union this is not true in general. Nevertheless, to any two $\perp\perp$-closed sets A, B in G there exists exactly one $\perp\perp$-closed set which contains both A and B as subsets and is a subset of each $\perp\perp$-closed set in G which contains A and B; we shall denote it by $A \vee B$ and call it the join of A and B. Evidently $A \vee B = (A \cup B)^{\perp\perp}$.

The set of all $\perp\perp$-closed sets in G with the operation of join and the operation of meet equal to the intersection forms a lattice. This lattice with the operation $A \mapsto A^\perp$ added is called the logic of G and denoted by $\mathscr{L}(G)$.

The logic $\mathscr{L}(G)$ of a graph G is called orthomodular, if each set $A \in \mathscr{L}(G)$ has the property that an arbitrary set $B \in \mathscr{L}(G)$ such that A is a proper subset of B has a non-empty intersection with $A^\perp$. If $\mathscr{L}(G)$ is a modular lattice, then it is orthomodular.

We shall prove two theorems.

Theorem 1. *Let G be a finite connected undirected graph without loops and multiple edges, let $V(G)$ be its vertex set, let $\mathscr{L}(G)$ be its logic. Let $\mathscr{L}(G)$ have the property that to each atom $A \in \mathscr{L}(G)$ exactly one element $\bar{A} \in \mathscr{L}(G)$ exists such that $A \wedge \bar{A} = \emptyset$, $A \vee \bar{A} = V(G)$. Then $\mathscr{L}(G)$ is a Boolean algebra and G is a complete n-partite graph, where n is such an integer that 2^n is the number of elements of $\mathscr{L}(G)$.*

Proof. Let A be an atom of $\mathscr{L}(G)$. Suppose that there exist two vertices x, y of A which are joined by an edge in G. Then $\{y\} \subseteq \{x\}^{\perp}$ and consequently $\{x\}^{\perp\perp} \subseteq \{y\}^{\perp}$. As $\{x\} \subseteq A$, we have $\{x\}^{\perp\perp} \subseteq A^{\perp\perp} = A$. As A is an atom of $\mathscr{L}(G)$ and $\{x\}^{\perp\perp} \neq \emptyset$, we must have $\{x\}^{\perp\perp} = A$. But then $A \subseteq \{y\}^{\perp}$, therefore $y \in \{y\}^{\perp}$, which is impossible, because G has no loops. Therefore A is an independent set in G.

Now consider $V(G) - A$. Obviously $(V(G) - A)^{\perp} \subseteq A$; as A is an independent set, so is $(V(G) - A)^{\perp}$ Therefore $(V(G) - A)^{\perp\perp} \cap A = \emptyset$ and $(V(G) - A)^{\perp\perp} \subseteq$ $\subseteq V(G) - A$, which implies $(V(G) - A)^{\perp\perp} = V(G) - A$ and $V(G) - A \in \mathscr{L}(G)$. We have $A \cap (V(G) - A) = \emptyset$, $A \cup (V(G) - A) = V(G)$, hence $V(G) - A$ is the complement of A in $\mathscr{L}(G)$. On the other hand, we have $A \wedge A^{\perp} = A \cap A^{\perp} = \emptyset$. The set $A \vee A^{\perp}$ is equal to $(A \cup A^{\perp})^{\perp\perp}$. If $(A \cup A^{\perp})^{\perp}$ is non-empty, then there exists $x \in (A \cup A^{\perp})^{\perp}$. This element x must be adjacent to all vertices of A, hence $x \in A^{\perp}$, but then ti is not adjacent to all elements of $A^{\perp}$. Hence $(A \cup A^{\perp})^{\perp} = \emptyset$ and $(A \cup A^{\perp})^{\perp\perp} = V(G)$. As the complement of A is unique (according to the assumption), we have $A^{\perp} = V(G) - A$. Hence each vertex of A is adjacent to all vertices of $V(G) - A$.

It remains to prove that each vertex of G is contained in an atom of $\mathscr{L}(G)$. Suppose that there exists a vertex y for which this is not true. Then $\{y\}^{\perp\perp}$ is not an atom of $\mathscr{L}(G)$ and there exists an atom A of $\mathscr{L}(G)$ such that A is a proper subset of $\{y\}^{\perp\perp}$. Denote $B = \{y\}^{\perp\perp} - A$. We have $B \subseteq V(G) - A$, therefore each vertex of A is adjacent to all vertices of B. Obviously $y \in B$, therefore $A \subseteq \{y\}^{\perp}$ and $\{y\}^{\perp\perp} \subseteq A^{\perp} =$ $= V(G) - A$, which is a contradiction, because A is a subset of $\{y\}^{\perp\perp}$.

Therefore the atoms of $\mathscr{L}(G)$ (being pairwise disjoint) form a partition of $V(G)$. Each of them is an independent set in G and all of its vertices are adjacent to all vertices not belonging to it. This implies that G is a complete n-partite graph for a positive integer n. But such a graph is the direct sum of n graphs, each of which consists only of isolated vertices. The logic of a graph consisting only of isolated vertices is a Boolean algebra with 2 elements. The logic of the complete n-partite graph is then the direct product of n such Boolean algebras, hence a Boolean algebra with 2^n elements.

Theorem 2. *Let G be a finite connected bipartite graph on the sets A, B. Then the logic $\mathscr{L}(G)$ of G is orthomodular if and only if G is a complete bipartite graph.*

Proof. If G is a complete bipartite graph, then by Theorem 1 its logic is a Boolean algebra, therefore it is orthomodular. Now suppose that G is not a complete bipartite graph. If M is a subset of $V(G) = A \cup B$ such that $M \cap A \neq \emptyset$, $M \cap B \neq \emptyset$, then no vertex is adjacent to all vertices of M, because a vertex of A (or of B) can be adjacent only to vertices of B (or A, respectively). Hence $M^{\perp} = \emptyset$, $M^{\perp\perp} = V(G)$. This implies that each element of $\mathscr{L}(G)$ different from $V(G)$ is a subset of A or of B. As G is connected, for each $a \in A$ the set $\{a\}^{\perp}$ is a non-empty subset of B and hence $\{a\}^{\perp\perp} \subseteq A$. Analogously $\{b\}^{\perp\perp} \subseteq B$ for each $b \in B$. Hence there exists at least one set A_0 which is a non-empty subset of A and belongs to $\mathscr{L}(G)$. The set $A_0^\perp$ is a non-empty subset of B. The subgraph G_0 of G induced by the set $A_0 \cup A_0^\perp$ is a complete bipartite graph on the sets $A_0, A_0^\perp$. As G is not a complete bipartite graph, the graph G_0 is a proper subgraph of G; as G is connected, there exists either a vertex of A_0 adjacent to a vertex of $B - A_0^\perp$, or a vertex of $A_0^\perp$ adjacent to a vertex of $A - A_0$. If there exists a vertex $x \in A_0$ adjacent to a vertex of $B - A_0^\perp$, then $A_0^\perp$ is a proper subset of $\{x\}^\perp$ and consequently $\{x\}^{\perp\perp}$ is a proper subset of A_0. Denote $X = \{x\}^{\perp\perp}$. Then $\{x\}^\perp = X^\perp \subseteq B$ and we have $A_0 \wedge X^\perp = \emptyset$ and X is a proper subset of A_0, hence $\mathscr{L}(G)$ is not orthomodular. If there exists a vertex $y \in A_0$ adjacent to a vertex of $A - A_0$, then A_0 is a proper subset of $\{y\}^\perp$ and consequently $\{y\}^{\perp\perp}$ is a proper subset of A_0. Denote $Y = \{y\}^{\perp\perp}$. Then $\{y\}^\perp = Y^\perp \subseteq A$ and we have $A_0^\perp \wedge Y^\perp = \emptyset$ and Y is a proper subset of A_0, hence again $\mathscr{L}(G)$ is not orthomodular.

Corollary. *Let G be a finite connected bipartite graph on the sets A, B. Then the logic of G is modular if and only if G is a complete bipartite graph.*

This follows immediately from Theorem 1 and Theorem 2.

Reference

[1] *D. J. Foulis, C. H. Randall:* Operational statistics. I. Basic concepts. Math. Physics, Vol. 13, No. 11.

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