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ON BOUNDARY ELEMENTS OF THE FOURTH KIND

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We use definitions and notation from [2]. Let Ω be a fixed subregion of the closed Gaussian plane $\mathbf{S}$, conformally equivalent to the unit circle $\mathbf{U}$; let $F : \Omega \xrightarrow{\text{onto}} \mathbf{U}$ be a fixed conformal mapping. As, if needed, we may apply a suitable homography, we suppose throughout the following text that $\partial\Omega$ does not contain the point ∞; in this way we simplify formal aspects while preserving the full generality of results.

By a cut in Ω we mean every one-one or Jordan curve $\varphi : \langle\alpha, \beta\rangle \to \bar{\Omega}$ with $(\varphi) (= \varphi((\alpha, \beta))) \subset \Omega$, $\varphi(\alpha), \varphi(\beta) \in \partial\Omega$, $(F \circ \varphi)(\alpha+) \neq (F \circ \varphi)(\beta-)$; let us note that the last inequality means the curves $\varphi \mid \langle\alpha, \frac{1}{2}(\alpha + \beta)\rangle$, $\dot{-}\varphi \mid \langle\frac{1}{2}(\alpha + \beta), \beta\rangle$ belong to two distinct bundles (of curves from $\partial\Omega$ into Ω – cf. [2]). Boundary elements of the region Ω are certain classes of "normal" (see [2]) sequences $\{\Omega_n\}_{n=1}^{\infty}$ of subregions of Ω; we denote by $\mathfrak{H}$ the set of all boundary elements of Ω.

1. Let $\mathscr{H} \in \mathfrak{H}$, $\{\Omega_n\}_{n=1}^{\infty} \in \mathscr{H}$. Suppose $\{z_k\}_{k=1}^{\infty}$ is a sequence of points from Ω and

$$(1) \qquad \varphi : (\alpha, \beta) \to \Omega \quad \text{is a continuous mapping}\,.$$

Write

$$(2') \qquad z_k \to \mathscr{H}\,,$$

iff for each n there is a $k(n)$ with $z_k \in \Omega_n$ for all $k > k(n)$; write

$$(2'') \qquad \varphi \to \mathscr{H}\,,$$

iff for each n is a $\delta_n > 0$ with $\varphi((\alpha, \alpha + \delta_n)) \subset \Omega_n$.[1])

As in [2], denote by $\gamma_F(\mathscr{H})$ the only element of the set $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$.[1]) We easily see that

$$(3') \qquad z_k \to \mathscr{H}\,, \quad \text{iff} \quad F(z_k) \to \gamma_F(\mathscr{H})\,,$$

[1]) As $\{\Omega_n\} \in \mathscr{H}$, $\{\Omega_m^*\} \in \mathscr{H}$ iff the (normal) sequences $\{\Omega_n\}$, $\{\Omega_m^*\}$ are mutually inscribed, the definition is independent of the choice of the sequence $\{\Omega_n\} \in \mathscr{H}$.

and

$$(3'') \qquad \varphi \to \mathscr{H}\,, \quad \text{iff} \quad (F \circ \varphi)(\alpha+) = \gamma_F(\mathscr{H})\,.$$

For each mapping (1) we denote (as in [3])

$$(4) \qquad \mathscr{P}(\varphi) = \bigcap_{n=1}^{\infty} \overline{\varphi((\alpha, \alpha + \delta_n))}\,,$$

where $\{\delta_n\}$ is an arbitrary strictly decreasing sequence of positive numbers converging to 0; evidently, the right-hand side of (4) is independent of the choice of such a sequence $\{\delta_n\}$. As is easy to see, the identities

$$(4') \qquad \mathscr{P}(\varphi) = \operatorname{Ls} \varphi((\alpha, \alpha + \delta_n)) = \operatorname{Ls} \varphi(\langle \alpha + \delta_{n+1}, \alpha + \delta_n \rangle)$$

(where Ls denotes the topological limes superior) hold.

We easily verify that

$$(5) \qquad \varphi \to \mathscr{H} \Rightarrow \mathscr{P}(\varphi) \subset \langle \mathscr{H} \rangle\,,$$

where $\langle \mathscr{H} \rangle$ is the geometrical image of the boundary element $\mathscr{H}$, i.e., the continuum $\bigcap_{n=1}^{\infty} \bar{\Omega}_n$ (see [2]).

Carathéodory (cf. [1]) distinguished four kinds of boundary elements; we denote by $\mathfrak{H}_j (1 \leqq j \leqq 4)$ the set of all elements of the j-th kind. (The classification may be realised, e.g., as follows: $\mathscr{H} \in \mathfrak{H}_1 \cup \mathfrak{H}_2$ means that there is a curve from $\partial\Omega$ into Ω with $\varphi \to \mathscr{H}$; then $\mathscr{H} \in \mathfrak{H}_1$ ($\mathscr{H} \in \mathfrak{H}_2$), iff $\langle \mathscr{H} \rangle$ is a one-point set (a proper continuum). $\mathscr{H} \in \mathfrak{H}_3 \cup \mathfrak{H}_4$ means that $\mathscr{H} \in \mathfrak{H} - (\mathfrak{H}_1 \cup \mathfrak{H}_2)$; then $\mathscr{H} \in \mathfrak{H}_3$ ($\mathscr{H} \in \mathfrak{H}_4$), iff the implication $\varphi \to \mathscr{H} \Rightarrow \mathscr{P}(\varphi) = \langle \mathscr{H} \rangle$ holds (does not hold). Thus, $\mathscr{H} \in \mathfrak{H}_1 \cup \mathfrak{H}_2$, iff there is a mapping (1) such that $\varphi \to \mathscr{H}$ and that $\mathscr{P}(\varphi)$ is a one-point set; further, $\mathscr{H} \in \mathfrak{H}_3 \cup \mathfrak{H}_4$, iff for each mapping (1) with $\varphi \to \mathscr{H}$ the set $\mathscr{P}(\varphi)$ is a proper continuum.)

We easily see that

(6) *for each $\mathscr{H} \in \mathfrak{H}$ there is a mapping* (1) *with $\varphi \to \mathscr{H}$ and $\mathscr{P}(\varphi) = \langle \mathscr{H} \rangle$;*

directly from the definition of boundary elements $\mathscr{H}$ of the second and the fourth kind it follows that there are mappings (1) with $\varphi \to \mathscr{H}$ and $\mathscr{P}(\varphi) \neq \langle \mathscr{H} \rangle$. If $\mathscr{H} \in \mathfrak{H}_2$, there is a point $a_{\mathscr{H}} \in \langle \mathscr{H} \rangle$ such that $a_{\mathscr{H}} \in \mathscr{P}(\varphi)$ for each $\varphi \to \mathscr{H}$; at the same time, there are mappings $\varphi \to \mathscr{H}$ with $\mathscr{P}(\varphi) = \{a_{\mathscr{H}}\}$. (In terms of definitions and notation from [2], the point $a_{\mathscr{H}}$ is the origin of the bundle $\mathscr{S}$ which determine the boundary element $\mathscr{H}$.)

Thus, if $\mathscr{H} \in \mathfrak{H}_1 \cup \mathfrak{H}_2 \cup \mathfrak{H}_3$, there are continuous mappings $\varphi \to \mathscr{H}$ with "minimal" $\mathscr{P}(\varphi)$. Our main goal is the proof of an analogous assertion for elements of the fourth kind:

Theorem. *If $\mathscr{H} \in \mathfrak{H}_4$, then there is a continuous mapping $\varphi_0 \to \mathscr{H}$ such that $\mathscr{P}(\varphi_0) \subset \mathscr{P}(\varphi)$ for each $\varphi \to \mathscr{H}$.*

We prove the theorem in § 3; before doing so we introduce several symbols and prove some auxiliary assertions.

If $\mathscr{H} \in \mathfrak{H}$, we write

$$(7') \qquad A(\mathscr{H}) = \{z \in \langle\mathscr{H}\rangle;\ z \in \mathscr{P}(\varphi) \text{ for each } \varphi \to \mathscr{H}\},$$

$$(7'') \qquad B(\mathscr{H}) = \{z \in \langle\mathscr{H}\rangle;\ \text{there is a } \varphi \to \mathscr{H} \text{ with } z \notin \mathscr{P}(\varphi)\}.$$

Evidently,

$$(8) \qquad \langle\mathscr{H}\rangle = A(\mathscr{H}) \cup B(\mathscr{H}),\quad A(\mathscr{H}) \cap B(\mathscr{H}) = \emptyset.$$

According to whether $\mathscr{H} \in \mathfrak{H}_1$, $\mathscr{H} \in \mathfrak{H}_2$, $\mathscr{H} \in \mathfrak{H}_3$, $\mathscr{H} \in \mathfrak{H}_4$, $A(\mathscr{H})$ is equal to the one-point set $\langle\mathscr{H}\rangle$ $(= \{a_{\mathscr{H}}\})$, to the one-point set $\{a_{\mathscr{H}}\}$ $(\neq \langle\mathscr{H}\rangle)$, to the proper continuum $\langle\mathscr{H}\rangle$, to the proper continuum $\mathscr{P}(\varphi_0)$ where φ_0 is as in the above theorem, respectively. Further, $\mathscr{H} \in \mathfrak{H}_1 \cup \mathfrak{H}_3$, iff $B(\mathscr{H}) = \emptyset$, and $\mathscr{H} \in \mathfrak{H}_2 \cup \mathfrak{H}_4$, iff $B(\mathscr{H}) \neq \emptyset$.

Example 1. Let Ω be the set-difference of the square $\{z; 0 < Re\, z < 1,\ 0 < Im\, z < 1\}$ and the union of all segments $\langle 2^{-2n}; 2^{-2n} + \frac{2}{3}i\rangle$, $\langle 2^{-2n+1} + i; 2^{-2n+1} + \frac{1}{3}i\rangle$ (where n is a positive integer). Then the segment $\langle 0; i\rangle$ is the geometrical image of precisely one boundary element $\mathscr{H}$ (of the region Ω); for this $\mathscr{H}$, $A(\mathscr{H})$ is the segment $\langle \frac{1}{3}i; \frac{2}{3}i\rangle$.

Remark 1. The connectedness of the set $A(\mathscr{H})$ is evident for each element $\mathscr{H} \in \mathfrak{H} - \mathfrak{H}_4$; by the theorem, $A(\mathscr{H})$ is a proper continuum for each $\mathscr{H} \in \mathfrak{H}_4$ as well. For each $\mathscr{H} \in \mathfrak{H}$, the set $A(\mathscr{H})$ is the intersection of all sets $\mathscr{P}(\varphi)$ where $\varphi \to \mathscr{H}$. This intersection being *always connected*, mappings $\varphi \to \mathscr{H}$, $\psi \to \mathscr{H}$ may exist with $\mathscr{P}(\varphi) \cap \mathscr{P}(\psi)$ *disconnected*; however, such a situation may occur only if $\mathscr{H} \in \mathfrak{H}_2 \cup \mathfrak{H}_4$. The following example confirms the possibility of the situation.

Example 2. On the left-hand (right-hand) figure, $\mathscr{H}$ is a boundary element of the second (fourth) kind with $\langle\mathscr{H}\rangle$ equal to the union of the segments B, D and the circumference C (of the segments B, D and the circumferences A, C).

With aid of continuous mappings φ, ψ we may "approach" the boundary element $\mathscr{H}$ "from the left" and "from the right", respectively, in such a way that the intersection $\mathscr{P}(\varphi) \cap \mathscr{P}(\psi)$ is the thickly marked disconnected set.

By easy modification, an example of Ω, $\mathscr{H}$, φ, ψ may be created in which $\mathscr{P}(\varphi) \cap \mathscr{P}(\psi)$ has uncountably many components.

2. In the proof of the theorem we shall need several auxiliary assertions.

Lemma 1. *If $\lambda : \langle 0, 1\rangle \to \bar{\Omega}$ is a Jordan curve with*

$$(9) \qquad \lambda(0), \lambda(1) \in \partial\Omega,\quad (\lambda) \subset \Omega,$$

(10) $\qquad \operatorname{Int} \lambda \cap \partial\Omega \neq \emptyset \neq \operatorname{Ext} \lambda \cap \partial\Omega ,$

then λ is a cut in Ω.

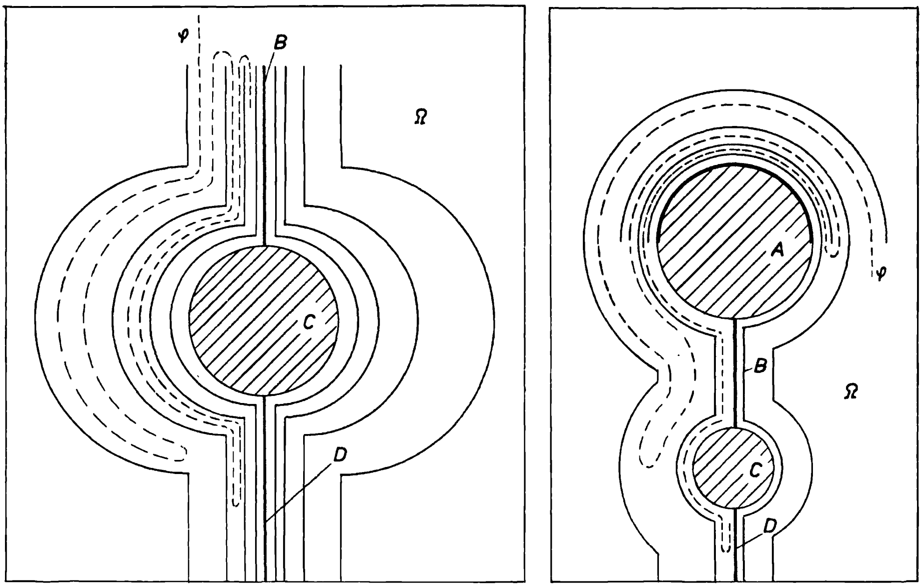

Fig. 1.

Proof. Suppose the assumptions of Lemma 1 are satisfied, but λ is no cut; then the F-image[2]) μ of the curve λ is a Jordan curve. By (10), there are curves λ_1, λ_2 such that $i.p.\lambda_1 \in \operatorname{Int} \lambda \cap \partial\Omega$, $i.p.\lambda_2 \in \operatorname{Ext} \lambda \cap \partial\Omega$, $\langle\lambda_1\rangle \subset \operatorname{Int} \lambda \cap \Omega$, $\langle\lambda_2\rangle \subset \operatorname{Ext} \lambda \cap \Omega$. The F-images μ_j of the curves λ_j are curves from $\partial\boldsymbol{U}$ into $\boldsymbol{U}$ and $\langle\mu_j\rangle \cap \langle\mu\rangle = \emptyset$ for $j = 1, 2$. Therefore, both end points $b_j = e.p.\mu_j$ must lie in the same component $U_1 = \boldsymbol{U} - \overline{\operatorname{Int} \mu}$ of the set $\boldsymbol{U} - \langle\mu\rangle$; as a consequence, $F_{-1}(U_1)$ is a component of the set $\Omega - \langle\lambda\rangle$ containing both the points $e.p.\lambda_j$ – a contradiction.

Lemma 2. (Carathéodory.) *For each $\mathscr{H} \in \mathfrak{H}$ there is a point $z_0 \in \langle\mathscr{H}\rangle$, a (strictly) decreasing sequence of positive numbers r_n with $r_n \to 0$, and a normal sequence $\{\Omega_n\} \in \mathscr{H}$ such that, for each n, $\Omega \cap \partial\Omega_n$ is a connected subset of the circumference $|z - z_0| = r_n$.*

Proof – see [1].

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[2]) If φ is as in (1) and if the limits $(F \circ \varphi)(\alpha+)$, $(F \circ \varphi)(\beta-)$ exist, then the F-image of φ is the curve ψ defined on $\langle\alpha, \beta\rangle$ as follows: $\psi(\alpha) = (F \circ \varphi)(\alpha+)$, $\psi(t) = F(\varphi(t))$ for $t \in (\alpha, \beta)$, $\psi(\beta) = (F \circ \varphi)(\beta-)$ (cf. [2]).

R e m a r k 2. Let the conditions of Lemma 2 hold. If $\varphi \to \mathscr{H}$, then $(\varphi) \cap \partial\Omega_n \neq \emptyset$ for all n sufficiently large. As a consequence, $z_0 \in \mathscr{P}(\varphi)$ (for each mapping (1) with $\varphi \to \mathscr{H}$); thus $z_0 \in A(\mathscr{H})$.

We see *the set $A(\mathscr{H})$ is non-empty (for each $\mathscr{H} \in \mathfrak{H}$).*

Given any (finite) complex number z and any number $\delta \in (0, \infty)$ we set

$$(11) \qquad Q(z, \delta) = \{z';\ |\mathrm{Re}\,(z' - z)| \leqq \delta,\ |\mathrm{Im}\,(z' - z)| \leqq \delta\}\,.$$

If $z \in \partial\Omega$, then the condition

$$(12_1) \qquad \partial\Omega - Q(z, \delta) \neq \emptyset\,,$$

and, as a consequence, also the condition

$$(12_2) \qquad \partial\Omega \cap \partial Q(z, \delta) \neq \emptyset\,,$$

hold for each sufficiently small $\delta > 0$. For each sufficiently small $\delta > 0$, moreover,

$$(12_3) \qquad F_{-1}(0) \in \Omega - Q(z, \delta)\,.$$

Suppose all these conditions hold and let

$$(13_1) \qquad \lambda : \langle 0, 1\rangle \xrightarrow{\text{onto}} \partial Q(z, \delta)$$

be a fixed Jordan curve with

$$(13_2) \qquad \lambda(0) = \lambda(1) \in \partial\Omega\,.$$

Then there is a finite or infinite sequence of disjoint open intervals

$$(14) \qquad I_1 = (u_1, v_1)\,,\quad I_2 = (u_2, v_2),\ \ldots$$

contained in $(0, 1)$ such that

$$(15) \qquad \Omega \cap \partial Q(z, \delta) = \bigcup_k \lambda(I_k)\,,\quad \lambda(u_k), \lambda(v_k) \in \partial\Omega\,.$$

We assert that then

(16) *the curve $\lambda_k = \lambda \mid \langle u_k, v_k\rangle$ is a cut in Ω (for each k).*

This is clear, if λ_k is one-one; if λ_k is not one-one, then $k = 1$, $\lambda_k = \lambda$, and λ_k is a cut by Lemma 1.

Denoting by μ_k the F-image of λ_k, μ_k is a one-one cut in $\boldsymbol{U}$. Evidently, the following two implications hold:

(17) *If $\mu_k(u_k) = \gamma_F(\mathscr{H})$ (for some $\mathscr{H} \in \mathfrak{H}$), then $\lambda_k \to \mathscr{H}$; if $\mu_k(v_k) = \gamma_F(\mathscr{H})$, then $\dot{-}\lambda_k \to \mathscr{H}$.*

As a consequence:

(18) *If either $\mu_k(u_k) = \gamma_F(\mathscr{H})$, or $\mu_k(v_k) = \gamma_F(\mathscr{H})$ (for some $\mathscr{H} \in \mathfrak{H}$), then $\mathscr{H} \in \in \mathfrak{H}_1 \cup \mathfrak{H}_2$.*

Each cut μ_k splits the circle $\boldsymbol{U}$ into two regions U_k, U_k^*; choose the notation so that $0 \in U_k$. Suppose now $\mathscr{H} \in \mathfrak{H}_3 \cup \mathfrak{H}_4$; then, by (18), $\gamma_F(\mathscr{H})$ is not equal to any point $\mu_k(u_k)$, $\mu_k(v_k)$. As a consequence, the point $\gamma_F(\mathscr{H})$ lies in the closure of precisely one of the regions U_k, U_k^*. Set

(19) $$C_1(\mathscr{H}) = \{k; \gamma_F(\mathscr{H}) \in \bar{U}_k\}, \quad C_2(\mathscr{H}) = \{k; \gamma_F(\mathscr{H}) \in \bar{U}_k^*\}.$$

Thus,

(20) $k \in C_1(\mathscr{H})$ $(k \in C_2(\mathscr{H}))$, *iff* $\langle\mu_k\rangle$ *does not separate (separates)* $\bar{U}$ *between the points* 0 *and* $\gamma_F(\mathscr{H})$.

Lemma 3. *For each* $\mathscr{H} \in \mathfrak{H}_4$ *and each* $z \in B(\mathscr{H})$, *there is a* $\Delta(z) > 0$ *such that the conditions* (12_1)–(12_3) *hold and* $C_2(\mathscr{H}) = \emptyset$ *for each* $\delta \in (0, \Delta(z)\rangle$.

Proof. Supposing the contrary there is an $\mathscr{H} \in \mathfrak{H}_4$, a $z \in B(\mathscr{H})$, a sequence of positive numbers δ_n with $\delta_n \to 0$, and cuts $\lambda^n : \langle u^n, v^n\rangle \to \partial Q(z, \delta_n)$ in Ω such that, denoting by μ^n the F-image of λ^n, each set $\langle\mu^n\rangle$ separates the circle $\bar{\boldsymbol{U}}$ between 0 and $\gamma_F(\mathscr{H})$.

As $z \in B(\mathscr{H})$, there is a continuous mapping $\varphi : (0, 1\rangle \to \Omega$ with $\varphi \to \mathscr{H}$, $z \notin \mathscr{P}(\varphi)$; we may suppose $\varphi(1) = F_{-1}(0)$. Denoting by ψ the F-image of φ we have $\psi(0) = \gamma_F(\mathscr{H})$, $\psi(1) = 0$. As a consequence, $(\mu^n) \cap (\psi) \neq \emptyset$, which implies $(\lambda^n) \cap (\varphi) \neq \emptyset$ (for all n). Choose numbers $t_n \in (0, 1)$ so that $\varphi(t_n) \in (\lambda^n)$. As $\operatorname{Ls}(\lambda^n) = \{z\} \in \partial\Omega$, we necessarily have $t_n \to 0$, and $\varphi(t_n) \to z$. This contradicts our premise $z \notin \mathscr{P}(\varphi)$; Lemma 3 is proved.

Lemma 4. *Suppose* $\mathscr{H} \in \mathfrak{H}_4$, $z \in B(\mathscr{H})$, $\varphi \to \mathscr{H}$, *and let* $\Delta(z)$ *be as in Lemma* 3. *Then there is a continuous mapping* $\psi \to \mathscr{H}$ *such that*

(21) $$\mathscr{P}(\psi) \subset \mathscr{P}(\varphi) - \operatorname{int} Q(z, \Delta(z)).$$

Proof. Let the assumptions of Lemma 4 hold. By a "slight" modification of the mapping φ we easily obtain a mapping $\varphi_0 : (0, 1\rangle \to \Omega$ with the following properties: The mapping φ_0 is not constant on any interval $I \subset (0, 1)$; for each $\eta \in (0, 1)$, the mapping $\varphi_0 \mid \langle\eta, 1\rangle$ is piece-wise linear; no segment contained in $\varphi_0((0, 1\rangle)$ is parallel to the real axis, nor to the imaginary one; $\mathscr{P}(\varphi_0) = \mathscr{P}(\varphi)$; $\varphi_0 \to \mathscr{H}$. Evidently, we may suppose $\varphi_0(1) = F_{-1}(0)$ as well.

Set $\delta = \Delta(z)$ (where $\Delta(z)$ is as in Lemma 3) and for the square $Q(z, \delta)$ construct the intervals (14) and the curves λ_k, μ_k (with the above properties); as above let U_k, U_k^* be the components of the set $\boldsymbol{U} - (\mu_k)$ $(0 \in U_k)$; set

(22) $$\Omega_k = F_{-1}(U_k), \quad \Omega_k^* = F_{-1}(U_k^*)$$

(so that Ω_k, Ω_k^* are (the only two) components of the set $\Omega - (\lambda_k)$). $C_j(\mathscr{H})$ $(j = 1, 2)$ being as in (19), we have $C_2(\mathscr{H}) = \emptyset$ by Lemma 3.

Two situations may occur: I. The mapping φ_0 has no substantial intersection point with $\partial Q(z, \delta)$ [3]; as $\varphi_0(1) = F_{-1}(0) \in \Omega - Q(z, \delta)$, we then have $\mathscr{P}(\varphi_0) \subset (\bar{\varphi}_0) \subset \subset \boldsymbol{S} - \operatorname{int} Q(z, \delta)$ and the mapping $\psi = \varphi_0$ satisfies (21).

II. The mapping φ_0 has at least one substantial intersection point with $\partial Q(z, \delta)$. Let t_1 be the maximal one and let $j_1 \in C_1(\mathscr{H})$ be the index with $\varphi_0(t_1) \in (\lambda_{j_1})$; as $\varphi_0(1) \in \Omega_{j_1}$ implies $\varphi_0((t_1, 1\rangle) \subset \Omega_{j_1}$, there is an $\eta > 0$ with $\varphi_0((t_1 - \eta, t_1)) \subset \Omega^*_{j_1}$. Relations $(F \circ \varphi_0)(0+) = \gamma_F(\mathscr{H}) \notin \bar{U}^*_{j_1}$ [4]) imply the existence of such an $\eta' > 0$ that $(F \circ \varphi_0)((0, \eta')) \subset U_{j_1}$, i.e., $\varphi_0((0, \eta')) \subset \Omega_{j_1}$. As a consequence, there is a minimal number $T_1 \in (0, t_1)$ with $\varphi_0(T_1) \in (\lambda_{j_1})$. Obviously, then $\varphi_0((0, T_1)) \subset \Omega_{j_1}$.

Define the mapping $h_1 : \langle T_1, t_1 \rangle \to (\lambda_{j_1})$ as follows: If $\varphi_0(T_1) = \varphi_0(t_1)$, then h_1 is constant, equal to $\varphi_0(T_1)$; if $\varphi_0(T_1) \neq \varphi_0(t_1)$, then h_1 is a one-one continuous mapping with $h_1(T_1) = \varphi_0(T_1)$, $h_1(t_1) = \varphi_0(t_1)$. The mapping

$$(23) \qquad \varphi_1(t) = \begin{cases} \varphi_0(t) & \text{for} \quad t \in (0, T_1\rangle \cup \langle t_1, 1\rangle\,, \\ h_1(t) & \text{for} \quad t \in \langle T_1, t_1 \rangle \end{cases}$$

is continuous on $(0, 1\rangle$, $\varphi_1 \to \mathscr{H}$, $\mathscr{P}(\varphi_1) = \mathscr{P}(\varphi)$, $\varphi_1((0, 1\rangle) \cap \Omega^*_{j_1} = \emptyset$.

Again, there are two possibilities: I′. The mapping φ_1 has no substantial intersection point with $\partial Q(z, \delta)$; then $\psi = \varphi_1$ satisfies (21). II′. The mapping φ_1 has at least one substantial intersection point with $\partial Q(z, \delta)$; then all such points lie in the interval $(0, T_1)$. Let t_2 be the maximal one; find the index $j_2 \in C_1(\mathscr{H})$ with $\varphi_1(t_2) \in (\lambda_{j_2})$. Evidently $j_2 \neq j_1$. For analogous reasons as above, there is a minimal number $T_2 \in (0, t_2)$ with $\varphi_1(T_2) \in (\lambda_{j_2})$, and $\varphi_1((0, T_2)) \cup \varphi_1((t_2, 1\rangle) \subset \Omega_{j_2}$. Analogously as above, construct the curve h_2 in (λ_{j_2}) with terminal points $h_2(T_2) = \varphi_0(T_2)$, $h_1(t_2) = \varphi_0(t_2)$, and with aid of it and of φ_1 define the mapping $\varphi_2 : (0, 1\rangle \to \Omega$ with the following properties: $\varphi_2((0, 1\rangle) \cap (\Omega^*_{j_1} \cup \Omega^*_{j_2}) = \emptyset$, $\varphi_2(t) = \varphi_0(t)$ on $(0, T_2)$, so that $\varphi_2 \to \mathscr{H}$ and $\mathscr{P}(\varphi_2) = \mathscr{P}(\varphi)$.

Continuing this process, we either construct, after a finite number of steps, a continuous mapping $\varphi_n : (0, 1\rangle \to \Omega$ with no substantial intersection point with $\partial Q(z, \delta)$ and such that $\varphi_n \to \mathscr{H}$, $\mathscr{P}(\varphi_n) = \mathscr{P}(\varphi)$, or the construction of mappings φ_n never ceases. In the former case we evidently have $(\varphi_n\rangle \cap \operatorname{int} Q(z, \delta) = \emptyset$, and the mapping $\psi = \varphi_n$ satisfies (21). In the latter case we obtain an infinite sequence of mappings $\varphi_n : (0, 1\rangle \to \Omega$, an infinite sequence of mutually distinct indices $j_n \in C_1(\mathscr{H})$, and an infinite sequence of numbers $1 > t_1 > T_1 > \ldots > t_n > T_n > \ldots > 0$ such that, for every integer $n \geqq 1$, the following conditions hold:

$$(24_1) \qquad \varphi_n(t) = \varphi_{n-1}(t) \quad \text{for each} \quad t \in \langle t_n, 1\rangle\,;$$

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[3]) We say a point t_0 of the set $M = \{t \in (0, 1\rangle;\ \varphi_0(t) \in \partial Q(z, \delta)\rangle$ is a substantial intersection point, iff there is an $\eta > 0$ such that one of the sets $\varphi_0((t_0 - \eta, t_0))$, $\varphi_0((t_0, t_0 + \eta))$ lies in the interior and the other one in the exterior of the square $Q(z, \delta)$. Note that, by properties of the mapping φ_0, the set M has no accumulation point in $(0, 1\rangle$.

[4]) As $C_2(\mathscr{H}) = \emptyset$, $\langle \mu_{j_1} \rangle$ does not separate $\bar{\boldsymbol{U}}$ between 0 and $\gamma_F(\mathscr{H})$.

$$(24_2) \qquad \varphi_n(t) \in (\lambda_{j_n}) \quad \text{for each} \quad t \in \langle T_n, t_n \rangle ;$$

$$(24_3) \qquad \varphi_n(t) = \varphi_0(t) \quad \text{for each} \quad t \in (0, T_n \rangle ;$$

$$(24_4) \qquad \varphi_n \mid \langle T_n, 1 \rangle \quad \text{does not intersect} \quad \partial Q(z, \delta) \quad \text{substantially} ;$$

$$(24_5) \qquad (\varphi_n \rangle \cap \bigcup_{k=1}^{n} \Omega^*_{j_k} = \emptyset ;$$

(24_3) implies

$$(24_6) \qquad \varphi_n \to \mathscr{H} , \quad \mathscr{P}(\varphi_n) = \mathscr{P}(\varphi) .$$

Let us show that $T_n \to 0$ (so that $t_n \to 0$ as well). As the indices j_n are mutually distinct, $(\lambda_{j_1}), (\lambda_{j_2}), \ldots, (\lambda_{j_n}), \ldots$ are disjoint open arcs contained in $\partial Q(z, \delta)$; as a consequence, diam $\langle \lambda_{j_n} \rangle \to 0$. As the terminal points of the arcs $\langle \lambda_{j_n} \rangle$ lie in $\partial\Omega$, it follows that Ls $\langle \lambda_{j_n} \rangle \subset \partial\Omega$. By (24_3) and (24_4), $\varphi_0(T_n) = \varphi_n(T_n) \in (\lambda_{j_n})$; therefore, Ls $\varphi_0(T_n) \subset \partial\Omega$ as well. As $(\varphi_0\rangle \subset \Omega$, this necessarily implies $T_n \to 0$.

By (24_1), identities

$$(25) \qquad \psi(t) = \varphi_n(t) \quad \text{for each} \quad t \in \langle t_{n+1}, 1 \rangle , \quad n = 0, 1, \ldots$$

define a (continuous) mapping $\psi : (0, 1\rangle \to \Omega$.

By (24_4), ψ has no substantial intersection point with $\partial Q(z, \delta)$; as $\psi(1) = F_{-1}(0) \in \Omega - Q(z, \delta)$, this implies $(\psi\rangle \cap \operatorname{int} Q(z, \delta) = \emptyset$ and, as a consequence,

$$(26) \qquad \mathscr{P}(\psi) \cap \operatorname{int} Q(z, \delta) = \emptyset$$

as well.

By (25), (24_3), (24_2), we have $\mathscr{P}(\psi) \subset \mathscr{P}(\varphi_0) \cup \text{Ls} \langle \lambda_{j_n} \rangle$; relations $\varphi_0(t_n) \in \langle \lambda_{j_n} \rangle$, diam $\langle \lambda_{j_n} \rangle \to 0$, $t_n \to 0$ imply Ls $\langle \lambda_{j_n} \rangle \subset \mathscr{P}(\varphi_0)$. It follows that

$$(27) \qquad \mathscr{P}(\psi) \subset \mathscr{P}(\varphi_0) = \mathscr{P}(\varphi) .$$

It remains to prove that $\psi \to \mathscr{H}$, i.e., $(F \circ \psi)(0+) = \gamma_F(\mathscr{H})$, i.e., $\mathscr{P}(F \circ \psi) = \{\gamma_F(\mathscr{H})\}$. However,

$$(28) \qquad \mathscr{P}(F \circ \psi) = \text{Ls}\,(F \circ \psi)(\langle t_{n+1}, t_n \rangle) = \text{Ls}\,(F \circ \psi)(\langle t_{n+1}, T_n \rangle) \cup \text{Ls}\,(F \circ \psi)(\langle T_n, t_n \rangle) \subset \text{Ls}\,(F \circ \varphi_0)(\langle t_{n+1}, T_n \rangle) \cup \text{Ls}\,(\mu_{j_n})$$

(by (25), (24_3), (24_2)). The relations $(F \circ \varphi_0)(0+) = \gamma_F(\mathscr{H})$, $T_n \to 0$, $t_n \to 0$ imply Ls $(F \circ \varphi_0)(\langle t_{n+1}, T_n \rangle) = \{\gamma_F(\mathscr{H})\}$. As $F(\varphi_0(t_n)) \in (\mu_{j_n})$, $F(\varphi_0(t_n)) \to \gamma_F(\mathscr{H})$, the equality Ls $(\mu_{j_n}) = \{\gamma_F(\mathscr{H})\}$ holds (and Lemma 4 is proved), if diam $(\mu_{j_n}) \to 0$.

Suppose the last relation is not correct. Then there is a subsequence $\{j_{n(k)}\}$ of $\{j_n\}$ and there are arcs $M_k \subset (\mu_{j_{n(k)}})$ with terminal points a_k, b_k such that $a_k \to a$, $b_k \to b \neq a$, and that Ls $\langle \lambda_{j_{n(k)}} \rangle$ is a one-point set[5]).

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[5]) As diam $\langle \lambda_{j_n} \rangle \to 0$ and **S** is compact, we can (by an appropriate choice of the subsequence $\{j_{n(k)}\}$) satisfy the last condition as well.

However, such a situation is impossible, for the mapping F_{-1} (inverse of F) would be, by a well known corollary of the Lindelöf Lemma[6]), constant.

This finishes the proof of Lemma 4.

3. Proof of the theorem. For each $z \in B(\mathscr{H})$ let us construct, by Lemma 3, the number $\Delta(z)$. By the inclusion $B(\mathscr{H}) \subset \bigcup_{z \in B(\mathscr{H})} U(z, \frac{1}{2}\Delta(z))$ and by separability, there is a sequence of points $z_n \in B(\mathscr{H})$ with

(29) $$B(\mathscr{H}) \subset \bigcup_{n=1}^{\infty} U(z_n, \tfrac{1}{2}\Delta(z_n)).$$

Put

(30) $$Q_n = Q(z_n, \Delta(z_n)), \quad Q_n^* = Q(z_n, \tfrac{1}{2}\Delta(z_n))$$

and construct a sequence of continuous mappings ψ_j, $j = 0, 1, \ldots$, in the following way: $\psi_0 : (0, 1\rangle \to \Omega$ is an arbitrary continuous mapping with $\psi_0 \to \mathscr{H}$. Further, suppose that, for an index n, a continuous mapping $\psi_n : (0, 1\rangle \to \Omega$ has already been constructed satisfying $\psi_n \to \mathscr{H}$ and

(31) $$\mathscr{P}(\psi_n) \subset \mathscr{P}(\psi_0) - \bigcup_{j=1}^{n} Q_j^*.$$

By Lemma 4, there is a continuous mapping $\psi_{n+1} : (0, 1\rangle \to \Omega$ with $\psi_{n+1} \to \mathscr{H}$ and $\mathscr{P}(\psi_{n+1}) \subset \mathscr{P}(\psi_n) - \operatorname{int} Q_{n+1} \subset \mathscr{P}(\psi_0) - \bigcup_{j=1}^{n+1} Q_j^*$.

Choose a decreasing sequence of numbers $\vartheta_n > 0$ such that $\vartheta_n \to 0$ and

(32) $$\overline{U(\mathscr{P}(\psi_n), \vartheta_n)} \cap \bigcup_{j=1}^{n} Q_j^* = \emptyset.$$

Let Ω_n, r_n, z_0 be as in Lemma 2 and set $R_n = \Omega \cap \partial\Omega_n$. As it is easily seen, there exist numbers $\delta_n > 0$ and an increasing sequence of indices k_n with

(33₁) $$\overline{\psi_n((0, \delta_n))} \subset U(\mathscr{P}(\psi_n), \vartheta_n),$$

(33₂) $$\psi_n(\delta_n) \in R_{k_n}, \quad r_{k_n} < \vartheta_n,$$

(33₃) $$\psi_n((0, \delta_n)) \subset \Omega_{k_n}.$$

Further, there exist numbers $\delta_n^* \in (0, \delta_n)$ with

(34) $$\psi_n(\delta_n^*) \in R_{k_{n+1}}.$$

Define a curve $\chi_n : \langle 0, \delta_n^* \rangle \to R_{k_{n+1}}$ as follows: If $\psi_{n+1}(\delta_{n+1}) = \psi_n(\delta_n^*)$, then χ_n is constant, equal to $\psi_n(\delta_n^*)$; if $\psi_{n+1}(\delta_{n+1}) \neq \psi_n(\delta_n^*)$, then χ_n is a one-one curve in $R_{k_{n+1}}$ satisfying $\chi_n(0) = \psi_{n+1}(\delta_{n+1})$, $\chi_n(\delta_n^*) = \psi_n(\delta_n^*)$.

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[6]) We mean the following corollary (see, e.g., [4]): Suppose Φ is meromorphic on U and $S - \Phi(U)$ contains a proper continuum; suppose there are curves ω_k in U with $\mathrm{Ls}\langle \omega_k \rangle \subset \partial U$, *i.p.* $\omega_k \to a$, *e.p.* $\omega_k \to b \neq a$, and with $\mathrm{Ls}\langle \Phi \circ \omega_k \rangle$ containing one point only. Then Φ is constant.

Let

$$v_n(t) = \begin{cases} \chi_n(t) & \text{for each} \quad t \in \langle 0, \delta_n^* \rangle, \\ \psi_n(t) & \text{for each} \quad t \in \langle \delta_n^*, \delta_n \rangle \end{cases} \tag{35}$$

and let $\omega_n : \langle 1/(n+1), 1/n \rangle \xrightarrow{\text{onto}} \langle 0, \delta_n \rangle$ be a continuous strictly increasing function. Putting

$$\varphi_0(t) = v_n(\omega_n(t)) \quad \text{for each} \quad t \in \langle 1/(n+1), 1/n \rangle, \quad n = 1, 2, \ldots \tag{36}$$

the mapping $\varphi_0 : (0, 1\rangle \to \Omega$ is continuous.

The inclusion $\langle v_n \rangle \subset \Omega_{k_n}$ implies $\varphi_0((0, 1/n)) = \bigcup_{j=n}^{\infty} \langle v_j \rangle \subset \bigcup_{j=n}^{\infty} \Omega_{k_j} = \Omega_{k_n}$, and, as a consequence, $\varphi_0 \to \mathscr{H}$. Moreover, as

$$\varphi_0((0, 1/n\rangle) = \bigcup_{j=n}^{\infty} \langle v_j \rangle \subset \bigcup_{j=n}^{\infty} U(\mathscr{P}(\psi_j), \vartheta_j) = U(\mathscr{P}(\psi_n), \vartheta_n) \subset U(\mathscr{P}(\psi_0), \vartheta_n), \tag{37}$$

we have

$$\begin{aligned} \mathscr{P}(\varphi_0) &= \bigcap_{n=1}^{\infty} \overline{\varphi_0((0, 1/n\rangle)} \subset \bigcap_{n=1}^{\infty} \overline{U(\mathscr{P}(\psi_n), \vartheta_n)} \subset \\ &\subset \bigcap_{n=1}^{\infty} (\overline{U(\mathscr{P}(\psi_0), \vartheta_n)} - \bigcup_{j=1}^{n} Q_j^*) = \mathscr{P}(\psi_0) - \bigcup_{j=1}^{\infty} Q_j^*, \end{aligned} \tag{38}$$

and therefore $\mathscr{P}(\varphi_0) \subset \langle \mathscr{H} \rangle - B(\mathscr{H}) = A(\mathscr{H})$. As the relation $\varphi_0 \to \mathscr{H}$ implies the inclusion $A(\mathscr{H}) \subset \mathscr{P}(\varphi_0)$, the identity $A(\mathscr{H}) = \mathscr{P}(\varphi_0)$ holds. Q.E.D.

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