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ON LINEARLY ORDERED SUBGROUPS OF A LATTICE ORDERED GROUP

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In this note the following conditions for a lattice ordered group G will be considered:

(a) G is singular.

(b) Each linearly ordered subgroup of G is cyclic.

(c) For each $0 < h \in G$ there exists a group which is the largest linearly ordered subgroup of G containing h.

The standard terminology and notation for lattices and lattice ordered groups will be used (cf. [1]).

Conrad and Montgomery [2] proved that if G is archimedean, then (a) implies (b), and they proposed the question whether (a) is implied by (b) for each archimedean lattice ordered group G. The author [3] proved the following result showing that the answer to this question is negative:

Proposition 1. *There exists an archimedean lattice ordered group G such that G fulfils* (b) *and G fails to be singular.*

Further, we have

Proposition 2. (Cf. [3].) *If G is a complete lattice ordered group and if G fulfils* (b), *then G is singular.*

In Proposition 2 the assumption of completeness cannot be replaced by assuming the σ-completeness (cf. Rotkovič [4]).

In [4] (Lemma 1) it was asserted that if a lattice ordered group G fulfils the condition (b) then G is archimedean. It will be proved below that this assertion does not hold. Also it will be shown that each abelian lattice ordered group fulfilling (b) must be archimedean.

The condition (c) was applied in [4] (proof of Lemma 1). It will be established that each archimedean lattice ordered group fulfils (c), but (c) need not hold for nonarchimedean lattice ordered groups. The class $\mathscr{B}$ of all lattice ordered groups fulfilling the condition (b) is closed with respect to direct products (but $\mathscr{B}$ fails to be closed with respect to homŏmorphic images).

We need the following examples of lattice ordered groups.

Example 1. Let N_0 be the set of all integers, $G = N_0 \times N_0 \times N_0$. For $g_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ we put $g_1 \leqq g_2$ if either $z_1 < z_2$, or $z_1 = z_2$ and $x_1 \leqq x_2$, $y_1 \leqq y_2$. The operation $+$ in G is defined componentwise. It is obvious that G is a nonarchimedean abelian lattice ordered group.

Example 2. Let F_0 be the set of all functions $f : N_0 \to N_0$ with finite supports. For $f \in F_0$ we denote by sf the support of f. In F_0 we define the operation $+$ and the lattice operations componentwise. If $z \in N_0$ and $f \in F_0$, then f^z denotes the function belonging to F_0, such that

$$f^z(i) = f(i - z)$$

is valid for each $i \in N_0$.

Let $H = F_0 \times N_0$. For $h_i = (f_i, z_i) \in H$ $(i = 1, 2)$ we put $h_1 \leqq h_2$ if either $z_1 < z_2$, or $z_1 = z_2$ and $f_1 \leqq f_2$. Further, we set

$$(f_1, z_1) + (f_2, z_2) = (f_1 + f_2^{z_1}, \; z_1 + z_2)\,.$$

It is not hard to verify that H is a nonarchimedean lattice ordered group.

Lemma 1. *Let $f \in F_0$, $f \neq 0$, $0 \neq z \in N_0$. Then f and f^z are incomparable.*

The proof is easy.

Lemma 2. *Let $h_1, h \in H$, $0 \neq h_1 = (f_1, 0)$, $h = (f, z)$, $z \neq 0$. Then the elements h_1 and $-h + h_1 + h$ are incomparable.*

Proof. We have $-h = (-f^{-z}, -z)$. Hence

$$-h + h_1 + h = (-f^{-z}, -z) + (f_1, 0) + (f, z) =$$
$$= (-f^{-z} + f_1^{-z}, -z) + (f, z) = (-f^{-z} + f_1^{-z} + f^{-z}, 0) = (f_1^{-z}, 0)\,.$$

Now it suffices to apply Lemma 1.

Lemma 3. *H fulfils the condition* (b).

Proof. Let $H_1 \neq \{0\}$ be a linearly ordered subgroup of H. Let Z be the set of all integers z' with the property that $(f', z') \in H_1$ for some $f' \in F_0$. We distinguish two cases.

a) $Z \neq \{0\}$. Let z be the least positive integer belonging to Z. Lemma 2 implies that for each $h_1 = (f_1, 0) \in H_1$ we must have $f_1 = 0$. There exists $f \in F_0$ with $h = (f, z) \in H_1$. Let $0 < h' = (f', z') \in H_1$. Hence $z' > 0$. Let k, $z_1 \in N_0$ with $z' =$

$= kz + z_1$, $k \geqq 0$, $0 \leqq z_1 < z$. Then $h' - kh \in H_1$ and there is $f_2 \in F_0$ with $h' - kh = (f_2, z_1)$. Thus $z_1 = 0$, implying $f_2 = 0$ and $h' = kh$. Therefore H_1 is cyclic (generated by h).

b) $Z = \{0\}$. Since the positive cone of F_0 fulfils the descending chain condition, the same is valid for the positive cone of H_1. Hence there exists $h_1 = (f_1, 0)$ in H_1 such that h_1 covers the element 0 in H_1. Let $0 < h = (f, 0) \in H_1$. Therefore $f_1 \leqq f$.

Let $i \in sf_1$. From $0 < f_1(i) \leqq f(i)$ it follows that there are $n_i, q_i \in N_0$ with $n_i > 0$, $0 \leqq q_i < n_i$ such that $f(i) = n_i f_1(i) + q_i$. Put $g_i = f - n_i f_1$. We have $(g_i, 0) \in H_1$ and $g_i(i) = q_i$. If $q_i > 0$, then $0 < g_i$ and $f_1 \not\leqq g_i$, which is a contradiction; hence $q_i = 0$.

Let $j \in sf_1$, $j \neq i$. Assume that $n_i < n_j$. Hence

$$g_i(j) = f(j) - n_i f_1(j) > f(j) - n_j f_1(j) = 0\,,$$

implying $g_i > 0$. Because of $g_i(i) = 0$ we infer that $f_1 \not\leqq g_i$, which is a contradiction. Thus there is a positive integer n such that $n_i = n$ for each $i \in sf_1$.

Now let $k \in sf$. Assume that k does not belong to sf_1. Put $g = f - nf_1$. Then $g \in H_1$ and $g(k) = f(k) > 0$, hence $g > 0$. For each $i \in sf_1$ we have $g(i) = 0$, thus $f_1 \not\leqq g$, a contradiction. Hence $sf = sf_1$ and $g = 0, f = nf_1$.

Corollary 1. *There exists a nonarchimedean lattice ordered group fulfilling the condition* (b).

Proposition 3. *Let G be an abelian lattice ordered group. Assume that G fulfils the condition* (b). *Then G is archimedean.*

Proof. Assume that G fails to be archimedean. Thus there are elements $g, h \in G$ such that $0 < ng < h$ is valid for each $0 < n \in N_0$. Let

$$H_1 = \{n_1 g + n_2 j : n_1, n_2 \in N_0\}\ .$$

Then H_1 is a linearly ordered subgroup of G and H_1 is not cyclic; this is a contradiction.

Let G_0 be an archimedean lattice ordered group, $0 < f \in G$, $0 < g \in G$. Let $0 < \alpha$ be a real number. Suppose that, whenever n_1, m_1, n_2, m_2 are positive integers with

$$n_1 m_1^{-1} < \alpha < n_2 m_2^{-1}\,,$$

then $n_1 f < m_1 g$ and $n_2 f > m_2 g$. Under these assumptions we write $\alpha f = g$.

Lemma 4. (Cf. [4].) *Let H be a linearly ordered subgroup of an archimedean lattice ordered group G_0. Let $0 < f \in H$, $0 < g \in H$. Then there is a real $\alpha > 0$ with $\alpha f = g$.*

For $0 < f \in G_0$ we denote by $l(f)$ the set of all elements $g \in G_0$ with the property that there exists a real $\alpha(g) > 0$ with $\alpha(g) f = g$. The set $l(f)$ is a linearly ordered subsemigroup of G_0 containing f. Hence $l(f) - l(f)$ is a linearly ordered subgroup of G_0 containing f. From this and from Lemma 4 we obtain:

Corollary 2. *Let $0 < f \in G_0$. Then $l(f) - l(f)$ is the largest linearly ordered subgroup of G_0 containing f.*

Corollary 3. *Each archimedean lattice ordered group fulfils the condition* (c).

A lattice ordered group G fulfilling the condition (c) need not satisfy (b) (e.g., let G be a noncyclic linearly ordered group).

Now let G be as in Example 1. Let $f = (0, 0, 1)$ and

$$H_1 = \{(x, 0, z) : x \in N_0,\ y \in N_0\}\ ,$$

$$H_2 = \{(0, y, z) : y \in N_0,\ z \in N_0\}\ .$$

Then both H_1 and H_2 are linearly ordered subgroups of G and $f \in H_1 \cap H_2$. Also, $h_1 = (1, 0, 0) \in H_1$, $h_2 = (0, 1, 0) \in H_2$ and the elements h_1, h_2 are incomparable. Hence there exists no linearly ordered subgroup H of G with $H_1 \subseteq H$, $H_2 \subseteq H$. Thus G does not fulfil the condition (c).

Let $\mathscr{B}$ be the class of all lattice ordered groups fulfilling the condition (b). If $G \in \mathscr{B}$ and if H is an l-subgroup of G, then obviously H belongs to $\mathscr{B}$ as well.

Proposition 4. *The class $\mathscr{B}$ is closed with respect to direct products.*

Proof. Let $\{G_i : i \in I\}$ be a system of lattice ordered groups belonging to $\mathscr{B}$ and let $G = \Pi_{i \in I} G_i$. Let $\{0\} \neq R$ be a linearly ordered subgroup of G. For $i \in I$ and $Y \subseteq G$ we denote by Y_i the projection of R into G_i, i.e., Y_i is the set of all $x \in G_i$ such that $x = y(i)$ for some $y \in Y$. Without loss of generality we can assume that card $I > 1$ and $R_i \neq \{0\}$ for each $i \in I$.

Let $j \in I$ be fixed. Assume that there exist distinct elements $r_1, r_2 \in R$ with $r_1(j) = r_2(j)$. Put $r = |r_1 - r_2|$. Then $0 < r \in R$ and $r(j) = 0$. Let $R_1 = \{g \in R : g(j) = 0\}$. Hence R_1 is a convex l-subgroup of R and $R_1 \neq \{0\}$, $R_1 \neq R$.

From $R_1 \neq \{0\}$ it follows that there exists $k \in K$ with $R_{1k} \neq \{0\}$. Clearly R_{1k} is a convex l-subgroup of R_k. Because R_k is a linearly ordered subgroup of G_k, R_k must be cyclic and thus $R_{1k} = R_k$. In view of $R_1 \neq R$ there is $0 < g$ with $g \in R \setminus R_1$. Hence $g > r_1$ for each $r_1 \in R_1$. There exists $x \in R_1$ such that $g(k) = x(k)$. We have $2x \in R_1$ and $2\,x(k) > g(k) > 0$, thus $g \not> 2x$, which is a contradiction. Therefore $r_1(j) \neq r_2(j)$. This implies that the mapping $\varphi_j : R \to R_j$ defined by $\varphi_j(r) = r(j)$ for each $r \in R$ is an isomorphism of R onto R_j. Because R_j is cyclic, so is R. This completes the proof.

Proposition 5. *The class $\mathscr{B}$ fails to be closed with respect to homomorphic images.*

Proof. Let I be an infinite set and for each $i \in I$ let $G_i = N_0$, $G = \Pi_{i \in I} G_i$. Since each G_i fulfils (b), according to Proposition 4, G satisfies (b) as well. There exists an l-ideal H of G such that G/H is not archimedean. Hence in view of Proposition 3, G/H does not fulfil the condition (b).

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