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EXISTENCE OF SCHÜTTE SEMIAUTOMORPHISMS

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The purpose of this paper is to discuss the existence of Schütte semiautomorphisms (i.e., semiautomorphisms of alternative division rings, satisfying Schütte condition of orthogonality, [2]). A natural classification of these semiautomorphisms is found and examples corresponding to each of the types of semiautomorphisms are constructed.

1.1. *An affine plane* is a triple $(\mathscr{P}, \mathscr{L}, \mathrm{I})$, where $\mathscr{P}$ is a set of points, $\mathscr{L}$ a set of lines and I is an incidence relation, satisfying

1) Any two distinct points $P_1, P_2 \in \mathscr{P}$ lie on exactly one line $l \in \mathscr{L}$ $(P_1 \mathrm{I}\, l,\ P_2 \mathrm{I}\, l$; denotation: $l = P_1 \sqcup P_2)$.
2) For every $P \in \mathscr{P}$ and $l_1 \in \mathscr{L}$ such that P non I l_1 there exists exactly one line $l_2 \in \mathscr{L}$ that passes through P and has no point on l_1 (l_1 and l_2 are parallel; denotation: $l_1 \parallel l_2$). If $P \mathrm{I}\, l_1$, then $l_1 = l_2$.
3) There exist three non colinear (not lying on the same line) points.

Herewith a binary relation of parallelity among lines is defined and this relation is reflexive, symmetric and transitive.

An isomorphism from an affine plane $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ onto an affine plane $(\mathscr{P}', \mathscr{L}', \mathrm{I}')$ is a couple (π, λ) of bijective mappings $\pi : \mathscr{P} \to \mathscr{P}'$, $\lambda : \mathscr{L} \to \mathscr{L}'$ such that $P \mathrm{I}\, l \Leftrightarrow P^\pi \mathrm{I}'\, l^\lambda$. The relation of isomorphism divides the class of all planes into disjoint classes of mutually isomorphic planes.

A binary relation on $\mathscr{L}$ is called *an orthogonality* (denoted by $\perp$) if it satisfies the following axioms:

1) If $l_1 \perp l_2$, then $l_2 \perp l_1$.
2) If $P \in \mathscr{P}$ and $l_1 \in \mathscr{L}$, then there is exactly one $l_2 \in \mathscr{L}$ such that $P \mathrm{I}\, l_2$ and $l_2 \perp l_1$.

We shall denote by $(\mathscr{P}, \mathscr{L}, \mathrm{I}; \perp)$ an affine plane with an orthogonality $\perp$. An isomorphism from $(\mathscr{P}, \mathscr{L}, \mathrm{I}; \perp)$ onto $(\mathscr{P}', \mathscr{L}', \mathrm{I}'; \perp')$ is a couple (π, λ) of bijective mappings $\pi : \mathscr{P} \to \mathscr{P}'$, $\lambda : \mathscr{L} \to \mathscr{L}'$ such that $P \mathrm{I}\, l \Leftrightarrow P^\pi \mathrm{I}'\, l^\lambda$ and $l_1 \perp l_2 \Leftrightarrow l_1^\lambda \perp' l_2^\lambda$.

The preceding definitions imply:

$$l_1 \perp l_2\,,\quad l_2 \parallel l_3 \Rightarrow l_1 \perp l_3\,.$$

The Fano condition for an affine plane has the following meaning: For every quadrangle (A_1, A_2, A_3, A_4) (an ordered quadruple of mutually distinct points), where $A_1 \sqcup A_2 \parallel A_3 \sqcup A_4$ and $A_1 \sqcup A_4 \parallel A_2 \sqcup A_3$, there exists exactly one point $B \in \mathscr{P}$ such that $(A_1 \sqcup A_3) \sqcap (A_2 \sqcup A_4) = B$. (The symbol $\sqcap$ denotes the point of intersection of two non-parallel lines.)

The *trapez condition*: Let (A_1, A_2, A_3, A_4) and (B_1, B_2, B_3, B_4) be two quadrangles, where $A_1 \sqcup A_2 \parallel A_3 \sqcup A_4$ and $B_1 \sqcup B_2 \parallel B_3 \sqcup B_4$, $A_i, B_i \in \mathscr{P}$. If five of the relations $A_i \sqcup A_k \perp B_i \sqcup B_k$ $(1 \leqq i < k \leqq 4)$ are satisfied, then the remaining sixth relation is also satisfied.

1.2. *An alternative divison ring* is a non-void set $\mathbf{T}$ together with two binary operations $+, \cdot$ on $\mathbf{T}$, where $(\mathbf{T}, +)$ is an Abelian group with a neutral element 0 (zero), $(\mathbf{T} \setminus \{0\}, \cdot)$ is a loop with a neutral element 1 (identity) and both distributive laws as well as both alternative laws are satisfied:

$$a(b + c) = ab + ac\,, \quad (a + b)\,c = ac + bc$$
$$(ab)\,b = ab^2\,, \qquad a^2b = a(ab)$$

for all $a, b, c \in \mathbf{T}$.

The *center* $\mathbf{C}$ of $\mathbf{T}$ is the set of all $p \in \mathbf{T}$, which commute and associate with all elements of $\mathbf{T}$:

$$\mathbf{C} = \{p \in \mathbf{T} \mid (px)\,y = p(xy),\ px = xp \text{ for every } x, y \in \mathbf{T}\}\,.$$

A one-to-one mapping $\sigma : \mathbf{T} \to \mathbf{T}$ satisfying $(x + y)^\sigma = x^\sigma + y^\sigma$ is called

1) *an automorphism* if $(xy)^\sigma = x^\sigma y^\sigma$ for all $x, y \in \mathbf{T}$,
2) *an antiautomorphism* if $(xy)^\sigma = y^\sigma x^\sigma$ for all $x, y \in \mathbf{T}$,
3) *an semiautomorphism* if one of the following pairwise mutually equivalent conditions is fulfilled:
   a) $(xyx)^\sigma = x^\sigma y^\sigma x^\sigma$ for all $x, y \in \mathbf{T}$,
   b) $(x^2)^\sigma = (x^\sigma)^2$ for all $x \in \mathbf{T}$,
   c) $(xy + yx)^\sigma = x^\sigma y^\sigma + y^\sigma x^\sigma$ for all $x, y \in \mathbf{T}$,
   d) $(y^{-1})^\sigma = (y^\sigma)^{-1}$ for $y \neq 0$, $y \in \mathbf{T}$.

Every automorphism or antiautomorphism is a special kind of semiautomorphism on $\mathbf{T}$. An alternative non-associative division ring admits semiautomorphisms which are not automorphisms nor antiautomorphisms.

1.3. Let $(\mathbf{T}, +, \cdot)$ be an alternative division ring. We put $\mathscr{P} := \mathbf{T} \times \mathbf{T}$, $\mathscr{L} := (\mathbf{T} \times \mathbf{T}) \cup \mathbf{T}$ and define $\mathrm{I} \subseteq \mathscr{P} \times \mathscr{L}$ as follows:

$(x, y)\, \mathrm{I}\, (u, v) \Leftrightarrow y = ux + v$ for all $x, y, u, v \in \mathbf{T}$,
$(x, y)\, \mathrm{I}\, u \Leftrightarrow x = u$ for all $x, y, u \in \mathbf{T}$.
Then $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ is an affine plane over $\mathbf{T}$. In this plane the Little Desargues condition holds. If $\mathbf{T}$ is associative, then the affine plane satisfies the Desargues condition ([1], p. 73).

Theorem (K. Schütte). *For every affine plane with an orthogonality $(\mathscr{P}, \mathscr{L}, \mathrm{I}; \perp)$ satisfying the trapez condition there exist an alternative division ring $\mathbf{T}$, a semiautomorphism $\sigma : \mathbf{T} \to \mathbf{T}$ and an element $k \in \mathbf{T}$ such that $(ka^\sigma)^\sigma = ak$ holds for every $a \in \mathbf{T}$. Then the affine plane over $\mathbf{T}$ with the orthogonality defined by $y = ax \perp y = (ka^\sigma)^{-1} x$ is isomorphic with the original affine plane.*

Conversely. *Let $\mathbf{T}$ be an alternative division ring, $\sigma : \mathbf{T} \to \mathbf{T}$ a semiautomorphism and $k \in \mathbf{T}$ an element satisfying $(ka^\sigma)^\sigma = ak$ for every $a \in \mathbf{T}$. Then the affine plane over $\mathbf{T}$ provided with the orthogonality $y = ax \perp y = (ka^\sigma)^{-1} x$ satisfies the trapez condition* ([2] – Theorem 9).

1.4. Let $\mathbf{F}$ be a field of characteristic $\neq 2$ and let $\mathbf{Q}$ be a quaternion division algebra over $\mathbf{F}$, consisting of elements of the form $x = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$; $a_0, a_1, a_2, a_3 \in \mathbf{F}$. The symbol $\bar{x}$ will denote the conjugate element to x, $\bar{x} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$.

A *Cayley (Cayley-Dickson) division algebra* $\mathbf{A}$ *over* $\mathbf{F}$ is a set of the form $\mathbf{A} = \mathbf{Q} + g\mathbf{Q}$ with elements $x = x_1 + gx_2$ $(x_i \in \mathbf{Q})$ and with the following operations:

a) addition is defined by the rule
$$(x_1 + gx_2) + (y_1 + gy_2) = (x_1 + y_1) + g(x_2 + y_2)$$
for every $x_i, y_i \in \mathbf{Q}$,

b) multiplication is defined by
$$(x_1 + gx_2)(y_1 + gy_2) = (x_1 y_1 + \gamma y_2 \bar{x}_2) + g(\bar{x}_1 y_2 + y_1 x_2)$$
for every $x_i, y_i \in \mathbf{Q}$, where $g^2 = \gamma \neq 0$, $\gamma \in \mathbf{F}$.

The following theorems are known ([1], p. 175, p. 302):

Theorem (L. A. Skornjakov, R. H. Bruck, E. Kleinfeld). *If $\mathbf{T}$ is an alternative division ring over $\mathbf{F}$, then either $\mathbf{T}$ is associative or $\mathbf{T}$ is a Cayley division algebra over the field $\mathbf{F}$.*

Theorem (Wedderburn). *A finite alternative division ring is a field.*

All automorphisms of an alternative division ring have been described by N. Jacobson ([5]).

Let $\mathbf{T}$ be an alternative non-associative division ring over a field with characteristic $\neq 2$. Then $\mathbf{T}$ is a Cayley algebra over its center $\mathbf{C}$ and there is a basis $1, e_1, \ldots, e_7$, where $e_ie_j = -e_je_i$ $(i \neq j)$, $e_i^2 = -\alpha_i$, $\alpha_i \in \mathbf{C}$. The following result was proved in [3], Theorems 5, 6:

Theorem (V. Havel). *Every semiautomorphism σ of an alternative division ring $\mathbf{T}$ over its center $\mathbf{C}$ has the following form:*

$$(1) \qquad e_i^\sigma = \sum_{k=1}^{7} a_{ik} e_k \; ; \quad i = 1, \ldots, 7 \, ,$$

where the constants $a_{ik} \in \mathbf{C}$ satisfy

$$(2) \qquad \alpha_i^\sigma = \sum_{k=1}^{7} \alpha_k a_{ik}^2 \quad \text{for every} \quad i = 1, \ldots, 7 \, ,$$

$$(3) \qquad \sum_{k=1}^{7} \alpha_k a_{ik} a_{jk} = 0 \quad \text{for every} \quad i, j = 1, \ldots, 7 \, , \quad i \neq j \, .$$

Conversely. *Every mapping σ with the properties (1), (2) and (3) is a semiautomorphism of $\mathbf{T}$. Furthermore, the restriction $\sigma_{\mathbf{C}}$ is an automorphism on $\mathbf{C}$ and if $x \in \mathbf{C}$, $y \in \mathbf{T}$, then $(xy)^\sigma = x^\sigma y^\sigma$.*

If $\mathbf{C}$ is the field $\mathbf{R}$ of real numbers, then $\sigma_{\mathbf{R}} = \text{id}$, $0^\sigma = 0$, $1^\sigma = 1$.

Now we shall investigate the condition

$$(4) \qquad (ka^\sigma)^\sigma = ak \, ,$$

where for $a = 1$ we obtain

$$(5) \qquad k^\sigma = k \, .$$

We shall investigate this condition in single cases.

2.1. Let $k \in \mathbf{C}$. Then (4) implies: $(ka^\sigma)^\sigma = ak \Rightarrow k^\sigma a^{\sigma^2} = ak \Rightarrow ka^{\sigma^2} = ka \Rightarrow$ $\Rightarrow a^{\sigma^2} = a \Rightarrow$

$$(6) \qquad \sigma^2 = \text{id} \, , \quad \text{but} \quad \sigma \neq \text{id} \, .$$

If We choose $a = e_i$ then from (1) we get

$$e_i^{\sigma^2} = (e_i^\sigma)^\sigma = (\sum_j a_{ij} e_j)^\sigma = \sum_j a_{ij}^\sigma e_j^\sigma = \sum_{j,m} a_{ij}^\sigma a_{jm} e_m = e_i$$

or

$$(7) \qquad \sum_j a_{ij}^\sigma a_{jm} = \delta_{im} \, .$$

Now we shall demonstrate on two examples that such a mapping $\sigma \neq \text{id}$ exists.

Example 1. Let $\mathbf{T}$ be a Cayley division algebra with a basis $1, e_1, \ldots, e_7$ and the multiplication table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_2	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_3	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_4	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_5	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Here $e_i^2 = -1$, $\alpha_i = 1$.

Let the mapping σ be given by the matrix $\|a_{ij}\|$:

$$\|a_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

where

$$\begin{Vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{Vmatrix} \neq \begin{Vmatrix} \pm 1 & 0 \\ 0 & \pm 1(\mp 1) \end{Vmatrix}.$$

Thus the mapping is neither an automorphism nor an antiautomorphism:

$$e_2 = e_2^\sigma = (e_1 e_3)^\sigma \neq (e_1^\sigma e_3^\sigma) = e_1(a_{33}e_3 + a_{34}e_4) = a_{33}e_2 - a_{34}e_5\,,$$
$$e_2 = e_2^\sigma = (e_1 e_3)^\sigma \neq (e_3^\sigma e_1^\sigma) = (a_{33}e_3 + a_{34}e_4)\,e_1 = -a_{33}e_2 + a_{34}e_5\,.$$

The mapping σ is just a semiautomorphism if the constants a_{ij} and their images a_{ij}^σ satisfy

(8) $$\sum_k a_{ik}^2 = 1\,, \quad \sum_k a_{ik}a_{jk} = 0\,, \quad i \neq j$$

and

(9) $$\sum_j a_{ij}^\sigma a_{jm} = \delta_{im}\,, \quad a_{ij}^{\sigma^2} = a_{ij}\,, \quad \sigma \neq id\,.$$

In our case (9) yields $a_{ij}^\sigma = a_{ij}$ for $i \neq 3, 4$ or $j \neq 3, 4$. For $i, j = 3, 4$ the following identities must be fulfilled:

(10) $$\begin{cases} a_{33}^\sigma a_{33} + a_{34}^\sigma a_{43} = 1 \\ a_{33}^\sigma a_{34} + a_{34}^\sigma a_{44} = 0 \end{cases}$$

(11) $$\begin{cases} a_{43}^\sigma a_{33} + a_{44}^\sigma a_{43} = 0 \\ a_{43}^\sigma a_{34} + a_{44}^\sigma a_{44} = 1 \end{cases}.$$

The determinants of the systems (10) and (11) are

$$D = \begin{vmatrix} a_{33} & a_{43} \\ a_{34} & a_{44} \end{vmatrix}, \quad D = \pm 1, \text{ because the matrix } \|a_{ij}\|$$

must be orthogonal. From (8) we get

$$(12) \qquad \begin{cases} a_{33}^2 + a_{34}^2 = 1 \\ a_{43}^2 + a_{44}^2 = 1 \\ a_{33}a_{43} + a_{34}a_{44} = 0\,. \end{cases}$$

We shall investigate the last system in detail:

$$a_{34}^2 = 1 - a_{33}^2\,, \quad a_{43}^2 = 1 - a_{44}^2$$
$$a_{33}^2 a_{43}^2 = a_{34}^2 a_{44}^2$$
$$a_{33}^2(1 - a_{44}^2) = (1 - a_{33}^2)\, a_{44}^2 \Rightarrow a_{33}^2 = a_{44}^2 \Rightarrow a_{34}^2 = a_{43}^2$$
$$D = a_{33}a_{44} - a_{34}a_{43} = \pm 1\,.$$

The solutions of the systems (10) and (11) are

$$a_{33}^\sigma = \frac{a_{44}}{D}\,, \quad a_{34}^\sigma = \frac{-a_{34}}{D}\,, \quad a_{43}^\sigma = \frac{-a_{43}}{D}\,, \quad a_{44}^\sigma = \frac{a_{33}}{D}\,.$$

We distinguish the following cases:

1) $a_{34} = a_{43}$

a) $a_{44} = a_{33}$

$D = a_{33}^2 - 1 + a_{33}^2 = \pm 1$

I) $D = 1 : 2a_{33}^2 = 2 \Rightarrow a_{33} = \pm 1 = a_{44}$, $a_{34} = a_{43} = 0$

$D = \begin{vmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{vmatrix} = 1$, $\sigma_C = id$, σ is either an automorphism or an anti-automorphism.

II) $D = -1 : a_{33}^2 = 0 \Rightarrow a_{33} = a_{44} = 0$, $a_{34} = a_{43} = \pm 1$

$D = \begin{vmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{vmatrix} = -1$, $\sigma_C = id$, $\sigma_T \neq id$,

$e_3^\sigma = \pm e_4$, $e_4^\sigma = \pm e_3$, σ is a semiautomorphism of $\boldsymbol{T} : (e_1 e_3)^\sigma \neq e_1^\sigma e_3^\sigma$, $(e_1 e_3)^\sigma \neq e_3^\sigma e_1^\sigma$

b) $a_{44} = -a_{33}$

$D = -a_{33}^2 - 1 + a_{33}^2 = -1$

$$D = \begin{vmatrix} a_{33} & \pm\sqrt{(1 - a_{33}^2)} \\ \pm\sqrt{(1 - a_{33}^2)} & -a_{33} \end{vmatrix} = -1$$

$\sigma_{\mathbf{C}} = id$, σ is a semiautomorphism of $\boldsymbol{T}$

$e_3^\sigma e_1^\sigma \neq (e_1 e_3)^\sigma \neq e_1^\sigma e_3^\sigma$

2) $a_{34} = -a_{43}$

a) $a_{44} = a_{33}$

$D = a_{33}^2 + 1 - a_{33}^2 = 1$

$$D = \begin{vmatrix} a_{33} & \pm\sqrt{(1 - a_{33}^2)} \\ \mp\sqrt{(1 - a_{33}^2)} & a_{33} \end{vmatrix} = 1$$

$a_{33}^\sigma = a_{44}^\sigma = a_{33} = a_{44}$, $a_{34}^\sigma = a_{43}$, $a_{43}^\sigma = a_{34}$

σ is a semiautomorphism of $\boldsymbol{T}$, $\sigma_{\mathbf{C}} \neq id$

b) $a_{44} = -a_{33}$

$D = -a_{33}^2 + 1 - a_{33}^2 = \pm 1$

I) $D = 1 \Rightarrow a_{33}^2 = 0 \Rightarrow a_{33} = a_{44} = 0$, $a_{34} = -a_{43} = \pm 1$

$$D = \begin{vmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{vmatrix} = 1$$

$\sigma_{\mathbf{C}} = id$, σ is a semiautomorphism of $\boldsymbol{T}$

II) $D = -1 : a_{33}^2 = 1 \Rightarrow a_{33} = \pm 1$, $a_{44} = \mp 1$, $a_{34} = a_{43} = 0$

$$D = \begin{vmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{vmatrix} = -1$$

$\sigma_{\mathbf{C}} = id$, σ is an automorphism or an antiautomorphism of $\boldsymbol{T}$.

It can be easily verified that $\sigma^2 = id$ in all the cases investigated. The determinants from 1)b) and 2)a) have sense only in $\mathbf{C}$, where $\sqrt{}$ is defined.

Example 2. Let $\boldsymbol{T}$ be a Cayley division algebra with the multiplication table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\alpha_1$	$-e_3$	$\alpha_1 e_2$	$-e_5$	$\alpha_1 e_4$	e_7	$-\alpha_1 e_6$
e_2	e_3	$-\alpha_2$	$-\alpha_2 e_1$	$-e_6$	$-e_7$	$\alpha_2 e_4$	$\alpha_2 e_5$
e_3	$-\alpha_1 e_2$	$\alpha_2 e_1$	$-\alpha_3$	$-e_7$	$\alpha_1 e_6$	$-\alpha_2 e_5$	$\alpha_3 e_4$
e_4	e_5	e_6	e_7	$-\alpha_4$	$-\alpha_4 e_1$	$-\alpha_4 e_2$	$-\alpha_4 e_3$
e_5	$-\alpha_1 e_4$	e_7	$-\alpha_1 e_6$	$\alpha_4 e_5$	$-\alpha_5$	$\alpha_4 e_3$	$-\alpha_5 e_2$
e_6	$-e_7$	$-\alpha_2 e_4$	$\alpha_2 e_5$	$\alpha_4 e_2$	$-\alpha_4 e_3$	$-\alpha_6$	$\alpha_6 e_1$
e_7	$\alpha_1 e_6$	$-\alpha_2 e_5$	$-\alpha_3 e_4$	$\alpha_4 e_3$	$\alpha_5 e_2$	$-\alpha_6 e_1$	$-\alpha_7$

It is known that we can choose e_i, $i = 1, \dots, 7$ in such a way that $\alpha_3 = \alpha_1 \alpha_2$, $\alpha_5 = \alpha_1 \alpha_4$, $\alpha_6 = \alpha_2 \alpha_4$, $\alpha_7 = \alpha_1 \alpha_2 \alpha_4$.

Let $\|a_{ij}\|$ be the matrix of the mapping $\sigma : \boldsymbol{T} \to \boldsymbol{T}$. We want to construct an example with $\alpha_i^\sigma \neq \alpha_i$ at least for one i. If we choose $a_{ii} = a_{jj} = a_{kk} = a_{qq} = 1$ and $a_{im} = a_{jm} = a_{km} = a_{qm} = 0$ for $1 \leqq m \leqq 7$ and i, j, k, q mutually diferent, then we necessarily get $\alpha_i^\sigma = \alpha_i$ for all i's, because every α_i is either directly some of $\alpha_1, \alpha_2, \alpha_4$ or some of the products $\alpha_1\alpha_2, \alpha_1\alpha_4, \alpha_2\alpha_4, \alpha_1\alpha_2\alpha_4$, and when we express α_i, $1 \leqq i \leqq 7$, in terms of $\alpha_1, \alpha_2, \alpha_4$, then each of the elements $\alpha_1, \alpha_2, \alpha_4$ occurs in every quadruple $(\alpha_i, \alpha_j, \alpha_k, \alpha_q)$ $(i, j, k, q$ mutually diferent). For example: if $a_{11} = a_{33} = a_{55} = a_{77} = 1$, $a_{1i} = a_{3i} = a_{5i} = a_{7i} = 0$ for $1 \leqq i \leqq 7$, then $\alpha_1^\sigma = \alpha_1$, $\alpha_3^\sigma = \alpha_3$, $\alpha_5^\sigma = \alpha_5$, $\alpha_7^\sigma = \alpha_7$. From

$$\alpha_3 = \alpha_1\alpha_2 \quad \text{we get} \quad \alpha_3^\sigma = \alpha_1^\sigma\alpha_2^\sigma \Rightarrow \alpha_3 = \alpha_1\alpha_2^\sigma \Rightarrow \alpha_2^\sigma = \alpha_2\,;$$

$$\alpha_5 = \alpha_1\alpha_4 \Rightarrow \alpha_5^\sigma = \alpha_1^\sigma\alpha_4^\sigma \Rightarrow \alpha_5 = \alpha_1\alpha_4^\sigma \Rightarrow \alpha_4^\sigma = \alpha_4 \quad \text{and}$$

$$\alpha_6 = \alpha_2\alpha_4 \Rightarrow \alpha_6^\sigma = \alpha_2^\sigma\alpha_4^\sigma \Rightarrow \alpha_6^\sigma = \alpha_2\alpha_4 \Rightarrow \alpha_6^\sigma = \alpha_6\,.$$

Therefore we choose a matrix $\|a_{ij}\|$ which contains at most three 1's in the main diagonal:

$$\|a_{ij}\| = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\|,$$

where

$$\left\| \begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix} \right\| \neq \left\| \begin{matrix} \pm 1 & 0 \\ 0 & \pm 1 (\mp 1) \end{matrix} \right\| \neq \left\| \begin{matrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{matrix} \right\|;$$

σ is neither an automorphism nor an antiautomorphism:

$$\alpha_1^\sigma = \alpha_1 \quad (\alpha_2\alpha_4)^\sigma = \alpha_2^\sigma\alpha_4^\sigma = \alpha_2\alpha_4$$
$$(\alpha_1\alpha_2\alpha_4)^\sigma = \alpha_1^\sigma\alpha_2^\sigma\alpha_4^\sigma = \alpha_1\alpha_2\alpha_4\,.$$

From (1), (2), (3) and (7) we obtain

$$(13) \qquad \begin{cases} \alpha_2 a_{22}^2 + \alpha_1\alpha_2 a_{23}^2 = \alpha_2^\sigma \\ \alpha_2 a_{32}^2 + \alpha_1\alpha_2 a_{33}^2 = \alpha_1\alpha_2^\sigma \\ \alpha_4 a_{44}^2 + \alpha_1\alpha_4 a_{45}^2 = \alpha_4^\sigma \\ \alpha_4 a_{54}^2 + \alpha_1\alpha_4 a_{55}^2 = \alpha_1\alpha_4^\sigma \end{cases}$$

and consequently

$$(13') \qquad \begin{cases} \alpha_1 a_{22}^2 + \alpha_1^2 a_{23}^2 = a_{32}^2 + \alpha_1 a_{33}^2 \\ \alpha_1 a_{44}^2 + \alpha_1^2 a_{45}^2 = a_{54}^2 + \alpha_1 a_{55}^2 \end{cases}$$

$$(14) \qquad \begin{cases} \alpha_2 a_{22} a_{32} + \alpha_1\alpha_2 a_{23} a_{33} = 0 \\ \alpha_4 a_{44} a_{54} + \alpha_1\alpha_4 a_{45} a_{55} = 0 \end{cases}$$

(14') $$\begin{cases} a_{22}a_{32} + \alpha_1 a_{23}a_{33} = 0 \\ a_{44}a_{54} + \alpha_1 a_{45}a_{55} = 0 \end{cases}$$

(15) $a^\sigma_{ij} = a_{ij}$ for $(i, j) \neq (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)$

(16) $$\begin{cases} a^\sigma_{22}a_{22} + a^\sigma_{23}a_{32} = 1 \\ a^\sigma_{22}a_{23} + a^\sigma_{23}a_{33} = 0 \\ a^\sigma_{32}a_{22} + a^\sigma_{33}a_{32} = 0 \\ a^\sigma_{32}a_{23} + a^\sigma_{33}a_{33} = 1 \end{cases}$$ (16') $$\begin{cases} a^\sigma_{44}a_{44} + a^\sigma_{45}a_{54} = 1 \\ a^\sigma_{44}a_{45} + a^\sigma_{45}a_{55} = 0 \\ a^\sigma_{54}a_{44} + a^\sigma_{55}a_{54} = 0 \\ a^\sigma_{54}a_{45} + a^\sigma_{55}a_{55} = 1 \end{cases}$$

The determinants of the systems (16) and (16') are

$$D_1 = \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

and

$$D_2 = \begin{vmatrix} a_{44} & a_{54} \\ a_{45} & a_{55} \end{vmatrix} = a_{44}a_{55} - a_{45}a_{54}, \text{ where } D_1 D_2 = \pm 1 .$$

We shall restrict ourselves to $i, j \in \{2, 3\}$. From (13') we get

$$a^2_{33} = a^2_{22} + \alpha_1 a^2_{23} - \frac{a^2_{32}}{\alpha_1} .$$

We substitute this result in (14'):

$$(a^2_{22} + \alpha_1 a^2_{23})\, a^2_{32} = \alpha_1^2 a^2_{23}(a^2_{22} + \alpha_1 a^2_{23}) .$$

Let $a^2_{22} + \alpha_1 a^2_{23} \neq 0 \Rightarrow a^2_{32} = \alpha_1^2 a^2_{23} \Rightarrow a_{32} = \pm\alpha_1 a_{23} \Rightarrow a^2_{33} = a^2_{22} \Rightarrow a_{33} = \pm a_{22}$.

The solution of the system (16) is

$$a^\sigma_{22} = \frac{a_{33}}{D_1}, \quad a^\sigma_{33} = \frac{a_{22}}{D_1}, \quad a^\sigma_{23} = \frac{-a_{23}}{D_1}, \quad a^\sigma_{32} = \frac{-a_{32}}{D_1} .$$

Now we shall investigate the possibilities $a_{33} = \pm a_{22}$, $a_{32} = \pm\alpha_1 a_{23}$. We distinguish four cases:

1) $a_{33} = a_{22}$, $a_{32} = \alpha_1 a_{23}$,
$D_1 = a^2_{22} - \alpha_1 a^2_{23}$.

In this case (14') reads $\alpha_1 a_{22}a_{23} + \alpha_1 a_{23}a_{22} = 0$, $2\alpha_1 a_{22}a_{23} = 0$, $\alpha_1 \neq 0$.

a) $a_{23} = a_{32} = 0$,

$$D_1 = \begin{vmatrix} a_{22} & 0 \\ 0 & a_{22} \end{vmatrix} = a^2_{22}, \quad \sigma_C \neq id \Rightarrow a_{22} \neq \pm 1 ,$$

$$a^\sigma_{22} = a^\sigma_{33} = \frac{1}{a_{22}}, \quad \alpha^\sigma_2 = \alpha_2 a^2_{22} ;$$

b) $a_{22} = a_{33} = 0$,

$$D_1 = \begin{vmatrix} 0 & a_{23} \\ \alpha_1 a_{23} & 0 \end{vmatrix} = -\alpha_1 a_{23}^2 ,$$

$$a_{23}^\sigma = \frac{1}{\alpha_1 a_{23}}, \quad a_{32}^\sigma = \frac{1}{a_{32}}, \quad \alpha_2^\sigma = \alpha_1 \alpha_2 a_{23}^2 .$$

2) $a_{22} = a_{33}$, $a_{32} = -\alpha_1 a_{23}$,

$$D_1 = \begin{vmatrix} a_{22} & a_{23} \\ -\alpha_1 a_{23} & a_{22} \end{vmatrix} = a_{22}^2 + \alpha_1 a_{23}^2 \neq 0 \text{ (as we have already assumed)}.$$

Now (14′) is satisfied trivially $(-a_{22}\alpha_1 a_{23} + \alpha_1 a_{23} a_{22} = 0)$.

$$a_{22}^\sigma = a_{33}^\sigma = \frac{a_{22}}{a_{22}^2 + \alpha_1 a_{23}^2}, \quad a_{23}^\sigma = \frac{-a_{23}}{a_{22}^2 + \alpha_1 a_{23}^2},$$

$$a_{32}^\sigma = \frac{\alpha_1 a_{23}}{a_{22}^2 + \alpha_1 a_{23}^2}, \quad \alpha_2^\sigma = \alpha_2 (a_{22}^2 + \alpha_1 a_{23}^2),$$

$\sigma_C \neq id \Leftrightarrow a_{23} \neq 0$ and at the same time $a_{22} \neq \pm 1$.

3) $a_{33} = -a_{22}$, $a_{32} = \alpha_1 a_{23}$,

$$D_1 = \begin{vmatrix} a_{22} & a_{23} \\ \alpha_1 a_{23} & -a_{22} \end{vmatrix} = -(a_{22}^2 + \alpha_1 a_{23}^2) \neq 0 .$$

(14′) is also satisfied trivially,

$$a_{22}^\sigma = \frac{-a_{22}}{D_1}, \quad a_{33}^\sigma = \frac{a_{22}}{D_1}, \quad a_{23}^\sigma = \frac{-a_{23}}{D_1}, \quad a_{32}^\sigma = \frac{-\alpha_1 a_{23}}{D_1},$$

$$\alpha_2^\sigma = \alpha_2 (a_{22}^2 + \alpha_1 a_{23}^2),$$

$\sigma_C \neq id \Leftrightarrow D_1 \neq -1$.

4) $a_{33} = -a_{22}$, $a_{32} = -\alpha_1 a_{23}$.

Now (14′) gives $\alpha_1 a_{22} a_{23} = 0$.

a) $a_{23} = a_{32} = 0$,

$$D_1 = \begin{vmatrix} a_{22} & 0 \\ 0 & -a_{22} \end{vmatrix} = -a_{22}^2 ,$$

$$a_{22}^\sigma = \frac{1}{a_{22}}, \quad a_{33}^\sigma = -\frac{1}{a_{22}}, \quad \alpha_2^\sigma = \alpha_2 a_{22}^2 ,$$

$\sigma_C \neq id \Rightarrow a_{22} \neq \pm 1$;

b) $a_{22} = a_{33} = 0$,

$$D_1 = \begin{vmatrix} 0 & a_{23} \\ -\alpha_1 a_{23} & 0 \end{vmatrix} = \alpha_1 a_{23}^2 ,$$

$$a_{23}^\sigma = \frac{-1}{\alpha_1 a_{23}}, \quad a_{32}^\sigma = \frac{1}{a_{23}}, \quad \alpha_2^\sigma = \alpha_1 \alpha_2 a_{23}^2 .$$

It can be verified by direct computation that $\sigma^2 = id$ in all the cases.

This completes the discussion of all possible choices for $a_{22}, a_{23}, a_{32}, a_{33}$ (6 cases) such that $\sigma_{\mathbf{C}} \neq id$ but $\sigma^2 = id$. The discussion for $a_{44}, a_{45}, a_{54}, a_{55}$ is similar, but the condition $D_1 D_2 = \pm 1$ must be then fulfilled, while in the preceding part we imposed no requirements on D_1. Now we shall choose concrete values for $a_{22}, a_{23}, a_{32}, a_{33}$ and the corresponding values for $a_{44}, a_{45}, a_{54}, a_{55}$ so that $\sigma_{\mathbf{C}} = id, \sigma^2 = id$:

$$D_1 = \begin{vmatrix} 0 & a_{23} \\ \alpha_1 a_{23} & 0 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} 0 & a_{45} \\ \alpha_1 a_{45} & 0 \end{vmatrix},$$

$a_{44} = a_{55} = 0,\ a_{54} = \alpha_1 a_{45},$

$$D_1 D_2 = \alpha_1^2 a_{23}^2 a_{45}^2 = 1 \Rightarrow a_{45}^2 = \frac{1}{\alpha_1^2 a_{23}^2} .$$

Let $a_{45} = \dfrac{1}{\alpha_1 a_{23}}, \quad a_{54} = \dfrac{1}{a_{23}};$ then $a_{45}^\sigma = a_{23} = \dfrac{1}{a_{54}},$

$$a_{54}^\sigma = \alpha_1^\sigma a_{45}^\sigma = \alpha_1 a_{23}, \quad \alpha_4^\sigma = \alpha_4 a_{44}^2 + \alpha_1 \alpha_4 a_{45}^2 = \frac{\alpha_4}{\alpha_1 a_{23}^2};$$

$a_{ij}^\sigma = a_{ij},\ \alpha_i^\sigma = \alpha_i$ for the remaining i, j.

The resulting matrix will be

$$\|a_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 a_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{\alpha_1 a_{23}} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{a_{23}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix} .$$

2.2. Let $k \notin \mathbf{C}$.

It is known that $b \in \mathbf{C} \Leftrightarrow b^\sigma \in \mathbf{C}$ ([4]). Let us apply this proposition to (4), replacing a by $b \in \mathbf{C}$:

$$(kb^\sigma)^\sigma = k^\sigma b^{\sigma^2} = b^{\sigma^2} k^\sigma = b^{\sigma^2} k = bk \Rightarrow b^{\sigma^2} = b \Rightarrow \sigma_{\mathbf{C}}^2 = id .$$

So if the given semiautomorphism satisfies (4), then the respective automorphism $\sigma_{\mathbf{C}}$ must satisfy $\sigma_{\mathbf{C}}^2 = id$. If $\mathbf{C} = \mathbf{R}$, then $\sigma_{\mathbf{R}} = id \Rightarrow \sigma_{\mathbf{R}}^2 = id$.

2.2.1. Let us suppose that σ is an automorphism of $\mathbf{T}$. Then (4) yields

$$(17) \qquad k^{\sigma} a^{\sigma^2} = ak \Rightarrow k a^{\sigma^2} = ak \Rightarrow$$
$$a^{\sigma^2} = k^{-1}(ak) \Rightarrow$$

$\sigma_{\mathbf{C}}^2 = id$, but $\sigma_{\mathbf{T}}^2 \neq id$. σ^2 is an inner automorphism determined by the element k ([5]). If $\sigma^2 = id$, then $a = k^{-1}(ak) \Rightarrow ka = ak$ for every $a \in \mathbf{T} \Rightarrow k \in \mathbf{C}$.

Example 3. Let σ be an automorphism of the type (17), $k \notin \mathbf{C}$. Let $\mathbf{T}$ be a Cayley algebra from Example 1, $\mathbf{C} = \mathbf{R}$. First we shall construct the automorphism σ^2 from the relation (17). We choose an element k,

$$k = k_0 + \sum_{i=1}^{7} k_i e_i, \quad k^{-1} = \frac{k_0 - \sum_{i=1}^{7} k_i e_i}{\sum_{j=0}^{7} k_j^2}.$$

The relation (17) must hold for all $a \in \mathbf{T}$. If we successively substitute e_1, e_2, e_3 for a in (17), we get

$$e_1^{\sigma^2} = \frac{1}{\sum_{j=0} k_j^2} [(k_0^2 + k_1^2 - k_2^2 - k_3^2 - k_4^2 - k_5^2 - k_6^2 - k_7^2)\, e_1 +$$
$$+ 2(k_0k_3 + k_1k_2)\, e_2 + 2(-k_0k_2 + k_1k_3)\, e_3 +$$
$$+ 2(k_0k_5 + k_1k_4)\, e_4 + 2(-k_0k_4 + k_1k_5)\, e_5 +$$
$$+ 2(-k_0k_7 + k_1k_6)\, e_6 + 2(k_0k_6 + k_1k_7)\, e_7],$$

$$e_2^{\sigma^2} = \frac{1}{\sum_{j=0} k_j^2} [2(-k_0k_3 + k_1k_2)\, e_1 + (k_0^2 - k_1^2 + k_2^2 - k_3^2 - k_4^2 - k_5^2 - k_6^2 - k_7^2)\, .$$
$$.\, e_2 + 2(k_0k_1 + k_2k_3)\, e_3 + 2(k_0k_6 + k_2k_4)\, e_4 + 2(k_0k_7 + k_2k_5)\, e_5 +$$
$$+ 2(-k_0k_4 + k_2k_6)\, e_6 + 2(-k_0k_5 + k_2k_7)\, e_7],$$

$$e_3^{\sigma^2} = \frac{1}{\sum_{j=0} k_j^2} [2(k_0k_2 + k_1k_3)\, e_1 + 2(-k_0k_1 + k_2k_3)\, e_2 +$$
$$+ (k_0^2 - k_1^2 - k_2^2 + k_3^2 - k_4^2 - k_5^2 - k_6^2 - k_7^2)\, e_3 +$$
$$+ 2(k_0k_7 + k_3k_4)\, e_4 + 2(-k_0k_6 + k_3k_5)\, e_5 +$$
$$+ 2(k_0k_5 + k_3k_6)\, e_6 + 2(-k_0k_4 + k_3k_7)\, e_7].$$

From the multiplication table we have $e_1 e_3 = e_2 \Rightarrow e_1^{\sigma^2} e_3^{\sigma^2} = e_2^{\sigma^2}$. We shall try to choose four the coordinates $k_0, \ldots, k_7$ being zero. The choices $k_0 = k_2 = k_4 = k_6 = 0$, $k_0 = k_1 = k_2 = k_3 = 0$, $k_1 = k_3 = k_5 = k_7 = 0$ are not possible. The choice $k_4 = k_5 = k_6 = k_7 = 0$ is suitable. Let us suppose further that $k_0 = k_1 = k_2 = k_3$. Then

$$e_1^{\sigma^2} = e_2\,, \quad e_2^{\sigma^2} = e_3\,, \quad e_3^{\sigma^2} = e_1\,,$$

$$e_4^{\sigma^2} = \tfrac{1}{2}(-e_4 + e_5 + e_6 + e_7)\,,$$

$$e_5^{\sigma^2} = \tfrac{1}{2}(-e_4 - e_5 - e_6 + e_7)\,,$$

$$e_6^{\sigma^2} = \tfrac{1}{2}(-e_4 + e_5 - e_6 - e_7)\,,$$

$$e_7^{\sigma^2} = \tfrac{1}{2}(-e_4 - e_5 + e_6 - e_7)\,.$$

If $e_i e_j = e_m$, then $e_i^{\sigma^2} e_j^{\sigma^2} = e_m^{\sigma^2}$ for all admissible triples (i, j, m), $i \neq j \neq m \neq i$. We denote by $\|\tilde{a}_{ij}\|$ the matrix of the automorphism σ^2. Then

$$\|\tilde{a}_{ij}\| = \left\| \begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{matrix} \right\|,$$

$$k^{\sigma^2} = k = k_0(1 + e_1 + e_2 + e_3)\,.$$

Now we shall find the matrix $\|a_{ij}\|$ of the automorphism σ. For $1 \leqq i, j \leqq 3$ we have

$$\|a_{ij}\| = \left\| \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right\|;$$

further, for $i, j \geqq 4$ the following implication must hold:

$$e_m^{\sigma^2} = (e_m^\sigma)^\sigma = \sum_{i=4}^{7} a_{mi} e_i^\sigma = \sum_{i,j=4}^{7} a_{mi} a_{ij} e_j = \sum_{j=4}^{7} \tilde{a}_{mj} e_j \Rightarrow \sum_{i=4}^{7} a_{mi} a_{ij} = \tilde{a}_{mj}\,;$$

moreover, $\sum_{j=4}^{7} a_{ij}^2 = 1$ and $\sum_{m=4}^{7} a_{im}a_{jm} = 0,\ i \neq j$. From this we can derive the matrix $\|a_{ij}\|$:

$$\|a_{ij}\| = \begin{Vmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}.$$

The matrices $\|a_{ij}\|$ and $\|\tilde{a}_{ij}\|$ have determinants equal to 1. It is easy to see that

$$\sum_j a_{ij}\tilde{a}_{ji} = -1, \qquad \sum_i a_{ij}\tilde{a}_{ji} = -1,$$
$$\sum_j a_{ij}\tilde{a}_{jm} = 0, \qquad \sum_j a_{ji}\tilde{a}_{mj} = 0 \quad \text{for} \quad i \neq m.$$

According to [5] every automorphism of a Cayley division algebra is always inner, thus there must exist an element $b \in \boldsymbol{T}$ such that $a^\sigma = b^{-1}(ab)$ for all $a \in \boldsymbol{T}$. Then

$$a^{\sigma^2} = (b^{-1})^2\,(ab^2) = k^{-1}(ak).$$

Thus the automorphism σ is determined by the element b for which $b^2 = k$. Let $b = b_0 + b_1e_1 + \ldots + b_7e_7$. If we write the relation $b^2 = k$ in coordinates we get

$$b = \pm\left(\frac{\sqrt{6k_0}}{2} + \sqrt{\frac{k_0}{6}}\,(e_1 + e_2 + e_3)\right).$$

2.2.2. Let σ be an antiautomorphism of $\boldsymbol{T}$. Then (4) yields

$$(ka^\sigma)^\sigma = ak \Rightarrow a^{\sigma^2}k^\sigma = ak \Rightarrow \sigma^2 = id.$$

Besides (1), (2), (3), the relation (7) must hold as well.

Example 4. Let σ be an antiautomorphism, $\boldsymbol{T}$ a Cayley division algebra from Example 1 with $\boldsymbol{C} = \boldsymbol{R}$ ($\sigma_{\boldsymbol{C}} = id$). The matrix $\|a_{ij}\|$ must represent an antiautomorphism, so that $e_ie_j = e_m$ for some triple (i, j, m) implies $e_j^\sigma e_i^\sigma = e_m^\sigma$. First we put $e_1^\sigma = e_2$, $e_2^\sigma = e_1$, $e_3^\sigma = e_3$. For the remaining e_4, e_5, e_6, e_7 the identities $e_7e_4 = e_3$,

$e_4^\sigma e_7^\sigma = e_3$, $e_5 e_6 = e_3$, $e_6^\sigma e_5^\sigma = e_3$ must hold and so on. Finally, we can choose $e_4^\sigma = e_7$, $e_7^\sigma = e_4$, $e_5^\sigma = e_5$, $e_6^\sigma = -e_6$ and we get the matrix

$$\|a_{ij}\| = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}, \quad \det \|a_{ij}\| = -1\,.$$

This antiautomorphism σ admits the corresponding k in the form $k = k_0 + k_1 e_1 + k_1 e_2 + k_3 e_3 + k_4 e_4 + k_5 e_5 + k_4 e_7$, with arbitrary k_0, k_1, k_3, k_4, k_5. Then $k^\sigma = k$, $a^{\sigma^2} = a$ for all $a \in \boldsymbol{T}$.

2.2.3. Let σ be neither an automorphism nor an antiautomorphism, but only a semiautomorphism with a fixed element $k = k^\sigma$ and $\sigma_{\mathbf{C}}^2 = id$. Then the fundamental relation $(ka^\sigma)^\sigma = ak$ must hold for all $a \in \boldsymbol{T}$.

Example 5. Let σ be a semiautomorphism, $\boldsymbol{T}$ a Cayley division algebra from Example 1 and $\mathbf{C} = \mathbf{R}$, $\sigma_{\mathbf{C}} = id$. We choose $k = 1 + e_1 + e_2 + e_3$ and $a_{1i} = a_{2i} = a_{3i} = 0$, $i \geqq 4$ for the elements of the matrix $\|a_{ij}\|$, so that

$$e_1^\sigma = a_{11}e_1 + a_{12}e_2 + a_{13}e_3\,,$$

$$e_2^\sigma = a_{21}e_1 + a_{22}e_2 + a_{23}e_3\,,$$

$$e_3^\sigma = a_{31}e_1 + a_{32}e_2 + a_{33}e_3\,.$$

We know that k and a_{ij} must satisfy (4), (5) and (8). From (4) we get

$$(ke_i^\sigma)^\sigma = e_i k \quad \text{for} \quad i \in \{1, 2, 3\}\,.$$

After the detailed analysis we see that the only solution different from identity is

$$\|a_{ij}\| = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \quad \text{for} \quad 1 \leqq i, j \leqq 3\,.$$

Similarly we choose $e_4^\sigma = e_5$, $e_5^\sigma = e_4$ and from (4) we get:

$$(ke_4^\sigma)^\sigma = e_4 k \Rightarrow ((1 + e_1 + e_2 + e_3)\, e_5)^\sigma = e_4(1 + e_1 + e_2 + e_3) \Rightarrow$$

$$\Rightarrow (e_5 + e_4 - e_7 + e_6)^\sigma = e_4 + e_5 + e_6 + e_7 \Rightarrow$$

$$\Rightarrow e_4 + e_5 - e_7^\sigma + e_6^\sigma = e_4 + e_5 + e_6 + e_7 \Rightarrow e_6^\sigma = e_6 \quad \text{and} \quad e_7^\sigma = -e_7\,.$$

Calculation shows that $(ke_j^\sigma)^\sigma = e_j k$ for $j \geqq 5$. The matrix $\|a_{ij}\|$ has the form

$$\|a_{ij}\| = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{Vmatrix}.$$

Let us examine whether σ is only a semiautomorphism.

a) If σ were an automorphism, then $(e_1 e_2)^\sigma = (-e_3)^\sigma$ would imply $e_1^\sigma e_2^\sigma = -e_3^\sigma$, but $e_2 e_1 \neq -e_3$ and we get that σ is not an automorphism.

b) If σ were an antiautomorphism, then $(e_3 e_4)^\sigma = (-e_7)^\sigma$ would imply $e_4^\sigma e_3^\sigma = -e_7^\sigma$, but $e_5 e_3 \neq e_7$ and we get that σ is not an antiautomorphism.

σ is a semiautomorphism satisfying (1), (2), (3), (4) and (5).

From Schütte's definition of orthogonality it follows that the line $y = x$ is orthogonal to the line $y = k^{-1}x$. In this example we have chosen $k = 1 + e_1 + e_2 + e_3$,

$$k^{-1} = \frac{1}{1 + e_1 + e_2 + e_3} = \tfrac{1}{4}(1 - e_1 - e_2 - e_3),$$

in such a way that the orthogonality is defined by

$$y = ax \perp y = (ka^\sigma)^{-1} x \quad \text{for all} \quad a \in \boldsymbol{T}.$$

2.3. As we have seen from the case 2.2.3, the conditions for a_{ij} which guarantee that σ is a Schütte semiautomorphism, depend on the multiplication table chosen for the Cayley division algebra $\boldsymbol{T}$ (relation (4)). The existence of Schütte semiautomorphisms is proved by Example 5. The determination of all Schütte semiautomorphisms for a given Cayley division algebra is still an open problem.

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