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A NOTE ON THE LOCAL STRUCTURE OF LEVELS OF A PLANE VECTOR FIELD

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Let $\boldsymbol{f} = (f_1, f_2)$ be a vector field of the class C_1 defined on a plane region Ω and satisfying the identities

$$\text{(1)} \qquad \operatorname{rot} \boldsymbol{f} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \equiv 0\,, \quad \operatorname{div} \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \equiv 0 \text{ in } \Omega\,.$$

Given any circle $U(z_0, \eta) = \{z;\ \|z - z_0\| < \eta\} \subset \Omega$, there exist two real-valued functions u, v defined on $U(z_0, \eta)$ with

$$\text{(2)} \qquad \operatorname{grad} u = \boldsymbol{f}\,, \quad \operatorname{grad} v = \boldsymbol{f}^* = (-f_2, f_1)\,;$$

we call them the *potential* and the *stream function* (of the field $\boldsymbol{f}$ on $U(z_0, \eta)$), respectively, and may choose them so that $u(z_0) = v(z_0) = 0$. If the first-order total differentials $\mathrm{d}u$ and $\mathrm{d}v$ at the point $z_0 = (x_0, y_0) \in \Omega$ are not zero, then for each sufficiently small $\delta > 0$ each of the sets

$$\text{(3)} \qquad \{z;\ \|z - z_0\| \leqq \delta,\ u(z) = 0\} \quad (\textit{the zero "potential-level"}),$$

$$\text{(4)} \qquad \{z;\ \|z - z_0\| \leqq \delta,\ v(z) = 0\} \quad (\textit{the zero "stream-level"})$$

is an analytic arc; the end-points of these arcs lie on the circumference $\|z - z_0\| = \delta$ and the arcs are perpendicular at their only intersection point z_0.

More generally, if $\mathrm{d}u = \mathrm{d}^2 u = \ldots = \mathrm{d}^{p-1} u = 0 \neq \mathrm{d}^p u$, $\mathrm{d}v = \mathrm{d}^2 v = \ldots$ $\ldots = \mathrm{d}^{p-1} v = 0 \neq \mathrm{d}^p v$ at z_0, the sets (3) and (4) equal respectively the union of p arcs $L_1, \ldots, L_p = L_0$ and $M_1, \ldots, M_p = M_0$ with end-points on the circumference $\|z - z_0\| = \delta$ and the only intersection point z_0. If $p > 1$ and if the arcs are numbered properly, then the angles between L_{j-1}, L_j and between M_{j-1}, M_j $(j = 1, \ldots, p)$ are equal to π/p, while the angles between L_{j-1}, M_{j-1} and M_{j-1}, L_1 equal $\pi/2p$.

A field $\boldsymbol{f}$ (of the class C_1) satisfies conditions (1), iff the function $F = f_1 - i f_2$ is holomorphic on Ω. (We do not distinguish between the point $z = (x, y)$ of the cartesian plane and the point $z = x + iy$ of the Gaussian plane; the region Ω is part of both the planes simultaneously.) The field $\boldsymbol{f}$ has a potential and a stream function on a region $\Omega_1 \subset \Omega$, iff the function F admits a primitive function on Ω_1. If $\Phi' = F$ on Ω_1, then $u = \operatorname{Re} \Phi$, $v = \operatorname{Im} \Phi$ are the potential and the stream function, respectively.

If no potential or no stream function on Ω exists, then instead of a primitive function Φ (which does not exist then), we may investigate a primitive analytic function $\mathscr{F}$, i.e. an analytic function $\mathscr{F}$ on Ω whose derivative $\mathscr{F}'$ on Ω equals F; it is called the *complex potential* of the field $\boldsymbol{f}$. Complex potentials exist for every field $\boldsymbol{f}$ with (1); their single-valued branches on subregions Ω_1 of Ω are primitive functions of F. (Speaking of analytic functions, we use the terminology and notation from [1] and [2]. An *analytic element* is any pair of the form (a, Φ), where a is a complex number and Φ a function holomorphic at the point a. If $\mathscr{E} = (a, \Phi)$ is an element, we write $s(\mathscr{E}) = a$ and $h(\mathscr{E}) = \Phi(a)$; its *derivative* is the element $\mathscr{E}' = (a, \Phi')$. The *derivative of an analytic function* $\mathscr{F}$ on Ω containing the element $\mathscr{E}$ is the analytic function $\mathscr{F}'$ on Ω containing the element $\mathscr{E}'$.)

If $\mathscr{F}$ is a complex potential of the field $\boldsymbol{f}$ containing an element of the form $\mathscr{E}_0 = (z_0, \Phi)$ where $\Phi(z_0) = 0$, we may investigate the local structure of the sets

$$(3') \qquad X = \{s(\mathscr{E});\ \mathscr{E} \in \mathscr{F},\ \operatorname{Re} h(\mathscr{E}) = 0\} \quad (\textit{the zero "potential-level"}),$$

$$(4') \qquad Y = \{s(\mathscr{E});\ \mathscr{E} \in \mathscr{F},\ \operatorname{Im} h(\mathscr{E}) = 0\} \quad (\textit{the zero "stream-level"})$$

containing the point z_0, which are generalizations of the sets (3) and (4).

Let Φ be holomorphic on the circle $U(z_0, \eta) = \{z; |z - z_0| < \eta\} \subset \Omega$; then $\Phi' = F$ on $U(z_0, \eta)$ and Φ is a single-valued branch of $\mathscr{F}$ on $U(z_0, \eta)$. According to the Monodromy Theorem, on $U(z_0, \eta)$ there are only single-valued branches of $\mathscr{F}$, since $\mathscr{F}$ is arbitrarily continuable on Ω (the notion of an arbitrarily continuable analytic function see in [1], p. 256); each of the branches is of the form Φ + const. (for $\mathscr{F}' = F$ on Ω).

If $\boldsymbol{f} \equiv 0$, then $X = Y = \Omega$ and there is nothing to investigate. Suppose, therefore, that $\boldsymbol{f} \not\equiv 0$. Then there is a natural number p such that $\Phi(z_0) = \Phi'(z_0) = \ldots = \Phi^{(p-1)}(z_0) = 0 \neq \Phi^{(p)}(z_0)$. According to [1], p. 161, (12.1), there exists an $\varepsilon_0 > 0$ with the following properties:

$$(5) \qquad z \in P(z_0, \varepsilon_0)\,^{1)} \Rightarrow \Phi(z) \neq 0 \neq \Phi'(z);$$

(6) for each $\varepsilon \in (0, \varepsilon_0\rangle$ there is a $\delta > 0$ such that for each $w \in P(0, \delta)$ there are precisely p points $z_1, \ldots, z_p \in U(z_0, \varepsilon)$ satisfying $w = \Phi(z_1) = \ldots = \Phi(z_p)$.

Let δ_0 be the number corresponding to $\varepsilon = \varepsilon_0$ in (6). Let $w \in P(0, \delta_0)$ be arbitrary and let $z \in U(z_0, \varepsilon_0)$ be a point with $\Phi(z) = w$. By (5), the element $\mathscr{E} = (z, \Phi)$ has an inverse element $\mathscr{E}_{-1}$. (See [1], p. 254.) All elements $\mathscr{E}_{-1}$ constructed in the above way constitute a p-valued analytic function $\mathscr{G}$ on $P(0, \delta_0)$ with the following properties

1) We denote $P(z_0, \varepsilon_0) = \{z; 0 < |z - z_0| < \varepsilon_0\}$.

(cf. [2], Theorem 232, p. 453):

(7') $\mathscr{G}$ is arbitrarily continuable in $P(0, \delta_0)$;

(7'') if the elements $\mathscr{E}_1, \mathscr{E}_2 \in \mathscr{G}$ are distinct[2]), then $h(\mathscr{E}_1) \neq h(\mathscr{E}_2)$.

Of course, $\mathscr{G}$ is the analytic function inverse to $\Phi \mid (\Phi_{-1}(P(0, \delta_0)))$; further, we have $\lim_{w \to 0} \mathscr{G}(w) = z_0$. (Cf. [2], p. 453.)

According to [1], Assertion (9.2), p. 262, there is a function Ψ holomorphic on $P(0, \delta_0^{1/p})$ such that

$$\mathscr{G}(w) = \Psi(\sqrt[p]{w}) \tag{8}$$

for every $w \in P(0, \delta_0)$; defining

$$\sqrt[p]{0} = 0\,, \quad \Psi(0) = z_0\,, \quad \mathscr{G}(0) = z_0\,, \tag{9}$$

we easily see that Ψ is holomorphic and one-one (and therefore conformal) on $U(0, \delta_0^{1/p})$, and the equality (8) holds for all $w \in U(0, \delta_0)$.

Now, let us choose a $\delta \in (0, \delta_0)$ arbitrarily. The function $\sqrt[p]{w}$ maps $U(0, \delta)$ onto $U(0, \delta^{1/p})$; the conformal mapping Ψ maps $U(0, \delta^{1/p})$ onto a Jordan region Z_δ containing the point z_0 and with an analytic boundary ∂Z_δ. Further, it is obvious that the set

$$\{z \in \bar{Z}_\delta;\ \operatorname{Im} \Phi(z) = 0\} \tag{10}$$

is the $\mathscr{G}$-image of the interval $\langle -\delta, \delta \rangle$. The analytic function $\sqrt[p]{w}$ (with $\sqrt[p]{0} = 0$) maps the interval $\langle -\delta, \delta \rangle$ onto the union of the segments

$$l_j = \overline{-\delta^{1/p} \exp \frac{ij\pi}{p}\,,\ \ \delta^{1/p} \exp \frac{ij\pi}{p}}\,, \tag{11}$$

where $j = 0, \ldots, p - 1$; the conformal mapping Ψ maps l_j onto the analytic arc

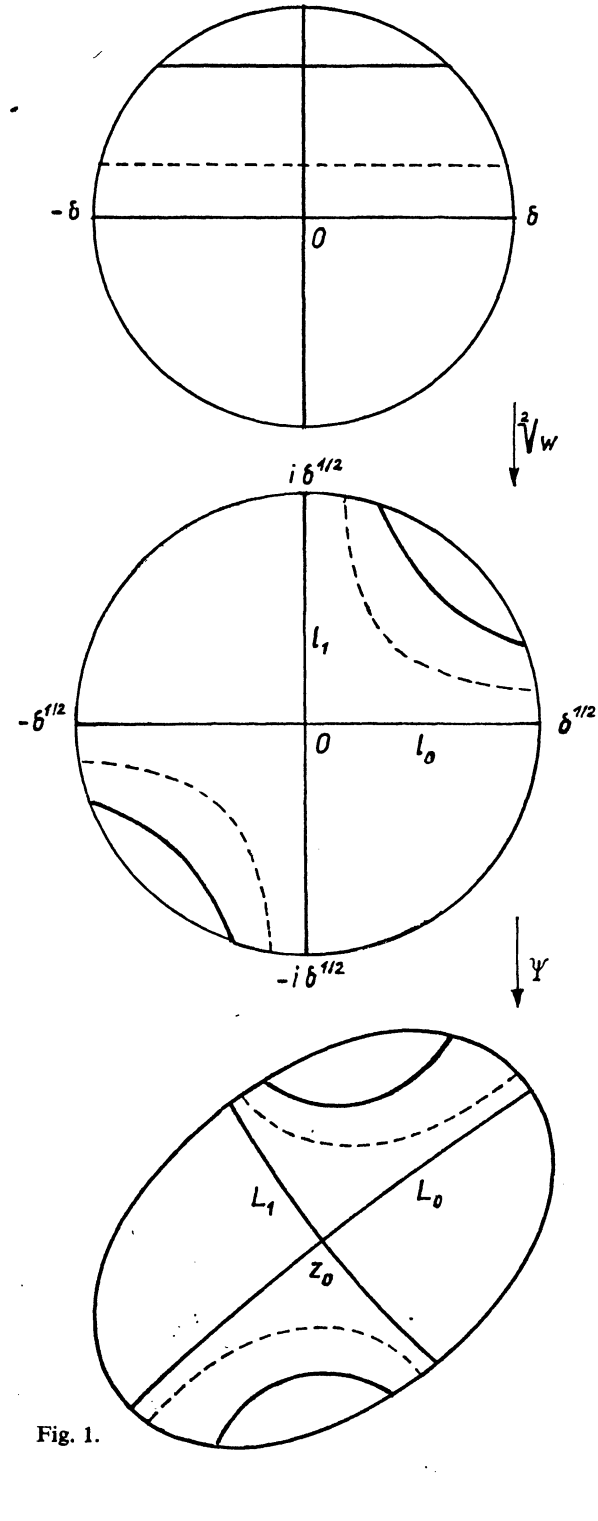
$$L_j = \Psi(l_j) \tag{12}$$

containing the point z_0 and having both end-points on ∂Z_δ. The set (10) is equal to the union

$$\bigcup_{j=0}^{p-1} L_j\,. \tag{13}$$

As the angle between the segments l_{j-1}, l_j is equal to π/p, the same angle is between their conformal images L_{j-1}, L_j at their (only) intersection point z_0.

[2]) Cf. [1], p. 239.



$-\delta$
δ
0
$\sqrt[2]{w}$
$i\delta^{1/2}$
l_1
$-\delta^{1/2}$
$\delta^{1/2}$
0
l_0
$-i\delta^{1/2}$
Ψ
L_1
L_0
z_0

Fig. 1.

The situation is, for the time being, analogous to the situation mentioned at the beginning of the paper. It is obvious that

$$(14) \qquad \{z \in \bar{Z}_\delta;\ \operatorname{Im} \Phi(z) = 0\} = \bigcup_{j=0}^{p-1} L_j \subset \bar{Z}_\delta \cap Y .$$

We do not know, however, if the equality may be written instead of the inclusion (at least, if $\delta > 0$ is sufficiently small).

Suppose that $z^* \in \bar{Z}_\delta \cap Y$. Then there is an element $\mathscr{E} = (z^*, \Phi^*) \in \mathscr{F}$ such that $\operatorname{Im} \Phi^*(z^*) = 0$. Φ^* is a single-valued branch of $\mathscr{F}$ on a neighbourhood $U(z^*) \subset \subset U(z_0, \eta)$. Therefore, $\Phi - \Phi^*$ is constant on $U(z^*)$, and Φ^* admits a (holomorphic) extension to $U(z_0, \eta)$. The difference $\Phi - \Phi^*$ remains constant. If $\Phi - \Phi^* = k$ on $U(z_0, \eta)$, then $\operatorname{Im} \Phi^*(z^*) = 0$, iff $\operatorname{Im} \Phi(z^*) = \operatorname{Im} k$. If $|\operatorname{Im} k| \leqq \delta$ holds, the set

$$(15) \qquad \{w \in \overline{U(0, \delta)};\ \operatorname{Im} w = \operatorname{Im} k\}$$

is non-empty, and its $\mathscr{G}$-image is part of the set $\bar{Z}_\delta \cap Y$. If $\operatorname{Im} k = 0$, then the set

$$(16) \qquad \mathscr{G}(\{w \in \overline{U(0, \delta)};\ \operatorname{Im} w = \operatorname{Im} k\})$$

is, of course, equal to $\mathscr{G}(\langle -\delta, \delta \rangle)$, i.e. to the set (10) and (13). However, if $0 < |\operatorname{Im} k| \leqq \delta$, then (by (7″)) the set (16) is disjoint with the set (10).

Consequently: The equality

$$(17) \qquad \{z \in \bar{Z}_\delta \cap Y;\ \operatorname{Im} \Phi(z) = 0\} = \bar{Z}_\delta \cap Y$$

holds, iff there is no single-valued branch Φ^* of $\mathscr{F}$ on $U(z_0, \eta)$ such that

$$(18) \qquad 0 < |\operatorname{Im} \Phi(z_0) - \operatorname{Im} \Phi^*(z_0)| \leqq \delta .$$

Suppose we have such a branch Φ^*. If $0 < |\operatorname{Im} k| < \delta$, then (15) is a chord of the circle $\overline{U(0, \delta)}$. The function $\sqrt[p]{w}$ maps it onto a union of p disjoint analytic arcs with end-points on the circumference $\partial U(0, \delta^{1/p})$; these arcs are disjoint with the arcs l_j. If $|\operatorname{Im} k| = \delta$, then the set (15) contains only one point which the function $\sqrt[p]{w}$ maps on a p-point set contained in $\partial U(0, \delta^{1/p})$. The conformal mapping Ψ maps the union of the arcs (resp. the p-point-set) onto the union of p disjoint analytic arcs (resp. a p-point-set) disjoint with the set (10). (Cf. Figure 1 where $p = 2$.)

If $\operatorname{Im} \tilde{k} \neq \operatorname{Im} k$, then (by (7″)) the set (16) is disjoint with the analogous set constructed for $\tilde{k}$.

According to the Poincaré-Volterra Theorem (see [1], p. 258) the analytic function $\mathscr{F}$ has in $U(z_0, \eta)$ only countably many single-valued branches. As a consequence,

(19) for each branch Φ^* of $\mathscr{F}$ in $U(z_0, \eta)$ with $\Phi - \Phi^* = k$, $0 < |\operatorname{Im} k| \leqq \delta$, the components of the set (16) are, at the same time, components of the set $\bar{Z}_\delta \cap Y$.

Thus, we have proved the following

Theorem 1. *Let* $\mathbf{f} = (f_1, f_2) \neq 0$ *be a plane vector field of the class* C_1 *on a region* Ω *satisfying conditions* (1). *Let* $z_0 \in \Omega$ *and let* Φ *be primitive to the function* $F = f_1 - if_2$ *on the circle* $U(z_0, \eta) \subset \Omega$; *let* $\Phi(z_0) = \Phi'(z_0) = \ldots = \Phi^{(p-1)}(z_0) = 0 \neq \Phi^{(p)}(z_0)$. *Let* $\mathcal{F}$ *be the complex potential of the field* $\mathbf{f}$ *containing* Φ *as a single-valued branch.*

Then there is a $\delta_0 > 0$ *such that for each* $\delta \in (0, \delta_0)$ *the set* $\bar{Z}_\delta = \Phi_{-1}(U(0, \delta))$ *is a Jordan region containing* z_0 *and with an analytic boundary. One of the components of the intersection* $\bar{Z}_\delta \cap Y$ *is the set* (10), *which is the union of* p *analytic arcs* $L_1, \ldots, L_p = L_0$ *with end-points on* ∂Z_δ. *The angle between the arcs* L_{j-1}, L_j *at their (only) intersection point* z_0 *is equal to* π/p.

The set $\bar{Z}_\delta \cap Y$ *has other components, iff the complex potential* $\mathcal{F}$ *has single-valued branches* Φ^* *in* $U(z_0, \eta)$ *with* $\Phi - \Phi^* = k$, $0 < |\mathrm{Im}\, k| \leqq \delta$. *To each of these branches, there correspond* p *distinct components of the set* $\bar{Z}_\delta \cap Y$. *According to whether* $0 < |\mathrm{Im}\, k| < \delta$ *or* $|\mathrm{Im}\, k| = \delta$, *these components are analytic arcs with end-points on* ∂Z_δ *or one-point sets in* ∂Z_δ. *There are only countably many such branches* Φ^*.

Let us show an example of a field $\mathbf{f}$ having the property that for each $\delta > 0$ there are single-valued branches Φ^* in $U(z_0, \eta)$ such that $0 < |\mathrm{Im}\, \Phi - \mathrm{Im}\, \Phi^*| < \delta$.

Example 1. Let $\mathbf{f} = (f_1, f_2)$ be a vector field for which the function $F = f_1 - if_2$ is defined by the equality

$$F(z) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n(n+z)} \tag{20}$$

on the region Ω which equals the plane without all negative integers. It is obvious that the series in (20) is locally uniformly convergent in Ω so that F is holomorphic on Ω. Further,

$$\mathrm{res}_{-q}\, F = \frac{1}{2\pi q} \quad \text{for every integer} \quad q > 0\,. \tag{21}$$

Let Φ be primitive to F on $U(0, 1)$, $\Phi(0) = 0$. As

$$\Phi'(0) = F(0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} > 0\,,$$

we have $p = 1$ in Theorem 1.

If $r = p/q$, where p, q are integers, $q > 0$, then, obviously, there exists a curve $\varphi : \langle 0, 1 \rangle \to \Omega$ with a finite length with $\varphi(0) = \varphi(1) = 0$ and $\mathrm{ind}_\varphi(-q) = p$, $\mathrm{ind}_\varphi(-n) = 0$, if $n \neq q$. The continuation $(0, \Phi^*)$ of the element $(0, \Phi)$ along the curve φ has the property that

$$\Phi^*(0) = \Phi^*(0) - \Phi(0) = \int_\varphi F = 2\pi i\, \mathrm{res}_{-q}\, F\, \mathrm{ind}_\varphi(-q) = \frac{ip}{q} = ir\,.$$

If $\delta > 0$ is arbitrarily chosen and if $0 < |r| < \delta$ (where $r = p/q$ as above), then $0 < |\mathrm{Im}\, \Phi^*(0) - \mathrm{Im}\, \Phi(0)| = |r| < \delta$.

The set $\bar{Z}_\delta \cap Y$ has, for each $\delta > 0$, infinitely many components. The analytic function $\mathscr{F}$, the branch of which in $U(0, 1)$ is Φ, has the following property: The set of all values of $\mathscr{F}$ at each point $z \in \Omega$ has no isolated points. Each component of the set $\bar{Z}_\delta \cap Y$ is part of the closure of the union of the remaining components of $\bar{Z}_\delta \cap Y$. The local structure of the set Y at any point $z \in \Omega$ is analogous to its local structure at the point 0.

For any positive integer p, the field $\tilde{\mathbf{f}}$ with $\tilde{F}(z) = z^p F(z)$ has analogous properties; for the primitive function $\tilde{\Phi}$ we have $\tilde{\Phi}(0) = \tilde{\Phi}'(0) = \ldots = \tilde{\Phi}^{(p-1)}(z_0) = 0 \neq \tilde{\Phi}^{(p)}(z_0)$ now.

Remark 1. Obviously, an analogous theorem holds for the set X. The component of the set $\bar{Z}_\delta \cap X$ containing z_0 is the union of p analytic arcs $M_1, \ldots, M_p = M_0$, where

$$M_j = \Psi(m_j) \tag{22}$$

and m_j are segments with end-points

$$\pm \delta^{1/p} \exp \frac{i(2j-1)\pi}{2p}. \tag{23}$$

The angle between M_{j-1}, M_j is π/p, the angle between L_{j-1}, M_{j-1} and M_{j-1}, L_j is $\pi/2p$ (at the only intersection point z_0).

If there are branches Φ^* with $0 < |\operatorname{Re} \Phi^* - \operatorname{Re} \Phi| \leqq \delta$, then there are other components of the set $\bar{Z}_\delta \cap X$ as well.

It is certainly not necessary to go into further details. ——

Now let us show that in two important cases the local structure of the set Y (and, analogously, of the set X) is simpler.

Theorem 2. *Let all assumptions of Theorem* 1 *hold; let us use the same notation. Then the following assertion holds*:

If either a stream function of the field $\mathbf{f}$ *exists* (*in* Ω), *or the region* Ω *is double connected*[3]), *then for each sufficiently small* $\delta > 0$ *the set* (10) *is the only component of the set* $\bar{Z}_\delta \cap Y$.

Proof. We have to prove there are no branches Φ^* of $\mathscr{F}$ in $U(z_0, \eta)$ with (18); let us use the above notation.

1. If a stream function of the field $\mathbf{f}$ exists, then there is a real-valued function v such that for each element (z, G) of $\mathscr{F}$ the equality $\operatorname{Im} G = v$ holds on a neighbourhood $U(z)$. This implies that any two single-valued branches of $\mathscr{F}$ in $U(z_0, \eta)$ have equal imaginary parts; the condition (18) does not hold for any branch Φ^* of $\mathscr{F}$ in $U(z_0, \eta)$.

[3]) This means the set $\mathbf{S} - \Omega$, where $\mathbf{S}$ is the Riemannian sphere, has exactly 2 components.

2. Let A be the bounded component of the set $\mathbf{S} - \Omega$. As is well known, there exists a positively oriented Jordan curve ω in Ω with a finite length such that $A \subset \subset \operatorname{Int} \omega$. If

$$\int_\omega F = d \text{ [4]} \tag{24}$$

and if $\zeta_0 \in A$ is an arbitrary point, then

$$\int_\varphi F = d \operatorname{ind}_\varphi \zeta_0 \tag{25}$$

for any closed curve φ in Ω with a finite length. Evidently, there is a positive number Δ such that $|\operatorname{Im}(nd)| \geqq \Delta$ for all integers n with $\operatorname{Im}(nd) \neq 0$.

For each single-valued branch Φ^* of $\mathscr{F}$ in $U(z_0, \eta)$, we have $\operatorname{Im} \Phi^* - \operatorname{Im} \Phi = \operatorname{Im}(nd)$ for an integer n; if $\operatorname{Im}(nd) \neq 0$, then $|\operatorname{Im}(nd)| \geqq \Delta$. As a consequence, if $\delta \in (0, \Delta)$, then (18) does not hold for any branch Φ^* of $\mathscr{F}$ in $U(z_0, \eta)$.

Remark 2. In fluid mechanics there often occur plane regions Ω such that $\mathbf{S} - \Omega$ has only countably many components, one of them being the one-point set $\{\infty\}$, and the other ones satisfying the following condition:

(26) If $A_1, A_2, \ldots$ is an infinite sequence of mutually distinct bounded components of $\mathbf{S} - \Omega$, then $\operatorname{Ls} A_n$[5]$) = \{\infty\}$.

Let us suppose that the region Ω is of this type; then, as may be shown, for each bounded component A of $\mathbf{S} - \Omega$ there is a Jordan curve ω_A in Ω with a finite length such that

$$(\mathbf{S} - \Omega) \cap \operatorname{Int} \omega_A = A\,. \tag{27}$$

Let $\mathbf{f}$, F, $\mathscr{F}$, etc. be as above and define

$$d_A = \int_{\omega_A} F \tag{28}$$

for each bounded component A of $\mathbf{S} - \Omega$.

As we easily see, each closed curve φ in Ω is homologous (in Ω) to a cycle[6]) $\Gamma =$

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[4]) In the terminology of fluid mechanics this integral is called the *circulation of the field round A*; by the Cauchy Theorem the number d is independent of the choice of ω with the above properties.

[5]) The *topological limes superior* of the sequence $\{A_n\}$ (defined as the set of all $z \in \mathbf{S}$ each neighbourhood $U(z)$ of which intersects an infinite number of sets A_n). The condition (26) (true, of course, if $\mathbf{S} - \Omega$ has only a finite number of components) means that each bounded component A of $\mathbf{S} - \Omega$ is "isolated", having a neighbourhood disjoint with the set $\mathbf{S} - (\Omega \cup A)$.

[6]) I.e., a finite sequence of closed curves.

$= \{\psi_1, \ldots, \psi_r\}$ containing only curves ψ_j equal either to the curves ω_A, or to the reversely oriented curves $\dot{-}\omega_A$. This implies

$$(29) \qquad \int_\varphi F = \sum_{j=1}^{s} d_{A_j} n_j$$

for each closed curve φ in Ω with a finite length, if $A_1, \ldots, A_s$ are properly chosen bounded components of $\mathbf{S} - \Omega$ and n_j properly chosen integers ($= \pm \operatorname{ind}_\varphi \zeta_A$, where $\zeta_A \in A$). Further, it follows that for each single-valued branch Φ^* of $\mathscr{F}$ in $U(z_0, \eta)$ the difference $\Phi^* - \Phi$ is of the form $\sum_{j=1}^{s} d_{A_j} n_j$.

Let

(30) $B = \{\operatorname{Im} \sum_{j=1}^{s} d_{A_j} n_j;\ A_1, \ldots, A_s$ are bounded component of $\mathbf{S} - \Omega, n_j$ integers$\}$.

As we see,

(31) condition (18) holds for some branch Φ^*, iff B contains at least one number c with $0 < |c| \leqq \delta$.

We are interested in the situation when for any $\delta > 0$ sufficiently small no such branch exists. This is equivalent to the statement that

(32) zero is an isolated point of the set B.

Thus we are led to the following number-theoretical problem:

(33) Given real non-zero numbers $c_1, c_2, \ldots$ (a finite or infinite sequence), under what conditions zero is an isolated point of the set $C = \{\sum_{j=1}^{s} c_j n_j;\ n_j$ integers$\}$?

We will prove that

(34) under the assumptions from (33), zero is an isolated point of the set C, iff there is a number $c > 0$ and integers p_j, q_j such that $c_j = p_j c/q_j$ for all j, where the sequence $q_1, q_2, \ldots$ is bounded.

Sufficiency: Let c_j be of the form given above and let q be a positive integer such that $|q_j| \leqq q$ for all j. As is easily seen, the inequality $\operatorname{dist}(0, C - \{0\}) \geqq c/q!$ holds then.

Necessity: Let us suppose that either (a) no $c > 0$ exists with all $c_j = p_j c/q_j$ for appropriately chosen integers p_j, q_j, or (b) all c_j are of the form $p_j c/q_j$, but for any such choice of integers p_j, q_j the sequence $\{q_j\}$ is unbounded.

Condition non (a) is equivalent to the statement that all quotients c_j/c_1 are rational numbers. If condition (a) holds, then, for instance, $c_2/c_1 = d$ is irrational. It is well known that then there are rational numbers of the form r/s (r, s integers, $s > 0$) with s arbitrarily large, such that $|d - r/s| < 1/s^2$. Given an arbitrary $\delta > 0$, choose r, s so that $|ds - r| < s^{-1} < \delta$; but $c_1(ds - r) \in C$, $c_1(ds - r) \neq 0$. Zero is, as a consequence, an accumulation point of the set C.

If condition (b) holds, we may suppose that $c_j = p_j c / q_j$, where $c > 0$, the greatest common divisor $(p_j, q_j) = 1$, and $q_j \to \infty$. Then $\alpha_j p_j + \beta_j q_j = 1$ for appropriately chosen integers α_j, β_j. It follows that

$$p_1 \alpha_j c_j + q_1 \beta_j c_1 = \frac{p_1 c}{q_j} (\alpha_j p_j + \beta_j q_j) = \frac{p_1 c}{q_j} ;$$

hence, the limit of the sequence of non-zero numbers $p_1 \alpha_j c_j + q_1 \beta_j c_1 \in C$ is equal to 0. Again, zero is an accumulation point of the set C.

Thus, we have proved the following

Theorem 3. *Let all assumptions of Theorem* 1 *hold; use the above notation. Let Ω be a region with $\mathbf{S} - \Omega = \{\infty\} \cup A_1 \cup A_2 \cup \ldots$ (a finite or infinite sequence), where $A_1, A_2, \ldots$ are disjoint bounded non-empty continua satisfying* (26).

For each set A_j let ω_j by a Jordan curve in Ω with a finite length for which

(35) $$(\mathbf{S} - \Omega) \cap \operatorname{Int} \omega_j = A_j$$

holds. For each set A_j let

(36) $$c_j = \operatorname{Im} \int_{\omega_j} F .$$

Then the condition

(37) *there is a positive number c, integers p_j and a bounded set of integers q_j such that $c_j = p_j c / q_j$ for all j*

is equivalent to the following condition:

(38) *for each sufficiently small $\delta > 0$ there are no single-valued branches Φ^* of $\mathcal{F}$ in $U(z_0, \eta)$ satisfying* (18).

Remark 3. If the complement $\mathbf{S} - \Omega$ of a region Ω has at least 2 distinct bounded components A_1, A_2, it may be proved that there are two Jordan curves ω_1, ω_2 in Ω with finite lengths such that

$$A_j \subset \operatorname{Int} \omega_j \quad \text{for} \quad j = 1, 2$$

and

$$A_1 \subset \operatorname{Ext} \omega_2 , \quad A_2 \subset \operatorname{Ext} \omega_1 .$$

Fix points $\zeta_j \in A_j$ $(j = 1, 2)$ and set

(39) $$F(z) = \frac{1}{2\pi} \left(\frac{c}{z - \zeta_1} + \frac{1}{z - \zeta_2} \right),$$

where c is a fixed irrational number. Then the corresponding field $\boldsymbol{f} = (\operatorname{Re} F, -\operatorname{Im} F)$ does not satisfy condition (37), and, consequently, nor the condition (38).

Thus we see that the second condition of Theorem 2 is an essential one.

Remark 4. It is clear now that we may expect a "simple" local structure of potential and stream levels (3'), (4') at any of their point z_0 (i.e. the connectivity of the sets $\bar{Z}_\delta \cap X$, $\bar{Z}_\delta \cap Y$ for each sufficiently small $\delta > 0$) only in two simple cases: 1. the field $\boldsymbol{f}$ has a potential or a stream function on the whole region Ω. 2. $\boldsymbol{S} - \Omega$ has at most 2 components (i.e., at most one bounded component).

In other cases it is "most probable" that the real or the imaginary parts of two integrals of the type occuring in (36) will have an irrational ratio, and, as a consequence, the sets $\bar{Z}_\delta \cap X$, $\bar{Z}_\delta \cap Y$ will be disconnected (and will have infinitely many components) for any $\delta > 0$ (at any point z_0 of the corresponding level).

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