Alexander Fischer

Existence of almost periodic solutions of systems of linear and quasilinear differential equations with time lag

Časopis pro pěstování matematiky, Vol. 106 (1981), No. 3, 256--268

Persistent URL: `http://dml.cz/dmlcz/118099`

Časopis pro pěstování matematiky, roč. 106 (1981), Praha

EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF SYSTEMS OF LINEAR AND QUASILINEAR DIFFERENTIAL EQUATIONS WITH TIME LAG

ALEXANDER FISCHER, Praha

(Received March 14, 1979)

INTRODUCTION

Applications of differential equations with time lag can be found in many various fields. This is the case especially in the theory of automatic control systems, telemechanics, radiodetection and ranging, radio navigation and in other branches. An important role is played here by periodic and almost periodic precesses and the corresponding periodic and almost periodic solutions of systems of differential equations with time lag.

This paper has been inspired by S. N. Šimanov's paper [4]. The author studies in [4] the equation (in matrix form)

$$x'(t) = a\,x(t) + b\,x(t - \tau) + f(t) \tag{1}$$

where a, b are constant square matrices of n-th order, the function $f(t)$ and its derivative $f'(t)$ are almost periodic vector functions (column with n components), and τ is a positive constant time lag. Provided that all roots of the characteristic equation of Equation (1) lie in the half-plane

$$\operatorname{Re} z \leqq -2\alpha < 0\,,$$

Equation (1) has a unique almost periodic solution $x_f(t)$ which satisfies the estimate

$$|x_f| \leqq AM\,.$$

Here A is a positive constant depending on a, b only, and

$$M = \max\{|f|, |f'|\}\,.$$

This result may be extended and generalized to a system of quasilinear equations. This is the aim of this paper.

The problem will be solved for one constant time lag. The method used for several constant time lags would be, however, quite analogous.

Let us consider the system (1) and introduce for all almost periodic functions $f(t)$ with an almost periodic derivative $f'(t)$ the norm

$$\|f\| = \max\{|f|, |f'|\},$$

where

$$|f| = \sup|f(t)|, \quad |f'| = \sup|f'(t)|$$

for $t \in J = (-\infty, +\infty)$.

Let

$$\Phi(z) = zE - a - be^{-z\tau},$$

where E is the unit matrix of n-th order, and let the characteristic quasipolynomial of Equation (1) be

$$\Delta(z) = \det \Phi(z).$$

$\Delta(z)$ is a transcendent entire function (in general) of complex variable z and, consequently, the equation

$$\Delta(z) = 0 \tag{2}$$

has an infinite number of roots without any finite limit point. Under $\sigma(\Delta(z))$ we understand the set of all roots of Equation (2) over the range C of all complex numbers. Let

$$P(\alpha) = \{z \in C : |\text{Re } z| \leqq \alpha\}$$

for a real positive number α. Each stripe $P(2\alpha)$, $\alpha > 0$, contains only a finite number of zeros of the characteristic quasi-polynomial $\Delta(z)$ because $\Phi(z)\,z^{-1}$ is arbitrarily close to the matrix E in the stripe $P(2\alpha)$ for z sufficiently large (in absolute value). Hence the matrix $\Phi(z)$ is a regular one for such z. Consequently the positive number α can be chosen so that the set

$$P(2\alpha) \cap \sigma(\Delta(z))$$

lies on the imaginary axis of the complex plane C. Let this set consist of points $i\omega_1, \ldots, i\omega_p$ with multiplicities $r_1, \ldots, r_p$ if regarded as roots of Equation (2).

Given a set $\mathcal{M} \subset J = (-\infty, +\infty)$ and a complex number β, we put

$$\beta\mathcal{M} = \{\beta\xi : \xi \in \mathcal{M}\}.$$

Theorem 1. *Let the distance*

$$d = \text{dist}\,[i\Lambda_f, \sigma(\Delta(z))]$$

of spectrum $i\Lambda_f$ *of an almost periodic function* $f(t)$ *(*Λ_f *is the set of all Fourier* *exponents of the function* $f(t)$) *from* $\sigma(\Delta(z))$ *in the complex plane C be positive. Then Equation* (1) *admits a unique almost periodic solution* $x_f(t)$ *which satisfies the inclusion*

$$(3) \qquad \Lambda_{x_f} \subset \Lambda_f$$

and the estimate

$$(4) \qquad \|x_f\| \leqq A\|f\|,$$

where the constant A depends on a, b, d, τ only.

Note. Equation (1) may admit an infinite number of almost periodic solutions, but only one of them has its spectrum contained in the spectrum of the function $f(t)$.

In order to prove Theorem 1, we need **Favard's theorem**:

If $f(t)$ *is an almost periodic function and if*

$$\Lambda_f \cap (-d, d) = \emptyset$$

where d is a positive number, then the primitive function

$$F(t) = \int_0^t f(s)\,\mathrm{d}s$$

is an almost periodic function, too, and the estimate

$$|F(t) - \underset{t}{M}\{F(t)\}| \leqq M|f|$$

is valid.

Here

$$\underset{t}{M}\{F(t)\} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T F(s)\,\mathrm{d}s$$

is the mean value of the almost periodic function $F(t)$ *and* $M = M(d)$ *is a positive constant depending, on d only.*

The proof of Favard's theorem was published in [1], [2], [3].

Proof of Theorem 1. Let

$$f(t) \sim \sum_\lambda \varphi(\lambda)\,e^{i\lambda t}, \quad \lambda \in \Lambda_f.$$

First we seek a formal almost periodic solution $x_f(t)$ of Equation (1) in the form

$$x_f(t) \sim \sum_\lambda c(\lambda)\,e^{i\lambda t}, \quad \lambda \in \Lambda_f$$

i.e. with the spectrum in $i\Lambda_f$. We obtain a system of algebraic linear equations for $c(\lambda)$:

$$(6) \qquad i\lambda\,c(\lambda) = a\,c(\lambda) + b\,c(\lambda)\,e^{-i\lambda\tau} + \varphi(\lambda), \quad \lambda \in \Lambda_f.$$

If $d > 0$, then this equation has the unique solution

(7) $$c(\lambda) = \Phi^{-1}(\mathrm{i}\lambda)\,\varphi(\lambda)\,, \quad \lambda \in \Lambda_f\,.$$

Thus the uniqueness of the required solution, if it exists, is verified.

Now let us prove existence of the required solution. First let $f(t)$ be a finite trigonometric polynomial. Under this assumption the formal solution of Equation (1)

(8) $$x_f(t) = \sum_{\lambda} \Phi^{-1}(\mathrm{i}\lambda)\,\varphi(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda t}\,, \quad \lambda \in \Lambda_f\,,$$

which is also a trigonometric polynomial, is the required almost periodic solution of Equation (1).

We prove Estimate (4) in this case. Denote by $K(z_0, \delta)$ the circle in the complex plane C with its centre at the point $z_0 \in C$ and with its radius δ, $\delta > 0$. Further, denote by $\varkappa(z_0, \delta)$ the open ring and by $\bar{\varkappa}(z_0, \delta)$ the closed ring with its centre at z_0 and with its radius δ. There exists $\delta > 0$ such that the circles $K_j = K(\mathrm{i}\omega_j, \delta)$, $j = 1, \ldots, p$ do not intersect and

$$0 < \delta < \min\{\alpha, d\}\,,$$

where $\alpha > 0$ is chosen so that

$$P(2\alpha) \cap \sigma(\Delta(z)) = \{\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_p\}\,.$$

If $R > 0$, denote by L_R the boundary of the closed region

$$P(\alpha) \cap \bar{\varkappa}(0, R)\,.$$

Further, denote by S_1 and S_2, respectively, the upper and lower arcs of the circle $K(0, R)$ lying in the stripe $P(\alpha)$. If we choose $R > 0$ sufficiently large, the bounded region with the boundary L_R contains all the circles $K_j = K(\mathrm{i}\omega_j, \delta)$, $j = 1, \ldots, p$ and at the same time the spectrum $\mathrm{i}\Lambda_f$ of the trigonometric polynomial $f(t)$. The solution x_f now can be expressed in the form of Cauchy's integral

(9) $$x_f(t) = \frac{1}{2\pi\mathrm{i}}\left[\oint_{L_R} \Phi^{-1}(z)\,F(t, z)\,\mathrm{d}z - \sum_{j=1}^{p} \oint_{K_j} \Phi^{-1}(z)\,F(t, z)\,\mathrm{d}z\right],$$

where

(10) $$F(t, z) = \sum_{\lambda} \frac{\mathrm{e}^{\mathrm{i}\lambda t}}{z - \mathrm{i}\lambda}\,\varphi(\lambda)\,, \quad \lambda \in \Lambda_f\,.$$

If $\operatorname{Re} z \neq 0$ and $t \in J$, we have

(11) $$F(t, z) = \int_0^{+\infty \operatorname{Re} z} f(t + s)\,\mathrm{e}^{-zs}\,\mathrm{d}s = \frac{1}{z}\left[f(t) + \int_0^{+\infty \operatorname{Re} z} f'(t + s)\,e^{-zs}\,\mathrm{d}s\right],$$

because

$$\frac{e^{i\lambda t}}{z - i\lambda} = e^{zt} \int_t^{+\infty \mathrm{Re} z} e^{(i\lambda - z)s}\, ds = \int_0^{+\infty \mathrm{Re} z} e^{i\lambda(t+s)} e^{-zs}\, ds .$$

Since

$$\lim_{|z| \to +\infty} |F(t, z)| = 0$$

uniformly for $t \in J$, there exists such $R_0 > 0$ that

$$(12) \qquad |F(t, z)| \leqq 1$$

holds for $|z| \geqq R_0$, $t \in J$.

We may write

$$\Phi(z) = z \left[E - \frac{1}{z} (a + be^{-z\tau}) \right]$$

for $z \neq 0$. With regard to this form and to the regularity of $\Phi(z)$ on $G \dot{-} \{0\}$, where

$$G = P(\alpha) \dot{-} \bigcup_{j=1}^{p} \varkappa(i\omega_j, \delta) ,$$

there exists such a positive constant C_1 that

$$(13) \qquad |\Phi^{-1}(z)| \leqq \frac{C_1}{|z|}$$

holds for all $z \in G \dot{-} \{0\}$. Inequality (13) and Estimate (12) imply that for $R \geqq R_0$ and for $R \to +\infty$

$$(14) \qquad \left| \left(\int_{S_1} + \int_{S_2} \right) \Phi^{-1}(z)\, F(t, z)\, dz \right| \leqq 4C_1 \arcsin \frac{\alpha}{R} \to 0$$

holds uniformly for $t \in J$.

If $|\mathrm{Re}\, z| = \alpha$, we obtain from (11) the inequality

$$(15) \qquad |F(t, z)| \leqq \frac{1 + \alpha}{\alpha |z|} \|f\|$$

which is valid uniformly for $t \in J$. Estimates (13), (14), (15) guarantee existence of the limit

$$I_0 = \lim_{R \to +\infty} \frac{1}{2\pi i} \oint_{L_R} \Phi^{-1}(z)\, F(t, z)\, dz = \frac{1}{2\pi i} \left(- \int_{-\alpha - i\infty}^{-\alpha + i\infty} + \int_{\alpha - i\infty}^{\alpha + i\infty} \right) \Phi^{-1}(z)\, F(t, z)\, dz ,$$

for the integrals on the right hand side converge (absolutely) according to these estimates. The following estimate holds:

$$|I_0| \leqq \frac{1+\alpha}{2\pi\alpha} C_1 \int_{-\infty}^{+\infty} \left[\frac{1}{|\alpha - \mathrm{i}s|^2} + \frac{1}{|\alpha + \mathrm{i}s|^2} \right] \mathrm{d}s \, \|f\| = A_0 \|f\| , \quad A = \frac{1+\alpha}{\alpha^2} C_1 .$$

We need to establish analogous estimates for integrals

$$I_j = -\frac{1}{2\pi\mathrm{i}} \oint_{K_j} \Phi^{-1}(z) F(t, z) \, \mathrm{d}z , \quad j = 1, \ldots, p .$$

Due to the properties of the roots of Equation (2) in the stripe $P(2\alpha)$, the identity

$$\Delta(z) = (z - \mathrm{i}\omega_j)^{r_j} \Delta_j(z) , \quad \Delta_j(\mathrm{i}\omega_j) \neq 0 ,$$

holds for $z \in \bar{\varkappa}(\mathrm{i}\omega_j, \delta)$, $j = 1, \ldots, p$. Hence

$$\Phi^{-1}(z) = (z - \mathrm{i}\omega_j)^{-r_j} \Delta_j^{-1}(z) \tilde{\Phi}(z)$$

for $z \in \bar{\varkappa}(\mathrm{i}\omega_j, \delta) \dot{-} \{\mathrm{i}\omega_j\}$, $j = 1, \ldots, p$, where $\tilde{\Phi}(z)$ is the matrix whose elements with subscripts j, k are equal to the algebraic complements of $\Phi(z)$ corresponding to the elements of $\Phi(z)$ with subscripts k, j; $j, k = 1, \ldots, n$. The matrix

$$\Gamma_j(z) = \Delta_j^{-1}(z) \tilde{\Phi}(z)$$

is analytic in the closed ring $\bar{\varkappa}(\mathrm{i}\omega_j, \delta)$, $j = 1, \ldots, p$. Notice that the development

$$\frac{1}{z - \mathrm{i}\lambda} = -\frac{1}{\mathrm{i}\lambda - \mathrm{i}\omega_j} \sum_{k=0}^{\infty} \left(\frac{z - \mathrm{i}\omega_j}{\mathrm{i}\lambda - \mathrm{i}\omega_j} \right)^k$$

is valid for arbitrary $\lambda \in \Lambda_f$ and $z \in K_j$ because

$$|\mathrm{i}\lambda - \mathrm{i}\omega_j| \geqq d > \delta = |z - \mathrm{i}\omega_j| , \quad j = 1, \ldots, p .$$

Hence

$$-\frac{1}{2\pi\mathrm{i}} \oint_{K_j} \Phi^{-1}(z) \frac{e^{\mathrm{i}\lambda t}}{z - \mathrm{i}\lambda} \, \mathrm{d}z =$$

$$= \frac{1}{\mathrm{i}\lambda - \mathrm{i}\omega_j} \sum_{k=0}^{r_j - 1} \frac{1}{2\pi\mathrm{i}} \oint_{K_j} (z - \mathrm{i}\omega_j)^{-r_j + k} \Gamma_j(z) \, \mathrm{d}z \frac{e^{\mathrm{i}\lambda t}}{(\mathrm{i}\lambda - \mathrm{i}\omega_j)^k} =$$

$$= e^{\mathrm{i}\omega_j t} \sum_{k=1}^{r_j} \frac{1}{(r_j - k)!} \Gamma_j^{(r_j - k)}(\mathrm{i}\omega_j) \frac{e^{\mathrm{i}(\lambda - \omega_j)t}}{(\mathrm{i}\lambda - \mathrm{i}\omega_j)^k} , \quad j = 1, \ldots, p .$$

Denote

$$g_{i,j}(t) = \sum_{\lambda} \frac{e^{\mathrm{i}(\lambda - \omega_j)t}}{(\mathrm{i}\lambda - \mathrm{i}\omega_j)^k} \varphi(\lambda) , \quad \lambda \in \Lambda_f , \quad k = 0, 1, \ldots, r_j ; \quad j = 1, \ldots, p .$$

These functions are trigonometric polynomials (i.e. almost periodic function). Since

$$|\lambda - \omega_j| \geqq d > 0 \quad \text{for} \quad \lambda \in \Lambda_f\,, \quad j = 1, \dots, p\,,$$

and since $g_{j,k}(t)$ is the almost periodic primitive function of $g_{j,k-1}(t)$ with zeromean value, we obtain estimates

$$|g_{j,k}| \leqq M^k |g_{j,0}| = M^k |f| \leqq M^k \|f\|\,, \quad k = 1, \dots, r_j\,; \quad j = 1, \dots, p\,,$$

by applying repeatedly Favard's theorem. Hence we get

$$|I_j| \leqq \sum_{k=1}^{r_j} \frac{\Gamma_j^{(r_j-k)}(\mathrm{i}\omega_j)}{(r_j - k)!} M^k \|f\| = A_j \|f\|\,, \quad j = 1, \dots, p\,.$$

Passing to the limit for $R \to +\infty$ in (9), we obtain

(16) $$x_f(t) = \frac{1}{2\pi \mathrm{i}} \left(- \int_{-\alpha - \mathrm{i}\infty}^{-\alpha + \mathrm{i}\infty} + \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \right) \Phi^{-1}(z)\, F(t, z)\, \mathrm{d}z - \\ - \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{p} \oint_{K_j} \Phi^{-1}(z)\, F(t, z)\, \mathrm{d}z\,.$$

If we set

$$\tilde{A} = \max \{1, \sum_{j=0}^{p} A_j\}\,,$$

then

$$|x_f| \leqq \tilde{A} \|f\|\,.$$

Further, we get the estimate

$$|x'_f| \leqq [|a| + |b|]\, |x_f| + |f| \leqq [|a| + |b| + 1]\, \tilde{A} \|f\| = A \|f\|$$

from Equation (1), where

$$A = [|a| + |b| + 1]\, \tilde{A}$$

depends on a, b, d, τ only (δ depends on a, b, d, τ only).

Note 1. The estimate of $|x'_f|$ can be also obtained by the method used above for $|x_f|$. But it is necessary to replace $f(t)$ and $F(t, z)$ by $f'(t)$ and $F'_t(t, z)$, respectively. In the general case of the almost periodic function $f(t)$ we must, however, suppose, that $f(t)$ has the almost periodic derivatives $f'(t), f''(t)$. ■

Finally, we get

(17) $$\|x_f\| \leqq A \|f\|$$

by combining the above estimates.

Consider now an arbitrary almost periodic function $f(t)$ satisfying the conditions of Theorem 1. Denote by $B_m(t)$, $m = 1, 2, \dots$, Bochner-Fejér's polynomials of $f(t)$

(uniformly converging to $f(t)$ on J), then their derivatives $B'_m(t)$, $m = 1, 2, \ldots$, are Bochner-Fejer's polynomials of $f'(t)$ (uniformly converging to $f'(t)$ on J). Let $x_m(t)$ be the unique almost periodic solution of the equation

$$x'(t) = a\,x(t) + b\,x(t - \tau) + B_m(t) \tag{1m}$$

whose spectrum is contained in $i\Lambda_f$, $m = 1, 2, \ldots$. These solutions are Bochner-Fejér's polynomials of the formal solution of Equation (1) and it is sufficient to prove their uniform convergence on J which is guaranteed, however, by the estimates

$$\|x_{m+p} - x_m\| \leqq A\|B_{m+p} - B_m\|\,,$$

$m = 1, 2, \ldots; p = 1, 2, \ldots$. We get the required solution of Equation (1) by passing to limit for $m \to \infty$:

$$x_f(t) = \lim x_m(t)\,.$$

With regard to (17) and to $\|B_m\| \to \|f\|$, $m \to \infty$, the required estimate

$$\|x_f\| \leqq A\|f\|$$

is valid.

II. QUASILINEAR EQUATION

Consider now a system of quasilinear differential equations with positive constant time lag τ and with a small complex parametr ε

$$x'(t) = a\,x(t) + b\,x(t - \tau) + f(t) + \varepsilon\, g(t, x(t), x(t - \tau), \varepsilon)\,. \tag{18}$$

Note 2. In order to keep simple record we shall study only one unique time lag as the method used for more constant time lags would be quite analogous. ■

The terms $a, b, f(t)$ in Equation (18) have the same meaning as in Part I. Assume that the function $g(t, u, v, \varepsilon)$ is defined, continuous on $J \times C^n \times C^n \times \bar{\varkappa}$, where $C^n = C \times \ldots \times C$ and $\bar{\varkappa} = \bar{\varkappa}(0, \delta_0)$, $\delta_0 > 0$, analytic in the variables u, v, ε and almost periodic in the variable t together with its partial derivative $\partial g/\partial t$ uniformly to others variables.

If we have two sets $\mathcal{M} \subset J$, $\mathcal{N} \subset J$, we define

$$\mathcal{M} + \mathcal{N} = \{\xi + \eta : \xi \in \mathcal{M},\ \eta \in \mathcal{N}\}\,.$$

Further, we define

$$\Lambda = \Lambda_f \cup \Lambda_g + S(\Lambda_g \cup \Lambda_f \cup \{0\}) = S(\Lambda_f \cup \Lambda_g)\,,$$

where $S(\Lambda_g \cup \Lambda_f \cup \{0\})$ is the smallest additive semigroup which contains the union $\Lambda_g \cup \Lambda_f \cup \{0\}$. Analogously $S(\Lambda_f \cup \Lambda_g)$.

Theorem 2. *Let* $d = \operatorname{dist}[\mathrm{i}\Lambda, \sigma(\Delta(z))]$ *be positive. Then there exists a real positive number* ε_0 *such that for each complex* $\varepsilon \in \bar{\varkappa}$, *satisfying* $|\varepsilon| < \varepsilon_0$, *Equation* (18) *has a unique almost periodic solution* $x_\varepsilon(t)$ *with its spectrum in* $\mathrm{i}\Lambda$.

Proof. Denote by B the Banach space of all almost periodic vector functions $f(t)$ (column with n components) with almost periodic derivatives $f'(t)$ and with their spectra in $\mathrm{i}\Lambda$. The norm in B is $\|\cdot\cdot\|$. Set for $R > 0$

$$B_R = \{f \in B : \|f\| \leqq R\},$$

which is a closed subset of the space B. For $R > 0$ we denote by $\|g\|_R$ the maximum value among the least upper bounds of magnitudes of the function $g(t, u, v, \varepsilon)$ and its first order derivatives with respect to variables t, u, v, its mixed second order derivatives with respect to variables t, u, v, and its second order derivatives with respect to variables u, v on the set

$$\{[t, u, v, \varepsilon] : t \in J,\ |u| \leqq R,\ |v| \leqq R,\ \varepsilon \in \bar{\varkappa}\}.$$

The least upper bounds considered exist under the assumption of analyticity in variables u, v, continuity in variables t, u, v, ε, and almost periodicity in variable t of functions g and $\partial g/\partial t$. If we denote

$$C_R^n = \{w \in C^n : |w| \leqq R\},$$

then the estimates

$$(19)\quad \begin{cases} |g(t, u, v, \varepsilon) - g(\tilde{t}, \tilde{u}, \tilde{v}, \varepsilon)| \leqq \|g\|_R\,[|t - \tilde{t}| + |u - \tilde{u}| + |v - \tilde{v}|] \\ |g_t(t, u, v, \varepsilon) - g_t(\tilde{t}, \tilde{u}, \tilde{v}, \varepsilon)| \leqq \|g\|_R\,[|t - \tilde{t}| + |u - \tilde{u}| + |v - \tilde{v}|] \\ |g_u(t, u, v, \varepsilon) - g_u(\tilde{t}, \tilde{u}, \tilde{v}, \varepsilon)| \leqq \|g\|_R\,[|t - \tilde{t}| + |u - \tilde{u}| + |v - \tilde{v}|] \\ |g_v(t, u, v, \varepsilon) - g_v(\tilde{t}, \tilde{u}, \tilde{v}, \varepsilon)| \leqq \|g\|_R\,[|t - \tilde{t}| + |u - \tilde{u}| + |v - \tilde{v}|] \end{cases}$$

are valid for arbitrary real $t, \tilde{t}$, for arbitrary $u, \tilde{u}, v, \tilde{v}$ from C_R^n, and for arbitrary $\varepsilon \in \bar{\varkappa}$. If $\xi(t) \in B_R$, then under our assumptions the function

$$\gamma(t) = \gamma(t, \varepsilon) = g(t, \xi(t), \xi(t - \tau), \varepsilon)$$

is an almost periodic function belonging to the space B for each $\varepsilon \in \bar{\varkappa}$ and

$$|\gamma| \leqq \|g\|_R\,,$$

$$|\gamma'| = \left|\frac{\partial g}{\partial t} + \frac{\partial g}{\partial u}\,\xi'(t) + \frac{\partial g}{\partial v}\,\xi'(t - \tau)\right| \leqq \|g\|_R\,[1 + 2\|\xi\|]\,.$$

Hence

$$\|\gamma\| \leqq \|g\|_R\,[1 + 2R]\,.$$

Define an operator $\mathscr{A} = \mathscr{A}(\varepsilon)$, $\varepsilon \in \bar{\varkappa}$, on the space B such that for $\xi(t) \in B$, $\mathscr{A}\,\xi(t)$

is the unique almost periodic solution belonging to the space B of the equation

$$x'(t) = a\, x(t) + b\, x(t - \tau) + f(t) + \varepsilon\, \gamma(t)$$

(uniqueness is guaranteed by Theorem 1). We get the inequality

$$\|\mathscr{A}\xi\| \leqq A\|f + \varepsilon\gamma\| \leqq A[\|f\| + |\varepsilon|\, \|\gamma\|] \leqq$$
$$\leqq A[\|f\| + |\varepsilon|\, \|g\|_R \,.\, (1 + 2R)]\,.$$

Consequently, the operator $\mathscr{A}$ maps the space B_R into itself if $R > A\|f\|$, $\varepsilon \in \bar{\varkappa}$ and

$$|\varepsilon| < \frac{R - A\|f\|}{(1 + 2R)\, A\|g\|_R}\,.$$

If the functions $\xi(t)$, $\eta(t)$ belong to B_R, we define for $\varepsilon \in \bar{\varkappa}$ the functions

$$\gamma_\xi(t) = g(t, \xi(t), \xi(t - \tau), \varepsilon)\,, \quad \gamma_\eta(t) = g(t, \eta(t), \eta(t - \tau), \varepsilon)\,,$$
$$x(t) = \mathscr{A}\, \xi(t)\,, \quad y(t) = \mathscr{A}\, \eta(t)\,, \quad w(t) = x(t) - y(t)\,.$$

The function $w(t)$ is the unique almost periodic solution belonging to B of the equation

$$w'(t) = a\, w(t) + b\, w(t - \tau) + \varepsilon[\gamma_\xi(t) - \gamma_\eta(t)]$$

and consequently

$$\|\mathscr{A}\xi - \mathscr{A}\eta\| = \|w\| \leqq A|\varepsilon|\, \|\gamma_\xi - \gamma_\eta\| \leqq$$
$$\leqq 2\, A|\varepsilon|\, \|g\|_R\, (1 + 2R)\, \|\xi - \eta\|\,,$$

since according to the definition of $\|g\|_R$ and (19) the estimates

$$|\gamma_\xi - \gamma_\eta| \leqq 2\|g\|_R\, \|\xi - \eta\|\,,$$
$$|\gamma'_\xi - \gamma'_\eta| \leqq 2\|g\|_R\, (1 + 2R)\, \|\xi - \eta\|$$

hold. In order to get a contractive operator $\mathscr{A}$ on B_R it is sufficient to put

$$|\varepsilon| < \frac{1}{2(1 + 2R)\, A\|g\|_R}\,.$$

The operator $\mathscr{A}$ maps the space B_R into itself and turns out to be a contraction on B_R for $|\varepsilon| < \varepsilon_0$, where

$$\varepsilon_0 = \min\left\{\frac{1}{2(1 + 2R)\, A\|g\|_R},\ \frac{R - A\|f\|}{(1 + 2R)\, A\|g\|_R},\ \delta_0\right\}.$$

Consequently, there exists a unique function $x_\varepsilon(t) \in B_R$, $|\varepsilon| < \varepsilon_0$, $R > A\|f\|$, such that

$$\mathscr{A}\, x_\varepsilon(t) = x_\varepsilon(t)\,,$$

i.e. there exists a unique almost periodic solution of Equation (18) with its spectrum in $i\Lambda$ for each ε if $|\varepsilon| < \varepsilon_0$.

III. A MORE GENERAL EQUATION

Consider a little more general form of the linear differential equation with time lag, namely

(20) $$x'(t) = a\,x(t) + b\,x(t-\tau) + c\,x'(t-\tau) + f(t)\,,$$

where a, b, c are constant square matrices of n-th order, $f(t)$ and its derivatives $f'(t), f''(t)$ are almost periodic vector functions (column with n components) and τ is a constant positive time lag. Denote

$$\Phi(z) = z(E - ce^{-z\tau}) - a - be^{-z\tau}$$

and

$$\Delta(z) = \det \Phi(z)\,.$$

Suppose again that the distance

$$d = \text{dist}\,[i\Lambda_f, \sigma(\Delta(z))]$$

is positive. Further, denote by

$$\Omega(z) = E - ce^{-z\tau}$$

(i.e. the matrix coefficient at the variable z in $\Phi(z)$) and

$$\omega(z) = \det \Omega(z)\,.$$

Suppose

(21) $$\sigma(c) \cap K(0, 1) = \emptyset\,,$$

where $\sigma(c)$ is the spectrum of the matrix c. Under this assumption there exists such a positive number $\delta < \frac{1}{2}$ that no characteristic number μ of the matrix c satisfies the inequality

$$1 - 2\delta \leqq |\mu| \leqq 1 + 2\delta\,.$$

If $\mu_1, \ldots, \mu_q$ are all mutually different characteristic numbers of the matrix c and if $k_1, \ldots, k_q$ are their multiplicities, then

$$\omega(z) = \prod_{j=1}^{q} (1 - \mu_j e^{-z\tau})^{k_j}\,.$$

Consequently, we may choose $\alpha > 0$ small enough so that

$$1 - \delta \leqq |e^{-z\tau}| \leqq 1 + \delta$$

and at the same time

$$|e^{z\tau} - \mu_j| \geqq \delta\,, \quad j = 1, \ldots, q\,,$$

for all z with $|\mathrm{Re}\, z| \leqq \alpha$. Hence it follows that in the stripe $|\mathrm{Re}\, z| \leqq \alpha$ there exists a bounded inverse matrix $\Omega^{-1}(z)$. Thus the matrices $\Phi(z)$ and $\Phi^{-1}(z)$ are arbitrarily close to the matrices $z\, \Omega(z)$ and $z^{-1}\, \Omega^{-1}(z)$ for z sufficiently large (in absolute value), since

$$\Phi(z) = z\, \Omega(z) \left[E - z^{-1}\, \Omega^{-1}(z) \left(a + b e^{-z\tau}\right)\right]$$

for $z \neq 0$. Consequently, the following modification of Theorem 1 holds for Equation (20) (with regard to Note 1):

Theorem 1'. *Let* $\sigma(c) \cap K(0, 1) = \emptyset$ *and let the distance* $d = \mathrm{dist}\,[\mathrm{i}\Lambda_f, \sigma(\Delta(z))]$ *be positive. Then Equation* (20) *has a unique almost periodic solution* $x_f(t)$ *such that*

$$\Lambda_{x_f} \subset \Lambda_f\,.$$

This solution satisfies the estimate

$$\|x_f\| \leqq A \|\!|f|\!\|\,,$$

where

$$\|\!|f|\!\| = \max\left\{\|f\|, \|f'\|\right\}$$

and A *depends on* a, b, c, d, τ *only.*

Note 3. Consider several constant lime lags. In order to guarantee the existence and uniqueness of the required type of solution we should demand the corresponding matrix coefficient $\Omega(z)$ at z in the expression for the corresponding matrix $\Phi(z)$ to have analogous properties as in the case of the single time lag, i.e. a uniformly bounded inverse matrix $\Omega^{-1}(z)$ should exist in a stripe $P(\alpha)$, $\alpha > 0$, for all z sufficiently large (in absolute value). ■

If $|c| < 1$, then Condition (21) is automatically fulfilled and Theorem 1' holds with the estimate

$$\|x_f\| \leqq A \|f\|\,.$$

Consequently, the following modification of Theorem 2 is valid for quasilinear equation

(22)
$$x'(t) = a\, x(t) + b\, x(t - \tau) + c\, x(t - \tau) + f(t) + \\ + \varepsilon\, g(t, x(t), x(t - \tau), \varepsilon)\,,$$

where the vector function $g(t, u, v, \varepsilon)$ is defined and continuous on

$$J \times C^n \times C^n \times \bar{\varkappa}$$

analytic in the variables u, v, ε and almost periodic together with $\partial g/\partial t$ in the variable t uniformly to the others variables.

Theorem 2'. *Let* $|c| < 1$ *and the distance*

$$d = d[\mathrm{i}\Lambda, \sigma(\Delta(z))]$$

be positive, where

$$\Lambda = S(\Lambda_f \cup \Lambda_g).$$

Then there is a real positive number ε_0 *such that for each* $\varepsilon \in \bar{\varkappa}$, $|\varepsilon| < \varepsilon_0$, *Equation* (22) *has a unique almost periodic solution* $x_\varepsilon(t)$ *with its spectrum in* $\mathrm{i}\Lambda$.

References

[1] *Amerio L., Prouse G.:* Almost periodic functions and functional equations. N. Y., Van Nostrand Reihold Company, 1971.

[2] *Levitan B. M.:* Almost periodic functions. G.I.T.T.L., Moscow 1953 (in Russian).

[4] *Levitan B. M., Žikov V. V.:* Almost periodic function and differential equations. I.M.U. Moscow 1978 (in Russian).

[5] *Šimanov S. N.:* Almost periodic solution nonhomogeneous linear differential equations with time lag. Izv. VUZ SSSR, Matematika, No 4 (5), 1958 (in Russian).

Author's address: 120 00 Praha 2, Karlovo nám. 13 (Strojní fakulta ČVUT).