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AN APPROXIMATE METHOD FOR DETERMINATION OF EIGENVALUES AND EIGENVECTORS OF SELF-ADJOINT OPERATORS

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1. The method (1) for the determination of eigenvalues and eigenvectors of linear self-adjoint operator A is investigated. The error estimates are derived in the following two cases: (i) λ_1 is only an extreme value of the spectrum $\sigma(A)$ of A, (ii) λ_1 is an isolated point of $\sigma(A)$. Moreover, it is shown that the method (1) can be used for the determination of an arbitrary eigenvalue of A and the corresponding eigenvector.

Let X be a real Hilbert space, $A : X \to X$ a linear self-adjoint and positive operator on X. By positivity of A we mean that $\langle Au, u \rangle > 0$ for each $u \in X$, $u \neq 0$ and $\langle Au, u \rangle = 0$ implies $u = 0$. Let m, λ_1 be the exact spectral bounds of the spectrum $\sigma(A)$ of A. Denote by $\sigma_p(A)$, $\sigma_c(A)$ the point spectrum and the continuous spectrum, respectively. The symbol $\{E_\lambda\}$ stands for the spectral resolution of identity corresponding to the self-adjoint operator A. We shall deal with the following procedure

$$\mu_{n+1} = \langle Au_n, u_n \rangle \, . \, \|u_n\|^{-2} \, , \quad u_{n+1} = \mu_{n+1}^{-1} A u_n \tag{1}$$

for finding the eigenvalues and eigenvectors of A. In (1) it is assumed that the initial approximation $u_0 \in X$ is different from zero. Our hypotheses on A imply that $\mu_n > 0$ and $\mu_n \neq 0$ for each n. In the sequel we assume that (μ_n), (u_n) are defined by (1), and $w_n = u_n \|u_n\|^{-1}$ for each n. For the recent results concerning the procedure (1), its variants, relations and for the bibliography see [1]–[3]. We refer the reader for instance to [4]–[11] for further methods.

2. We start with the following

Theorem 1. *Let X be a real Hilbert space, $A : X \to X$ a linear self-adjoint and positive operator. Assume that the starting approximation u_0 of (1) is such that $E_\lambda u_0 \neq u_0$ for each $\lambda < \lambda_1$.*

Then $\|Aw_n\| \nearrow \lambda_1$ as $n \to \infty$. Moreover,

$$\|A^2 w_n\| \, . \, \|Aw_n\|^{-1} = \|Aw_{n+1}\| \leqq \mu_{n+1}^{-1} \lambda_1 c_n \|Aw_n\| \, ,$$

where

$$c_n = \|u_n\| \, . \, \|u_{n+1}\|^{-1} \leqq 1 \, , \quad (n = 0, 1, 2, \ldots) \, .$$

Proof. First of all, $\mu_{n+1} \leqq \|Aw_n\|$ and $\mu_n \leqq \mu_{n+1}$ for each n ([1], Lemma 1). Since A is positive and self-adjoint, $\|Au\|^2 \leqq \|A\| \langle Au, u\rangle$, $u \in X$. Indeed, assuming that $\|A\| = 1$, this inequality follows from

$$\|Au\|^2 = \langle Au, u\rangle - \{\langle A(u - Au), u - Au\rangle + \\ + \|Au\|^2 - \langle A^2u, Au\rangle\},$$

the fact that A is positive and the inequality

$$\langle A^2u, Au\rangle \leqq \|Au\|^2, \quad u \in X.$$

Furthermore, $\lambda_1 = \|A\|$ and

$$0 \leqq \|Aw_n\|^2 - \mu_{n+1}^2 = \|Aw_n\|^2 - \langle Aw_n, w_n\rangle^2 \leqq \\ \leqq \|A\| \langle Aw_n, w_n\rangle - \langle Aw_n, w_n\rangle^2 = \langle Aw_n, w_n\rangle (\lambda_1 - \langle Aw_n, w_n\rangle) \to 0$$

as $n \to \infty$ for $\mu_{n+1} = \langle Aw_n, w_n\rangle \nearrow \lambda_1$ by Theorem 1 [1]. Hence

$$0 \leqq \lambda_1^2 - \|Aw_n\|^2 \leqq (\lambda_1^2 - \mu_n^2) + |\mu_n^2 - \|Aw_n\|^2| \to 0$$

as $n \to \infty$. We shall prove that $(\|Aw_n\|)_{=n1}^{\infty}$ is monotone. It follows from (1) that $w_{n+1} = \mu_{n+1}^{-1} c_n Aw_n$, where $c_n = \|u_n\|/\|u_{n+1}\| \leqq 1$ for each $n \geqq 0$. Hence

$$\|Aw_{n+1}\| = c_n\mu_{n+1}^{-1}\|A^2w_n\| = \mu_{n+1}^{-1} \frac{\|u_n\| \cdot \|u_{n+1}\|}{\|u_{n+1}\|^2} \|A^2w_n\| .$$

By our hypotheses $u_n \neq 0$, $Au_n \neq 0$, $A^2u_n \neq 0$, $(n = 0, 1, 2, \ldots)$. In view of (1) we obtain

$$\|Aw_{n+1}\| = \mu_{n+1}^{-1}\mu_{n+1}^2 \frac{\|u_n\| \cdot \|u_{n+1}\|}{\langle A^2u_n, u_n\rangle} \|A^2w_n\| \geqq \\ \geqq \mu_{n+1} \frac{\|u_n\| \cdot \|u_{n+1}\|}{\|A^2u_n\| \cdot \|u_n\|} \|A^2w_n\| = \mu_{n+1} \frac{\|u_{n+1}\|}{\|u_n\|} = \|Aw_n\| .$$

Hence $\|Aw_n\| \leqq \|Aw_{n+1}\|$ for each n and we have that $\|Aw_n\| \nearrow \lambda_1$ as $n \to \infty$.

Put $z_n = A^2w_n$, then

$$\|Aw_{n+1}\|^2 = \mu_{n+1}^{-2}c_n^2\langle A^2w_n, z_n\rangle = \\ = \mu_{n+1}^{-2}c_n^2 \int_m^{\lambda_1} \lambda^2 \, d\langle E_\lambda w_n, z_n\rangle \leqq \left(\frac{\lambda_1}{\mu_{n+1}}\right)^2 \cdot c_n^2 \int_m^{\lambda_1} d\langle E_\lambda w_n, z_n\rangle = \\ = \left(\frac{\lambda_1}{\mu_{n+1}}\right)^2 c_n^2\langle w_n, z_n\rangle = \left(\frac{\lambda_1}{\mu_{n+1}}\right)^2 c_n^2 \|Aw_n\|^2 .$$

On the other hand,

$$\|Aw_{n+1}\| = \mu_{n+1}^{-1}\|u_n\| \cdot \|u_{n+1}\|^{-1} \cdot \|A^2w_n\| \leqq$$
$$= \mu_{n+1}^{-1} \frac{\|u_n\|}{\mu_{n+1}^{-1}\|Au_n\|} \|A^2w_n\| = \|A^2w_n\| \cdot \|Aw_n\|^{-1},$$

for each n $(n = 0, 1, 2, \ldots)$, which completes the proof.

Remark 1. In addition to the assumptions of Theorem 1 assume that A is positive definite (i.e. $m > 0$). Then

$$mc_n\mu_{n+1}^{-1}\|Aw_n\| \leqq \|Aw_{n+1}\| \leqq c_n\lambda_1\mu_{n+1}^{-1}\|Aw_n\|$$

for each n.

Theorem 2. *Let X be a real Hilbert space, $A : X \to X$ a linear positive and self-adjoint operator on X. Assume that λ_1 is an eigenvalue of A and that the initial approximation u_0 of the procedure (u_n) is not orthogonal to* $\ker(A - \lambda_1 I)$.

Then $\|Aw_n\| \nearrow \lambda_1$ as $n \to \infty$.

Proof. Use Theorem 3 [2] and the arguments of the proof of Theorem 1.

Theorem 3. *In addition to the assumptions of Theorem* 1 *suppose that (w_n) contains a subsequence converging weakly to an element $w \in X$, $w \neq 0$.*

Then λ_1 is an eigenvalue of A and w is the corresponding eigenvector of A.

Proof. According to Theorem 1 [1] $\mu_n \nearrow \lambda_1$ and by Theorem 2 we have that $\|Aw_n\| \nearrow \lambda_1$. Hence

$$\|Aw_n\|^2 - \langle Aw_n, w_n\rangle^2 = \|Aw_n - \mu_{n+1}w_n\|^2 \to 0$$

as $n \to \infty$. Without loss of generality one can assume that $w_n \to w$ weakly, where $w \in X$, $w \neq 0$. Therefore $Aw_n - \mu_{n+1}w_n \to Aw - \lambda_1 w$ weakly and $Aw = \lambda_1 w$, which concludes the proof.

Corollary. *In addition to the assumptions of Theorem* 1 *assume that the sequence (w_n) contains a subsequence converging to an element $w \in X$.*

Then λ_1 is an eigenvalues of A and w is the corresponding eigenvector of A.

Theorem 4. *Let X be a real Hilbertspace, $B : X \to X$, $C : X \to X$ linear self-adjoint operators on X. Assume that λ_0 is an eigenvalue of B, $e_0 \in \ker(B - \lambda_0 I)$, $\|e_0\| = 1$ and that $\lambda_0 \notin \sigma(C)$. Let λ^* be an eigenvalue of C such that λ^* is nearest to λ_0 from the both sides. If* $\dim \ker (C - \lambda^* I) = 1$ *and* $e_0 \notin \ker(C - \lambda^* I)^{\perp}$, *then*

$$|\lambda^* - \lambda_0| \leqq \|(C - \lambda_0 I)\, w_n\| \leqq \|(C - \lambda_0 I)\, w_{n-1}\| \leqq \|B - C\|$$

for each n $(n = 1, 2, \ldots)$, where w_n is defined by (1) with $A = \alpha I - (C - \lambda_0 I)^2$, $u_0 = e_0$ and α is an arbitrary constant such that $\alpha > \|(C - \lambda_0 I)^2\|$.

Proof. Since the operators B, C are linear self-adjoint and defined on X, B, C are both bounded by the closed-graph theorem. Put $A = \alpha I - (C - \lambda_0 I)^2$, where $\alpha > \|(C - \lambda_0 I)^2\|$. Then A is linear self-adjoint bounded and positive definite with the greatest eigenvalue $\lambda_1 = \alpha - (\lambda^* - \lambda_0)^2$. Put $C_1 = C - \lambda_0 I$, $\lambda = \lambda^* - \lambda_0$, $C_2 = C - \lambda^* I$. We show that $\ker (C_1^2 - \lambda^2 I) = \ker C_2$. Suppose that $u \in \ker C_2$; this condition is equivalent to $C_1 u = \lambda u$. But $C_1^2 u = C_1(\lambda u) = \lambda^2 u$. Hence $u \in \ker (C_1^2 - \lambda^2 I)$ and $\ker C_2 \subset \ker (C_1^2 - \lambda^2 I)$. Assume that there exists an element $\tilde{u} \in X$ such that $\tilde{u} \in \ker (C_1^2 - \lambda^2 I)$ and $\tilde{u} \notin \ker C_2$, i.e. $C_1 \tilde{u} \neq \lambda \tilde{u}$, which contradicts the fact that $\tilde{u} \in \ker (C_1^2 - \lambda_1^2 I)$. Hence $\ker (C_1^2 - \lambda_1^2 I) = \ker C_2$ and this implies $\ker (A - \lambda_1 I) = \ker (C - \lambda^* I)$. According to our hypothesis $\langle u_0, w \rangle \neq 0$ for each $w \in \ker (C - \lambda^* I)$. Hence $\langle u_0, w \rangle \neq 0$ for each $w \in \ker (A - \lambda_1 I)$ and therefore $u_0 \notin \ker (A - \lambda_1 I)^{\perp}$. Thus all the assumptions of Theorem 3 [2] are satisfied. According to this theorem $\mu_{n+1} = \langle A w_n, w_n \rangle \nearrow \lambda_1 = \alpha - (\lambda^* - \lambda_0)^2$, where $w_n = u_n / \|u_n\|$ and (u_n) is defined by $u_{n+1} = \mu_{n+1}^{-1} A u_n$. Hence $\langle (C - \lambda_0 I) w_n, w_n \rangle \searrow (\lambda^* - \lambda_0)^2$ as $n \to \infty$. This conclusion implies that

$$\begin{aligned} |\lambda^* - \lambda_0| &\leqq \langle (C - \lambda_0 I)^2 w_n, w_n \rangle^{1/2} = \|(C - \lambda_0 I) w_n\| \leqq \\ &\leqq \langle (C - \lambda_0 I) w_{n-1}, w_{n-1} \rangle^{1/2} = \|(C - \lambda_0 I) w_{n-1}\| \leqq \\ &\leqq \ldots \leqq \|(C - \lambda_0 I) w_0\| = \|(C - \lambda_0 I) e_0 = \| \\ &= \|(C - B) e_0\| \leqq \|C - B\| \, \|e_0\| = \|C - B\| , \end{aligned}$$

because $e_0 \in \ker (B - \lambda_0 I)$, $\|e_0\| = 1$ and $w_0 = e_0$. The theorem is proved.

Remark 2. The estimate $|\lambda^* - \lambda_0| \leqq \|C - B\|$ for completely continuous linear operators C, B was derived by H. Weyl [10]. This estimate can be obtained in a more general setting also in the following way. Compare also [7], [11].

Proposition 1. *Let X be a real Hilbert space, $B : X \to X$, $C : X \to X$ linear self-adjoint operators. Assume that $\sigma_c(C) = \emptyset$ and that $\lambda_0 \notin \sigma(C)$ is an eigenvalue of B. If λ^* is an eigenvalue of C such that λ^* is nearest to λ_0 from the both sides, then $|\lambda^* - \lambda_0| \leqq \|B - C\|$.*

Proof. First of all, B, C are bounded by the closed-graph theorem. Since $\lambda_0 \notin \sigma(C)$, there exists a bounded linear operator $R_{\lambda_0} = (C - \lambda_0 I)^{-1}$ and R_{λ_0} is defined on the whole space X. Moreover, R_{λ_0} is self-adjoint. Since the function $f(\lambda) = 1/|\lambda - \lambda_0|$ is continuous on the compact set $\sigma(C)$, the spectral mapping theorem implies that

$$\|R_{\lambda_0}\| = \max_{\lambda \in \sigma(C)} \frac{1}{|\lambda - \lambda_0|} = \max_{\lambda \in \sigma_p(C)} \frac{1}{|\lambda - \lambda_0|} = \frac{1}{|\lambda^* - \lambda_0|} .$$

Let $e_0 \in \ker(B - \lambda_0 I)$, $\|e_0\| = 1$. As $\lambda_0 \notin \sigma(C)$, we have $\|(C - \lambda_0 I) e_0\| > 0$ and

$$|\lambda^* - \lambda_0| = \frac{1}{\|R_{\lambda_0}\|} \frac{\|(C - \lambda_0 I) e_0\|}{\|(C - \lambda_0 I) e_0\|},$$

$$1 = \|e_0\| = \|R_{\lambda_0}(C - \lambda_0 I) e_0\| \leqq \|R_{\lambda_0}\| \cdot \|(C - \lambda_0 I) e_0\| .$$

From the above relations we conclude that $|\lambda^* - \lambda_0| \leqq \|(C - \lambda_0) e_0\|$. Since $e_0 \in \ker(B - \lambda_0 I)$ and $\|e_0\| = 1$, we have

$$|\lambda^* - \lambda_0| \leqq \|Ce_0 - Be_0\| \leqq \|C - B\| \|e_0\| = \|C - B\|$$

as required.

Theorem 5. *Let X be a real Hilbert space, $B : X \to X$ a linear self-adjoint operator, λ^* an eigenvalue of B. Let λ_0 be a real number, $\lambda_0 \notin \sigma(B)$, and λ^* be nearest to λ_0 from the both sides. Suppose that the initial approximation u_0 of (u_n), where (u_n) is defined by (1) with $A = \alpha I - (B - \lambda_0 I)^2$, $\alpha > \|(B - \lambda_0 I)^2\|$ is not orthogonal to* $\ker(B - \lambda^* I)$.

Then $\|(B - \lambda_0 I) w_n\| \searrow |\lambda^* - \lambda_0|$, $\|u_n - Ne_0\| \to 0$, $\|w_n - e_0\| \to 0$ *as* $n \to \infty$, *where*

$$N = \sup_n \|u_n\| , \quad e_0 \in \ker(B - \lambda^* I) , \quad \|e_0\| = 1 .$$

Proof. Put $A = \alpha I - (B - \lambda_0 I)^2$, where α is an arbitrary positive number such that $\alpha > \|(B - \lambda_0 I)^2\|$. Then A is a linear positive definite self-adjoint operator with the greatest eigenvalue $\lambda_1 = \alpha - (\lambda^* - \lambda_0)^2$, while $\ker(A - \lambda_1 I) = \ker(B - \lambda^* I)$. Since $u_0 \notin \ker(B - \lambda^* I)^\perp$, we have $u_0 \notin \ker(A - \lambda_1 I)^\perp$. According to Theorem 3 [2] we have $\langle Aw_n, w_n \rangle \nearrow \lambda_1$ and $\|u_n - Ne_0\| \to 0$ as $n \to \infty$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $N = \sup_n \|u_n\| < \infty$. Hence $\|(B - \lambda_0 I) w_n\| \searrow |\lambda^* - \lambda_0|$, $\|w_n - e_0\| \to 0$ as $n \to \infty$, while $e_0 \in \ker(B - \lambda^* I)$, $\|e_0\| = 1$.

Indeed, since (u_n) is bounded monotone increasing ([1]), $\|u_n\| \to N$ as $n \to \infty$ and $u_0 \neq 0$, we have the

$$\|w_n - e_0\| = \frac{\|u_n - \|u_n\| e_0\|}{\|u_n\|} \leqq$$

$$\leqq \|u_0\|^{-1} (\|u_n - Ne_0\| + \|Ne_0 - \|u_n\| e_0\|) \to 0$$

as $n \to \infty$, which concludes the proof.

Theorem 6. *In addition to the assumptions of Theorem 2 assume that λ_1 is an isolated point of the spectrum $\sigma(A)$ of A (i.e. there exists a constant $M > 0$ such that $\sigma(A) - \{\lambda_1\} \subset [m, M]$).*

Then there exists an integer n_0 such that

$$\mu_{n+1} \leqq \lambda_1 \leqq \mu_{n+1} + (\|Aw_n\|^2 - \mu_{n+1}^2)^{1/2}$$

holds for each $n \geqq n_0$.

Proof. Since λ_1 is an isolated point of $\sigma(A)$, then λ_1 is an eigenvalue of A. By Theorem 3 [2] and Theorem 2 we have that $\mu_n \nearrow \lambda_1$ and $\|Aw_n\|^2 - \mu_{n+1}^2 \to 0$ as $n \to \infty$. Furthermore,

$$\|Aw_n\|^2 = \langle A^2 w_n, w_n \rangle = \int_m^{\lambda_1} \lambda^2 \, \mathrm{d}\langle E_\lambda w_n, w_n \rangle ,$$

$$\mu_{n+1} = \langle Aw_n, w_n \rangle = \int_m^{\lambda_1} \lambda \, \mathrm{d}\langle E_\lambda w_n, w_n \rangle ,$$

$$\|w_n\|^2 = \int_m^{\lambda_1} \mathrm{d}\langle E_\lambda w_n, w_n \rangle .$$

Hence

$$\|Aw_n\|^2 - \mu_{n+1}^2 = \|Aw_n - \mu_{n+1} w_n\|^2 = \int_m^{\lambda_1} \lambda^2 \, \mathrm{d}\|E_\lambda w_n\|^2 -$$

$$-2\mu_{n+1} \int_m^{\lambda_1} \lambda \, \mathrm{d}\|E_\lambda w_n\|^2 + \mu_{1+n}^2 \int_m^{\lambda_1} \mathrm{d}\|E_\lambda w_n\|^2 = \int_m^{\lambda_1} (\lambda - \mu_{n+1})^2 \, \mathrm{d}\|E_\lambda w_n\|^2 .$$

Since λ_1 is an isolated point of $\sigma(A)$ and $\mu_n \nearrow \lambda_1$, there exists an integer n_0 such that $\mu_n \in [\frac{1}{2}(M + \lambda_1), \lambda_1]$ for each $n \geqq n_0$. Hence we have for each fixed $n \geqq n_0$

$$\|Aw_n\|^2 - \mu_{n+1}^2 = \int_m^{\lambda_1} (\lambda - \mu_{n+1})^2 \, \mathrm{d}\langle E_\lambda w_n, w_n \rangle \geqq$$

$$\geqq \inf_{\lambda \in \sigma(A)} |\lambda - \mu_{n+1}|^2 \int_m^{\lambda_1} \mathrm{d}\langle E_\lambda w_n, w_n \rangle =$$

$$= \inf_{\lambda \in \sigma(A)} |\lambda - \mu_{n+1}|^2 \geqq$$

$$\geqq \inf \{(\lambda_1 - \mu_{n+1})^2, |M - \mu_{n+1}|^2\} = (\lambda_1 - \mu_{n+1})^2 .$$

The desired inequalities follow at once from the fact that $\mu_n \nearrow \lambda_1$ and the last relation. The theorem is proved.

Proposition 2. *Let X be a real Hilbert space, $A : X \to X$ a linear positive definite and self-adjoint operator. Assume that the starting approximation u_0 of (1) is such that $E_\lambda u_0 \neq u_0$ for each $\lambda < \lambda_1$. If ε is such that $0 < \varepsilon < \lambda_1 - m$, then*

$$\lambda_1 \geqq m^{3/2} \frac{\|E_{\lambda_1 - \varepsilon} u_{n-1}\|}{\langle Au_n, u_n \rangle^{1/2}} , \quad n = 1, 2, \ldots .$$

Moreover, there exists an integer n_0 such that

$$\lambda_1 < a_0^{-2} m^{-2n} \langle Au_n, u_n \rangle \prod_{k=1}^{n} \mu_k^2$$

holds for each $n \geqq n_0$, where $a_0^2 = \|u_0\|^2 - \|E_{\lambda_1 - \varepsilon} u_0\|^2 > 0$.

Proof. Assume that $0 < \varepsilon < \lambda_1 - m$. Then according to our hypothesis $E_{\lambda_1-\varepsilon}u_0 \neq u_0$. Applying the projector $E_{\lambda_1-\varepsilon}$ to the equality (1) we obtain that

$$\|E_{\lambda_1-\varepsilon}u_n\|^2 = \mu_n^{-2}\|E_{\lambda_1-\varepsilon}Au_{n-1}\|^2 = \mu_n^{-2}\|AE_{\lambda_1-\varepsilon}u_{n-1}\|^2 .$$

Since

$$\|AE_{\lambda_1-\varepsilon}u_{n-1}\|^2 = \langle A^2 E_{\lambda_1-\varepsilon}u_{n-1}, u_{n-1}\rangle =$$

$$= \int_m^{\lambda_1} \lambda^2 \, \mathrm{d}\langle E_\lambda E_{\lambda_1-\varepsilon}u_{n-1}, u_{n-1}\rangle =$$

$$= \int_m^{\lambda_1-\varepsilon} \lambda^2 \, \mathrm{d}\|E_\lambda u_{n-1}\|^2 \geqq m^2 \|E_{\lambda_1-\varepsilon}u_{n-1}\|^2 ,$$

we obtain that

$$\|E_{\lambda_1-\varepsilon}u_n\| \geqq \frac{m}{\lambda_1}\|E_{\lambda_1-\varepsilon}u_{n-1}\| . \tag{2}$$

On the other hand,

$$\|E_{\lambda_1-\varepsilon}u_n\| \leqq \|u_n\| \leqq m^{-1/2}\langle Au_n, u_n\rangle^{1/2} . \tag{3}$$

The relations (2), (3) immediately yield the first assertion.

We prove the second estimate in our theorem. Let $R(E_{\lambda_1-\varepsilon})$ be the range of $E_{\lambda_1-\varepsilon}$, where $0 < \varepsilon < \lambda_1 - m$. Since $E_{\lambda_1-\varepsilon}$ is a continuous projector, $R(E_{\lambda_1-\varepsilon})$ is a closed subspace of X. Denote by $R(E_{\lambda_1-\varepsilon})^\perp$ the orthogonal complement to $R(E_{\lambda_1-\varepsilon})$. Put $P_\varepsilon = I - E_{\lambda_1-\varepsilon}$, i.e. $P_\varepsilon = E_{\lambda_1} - E_{\lambda_1-\varepsilon}$, $w_n = u_n/\|u_n\|$. We shall show that

$$\lambda_1 \leqq \langle Au_n, u_n\rangle \, \|P_\varepsilon u_n\|^{-2} + \varepsilon \tag{4}$$

for sufficiently large n and a fixed ε satisfying the inequality $0 < \varepsilon < \lambda_1 - m$. Each element w_n of the sequence (w_n) can be uniquely expressed in the form $w_n = a_n^{(\varepsilon)}g_n + b_n^{(\varepsilon)}\tilde{z}_n$, where $g_n \in R(E_{\lambda_1-\varepsilon})^\perp$, $\tilde{z}_n \in R(E_{\lambda_1-\varepsilon})$ and $\|g_n\| = \|\tilde{z}_n\| = 1$, $(a_n^{(\varepsilon)})^2 + (b_n^{(\varepsilon)})^2 = 1$. Then $P_\varepsilon w_n = a_n^{(\varepsilon)}g_n$ and $\|P_\varepsilon w_n\|^2 = (a_n^{(\varepsilon)})^2$ and

$$\lambda_1 \geqq \mu_n = \langle Aw_n, w_n\rangle = (a_n^{(\varepsilon)})^2 \langle Ag_n, g_n\rangle + (b_n^{(\varepsilon)})^2 \langle A\tilde{z}_n, \tilde{z}_n\rangle \geqq$$

$$\geqq (a_n^{(\varepsilon)})^2 \langle Ag_n, g_n\rangle \geqq \|P_\varepsilon w_n\|^2 (\lambda_1 - \varepsilon) .$$

(See the proof of Theorem 6 [2].) Moreover, it has been shown [2] that $\lim_{n\to\infty} (b_n^{(\varepsilon)})^2 = 0$ for each fixed ε, $0 < \varepsilon < \lambda_1 - m$. Therefore $(a_n^{(\varepsilon)})^2 = \|P_\varepsilon w_n\|^2 \to 1$ as $n \to \infty$

and therefore there exists an integer n_0 such that $\|P_\varepsilon w_n\| > 0$ for each $n \geqq n_0$. Hence (4) is valid for each $n \geqq n_0$.

Now we estimate $\|P_\varepsilon u_n\|$. By the definition of P_ε we have

$$\|P_\varepsilon u_n\|^2 = \|u_n - E_{\lambda_1-\varepsilon} u_n\|^2 = \|u_n\|^2 - \|E_{\lambda_1-\varepsilon} u_n\|^2 .$$

By (1) we get $\|u_n\|^2 = \mu_n^{-2}\|Au_{n-1}\|^2$ and

$$\|E_{\lambda_1-\varepsilon} u_n\|^2 = \mu_n^{-2}\|AE_{\lambda_1-\varepsilon} u_{n-1}\|^2 . \tag{5}$$

Now

$$\|AE_{\lambda_1-\varepsilon} u_{n-1}\|^2 = \langle A^2 E_{\lambda_1-\varepsilon} u_{n-1}, u_{n-1}\rangle = \tag{6}$$

$$= \int_m^{\lambda_1-\varepsilon} \lambda^2 \, \mathrm{d}\langle E_\lambda u_{n-1}, u_{n-1}\rangle .$$

Hence

$$\|u_n\|^2 - \|E_{\lambda_1-\varepsilon} u_n\|^2 = \mu_n^{-2}(\langle A^2 u_{n-1}, u_{n-1}\rangle - \langle A^2 E_{\lambda_1-\varepsilon} u_{n-1}, u_{n-1}\rangle) =$$

$$= \mu_n^{-2}\int_{\lambda_1-\varepsilon}^{\lambda_1} \lambda^2 \, \mathrm{d}\langle E_\lambda u_{n-1}, u_{n-1}\rangle \geqq$$

$$\geqq (\lambda_1 - \varepsilon)^2 \mu_n^{-2}(\|u_{n-1}\|^2 - \|E_{\lambda_1-\varepsilon} u_{n-1}\|^2) > m^2 \mu_n^{-2}\|P_\varepsilon u_{n-1}\|^2 .$$

Therefore

$$\|P_\varepsilon u_n\|^2 > m^2\mu_n^{-2}\|P_\varepsilon u_{n-1}\|^2 > \ldots > m^{2n}\mu_n^{-2}\mu_{n-1}^{-2} \ldots \mu_1^{-2}\|P_\varepsilon u_0\|^2 =$$

$$= m^{2n}\|u_0 - E_{\lambda_1-\varepsilon} u_0\|^2 \prod_{k=1}^{n} \mu_k^{-2} = m^{2n}(\|u_0\|^2 - \|E_{\lambda_1-\varepsilon} u_0\|^2) \prod_{k=1}^{n} \mu_k^{-2} > 0 ,$$

for $\|(I - E_{\lambda_1-\varepsilon}) u_0\| > 0$. This inequality together with the relation (4) give our estimate.

Remark 3. Let us point out that the asymptotic estimates corresponding to that of Proposition 2 are not efficient. Under the conditions of Proposition 2 the estimate

$$m^{1+1/2n} \frac{\|E_{\lambda_1-\varepsilon} u_0\|^{1/n}}{\langle Au_n, u_n\rangle} \leqq \lambda_1$$

is valid for each n ($n = 0, 1, 2, \ldots$). Indeed, (2) implies that

$$\|E_{\lambda_1-\varepsilon} u_n\| \geqq \frac{m}{\lambda_1} \|E_{\lambda_1-\varepsilon} u_{n-1}\| \geqq \ldots \geqq \left(\frac{m}{\lambda_1}\right)^n \|E_{\lambda_1-\varepsilon} u_0\| .$$

Hence the last inequalities and (3) give the desired result. Moreover, there exists an integer n_0 such that $\|E_{\lambda_1-\varepsilon} u_{n+1}\| \leqq \|E_{\lambda_1-\varepsilon} u_n\|$ for each $n \geqq n_0$. Indeed, from (6) we have that

$$\|AE_{\lambda_1-\varepsilon} u_{n+1}\| \leqq (\lambda_1 - \varepsilon) \|E_{\lambda_1-\varepsilon} u_n\| , \quad n = 0, 1, 2, \ldots .$$

According to (5),

$$\|E_{\lambda_1-\varepsilon}u_{n+1}\| \leqq \frac{\lambda_1 - \varepsilon}{\mu_{n+1}} \|E_{\lambda_1-\varepsilon}u_n\|$$

for each n $(n = 0, 1, 2, \ldots)$. By Theorem 1 [1], $\mu_n \nearrow \lambda_1$. Therefore there exists an integer n_0 such that $(\lambda_1 - \varepsilon)\mu_n^{-1} \leqq 1$ for each $n \geqq n_0$. Hence $\|E_{\lambda 1-\varepsilon}u_{n+1}\| \leqq \leqq \|E_{\lambda_1-\varepsilon}u_n\|$ for each $n \geqq n_0$.

To establish further estimates we use Lemma 1 [2] which reads if the initial approximation u_0 of (1) is not orthogonal to $\ker(A - \lambda_1 I) \neq (0)$, then each element u_n of the sequence (u_n) defined by (1) is of the form $u_n = a_n e_0 + z_n$, where $z_n \in \in \ker(A - \lambda_1 I)^\perp$ and $a_n > 0$ for each n $(n = 0, 1, 2, \ldots)$, $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$.

Theorem 7. *Let X be a real Hilbert space, $A : X \to X$ a linear positive and self-adjoint operator such that λ_1 is an isolated point of $\sigma(A)$ (i.e. there exists a constant $M > 0$ such that $\sigma(A) - \{\lambda_1\} \subset [m, M]$). Assume that the starting approximation u_0 of the procedure (1) is not orthogonal to* $\ker(A - \lambda_1 I)$.

Then

$$(8)\quad (\lambda_1 - M)\, m\mu_{n+1}^{-2} \|z_{n-1}\|^2 \leqq \lambda_1 - \mu_{n+1} \leqq \alpha_n^2\alpha_{n-1}^2 \ldots \alpha_0^2(\lambda_1 - m)\, \|z_0\|^2 \|u_0\|^{-2},$$

$$(9)\qquad \|w_{n+1} - \langle w_{n+1}, e_0\rangle e_0\| < \alpha_n\alpha_{n-1} \ldots \alpha_0 \|w_0 - \langle w_0, e_0\rangle e_0\|$$

for each n, where

$$\alpha_n = \left[1 - \frac{a_n^2}{\|u_n\|^2}\left(1 - \frac{M}{\lambda_1}\right)\right]^{1/2},$$

$0 < \alpha_n < \alpha_{n-1} < \ldots < \alpha_0 < 1$, a_n, z_n are elements from the representation of u_n, $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$ and $\alpha_n \leqq [1 - (1 - (M/\mu_n)^2)(1 - M/\lambda_1)]^{1/2}$ for sufficiently large n.

Proof. First of all we derive (9). Since λ_1 is an isolated point of $\sigma(A)$, λ_1 is an eigenvalue of A. According to Lemma 1 [2] each element u_n defined by (1) can be represented in the form $u_n = a_n e_0 + z_n$, where $\|e_0\| = 1$, $e_0 \in \ker(A - \lambda_1 I)$, $z_n \in \in \ker(A - \lambda_1 I)^\perp$ and the constants a_n are positive. Put

$$v_n = z_n/\|u_n\|, \quad c_n = a_n/\|u_n\|, \quad u_{n+1}^{(1)} = u_{n+1}/\|u_n\|.$$

Then $w_n = c_n e_0 + v_n$, $\mu_{n+1} = \langle Aw_n, w_n\rangle$ and

$$u_{n+1}^{(1)} = \mu_{n+1}^{-1} A w_n = \mu_{n+1}^{-1}(\lambda_1 c_n e_0 + A v_n).$$

Set $\beta_{n+1} = \mu_{n+1}^{-1}\lambda_1$, $h_{n+1} = \mu_{n+1}^{-1} A v_n$. Then $u_{n+1}^{(1)} = \beta_{n+1} c_n e_0 + h_{n+1}$ and $a_{n+1} = = \beta_{n+1} c_n \|u_n\|$, $z_{n+1} = \|u_n\|\, h_{n+1}$. Since $c_n^2 = 1 - \|v_n\|^2$, we have

$$\mu_{n+1} = c_n^2 \lambda_1 + \langle A v_n, v_n\rangle = \lambda_1 - r_n,$$

where $r_n = \langle(\lambda_1 I - A) v_n, v_n\rangle$, $(n = 0, 1, 2, \ldots)$. Hence $\beta_{n+1} = \lambda_1(\lambda_1 - r_n)^{-1}$, $h_{n+1} = (\lambda_1 - r_n)^{-1} Av_n$ for each n $(n = 0, 1, 2, \ldots)$. We shall estimate the quantity

$$(10) \qquad J = \frac{\|h_{n+1}\|^2}{\|u_{n+1}^{(1)}\|^2 \|v_n\|^2} = 1 - \frac{\|u_{n+1}^{(1)}\|^2 \|v_n\|^2 - \|h_{n+1}\|^2}{\|u_{n+1}^{(1)}\|^2 \|v_n\|^2},$$

where $\|u_{n+1}^{(1)}\|^2 = \beta_{n+1}^2 c_n^2 + \|h_{n+1}\|^2$. Using again $c_n^2 = 1 - \|v_n\|^2$ and simple calculations, we get that

$$(11) \qquad J = 1 - \frac{b_{n+1} c_n^2}{(\beta_{n+1}^2 - b_{n+1}) \|v_n\|^2},$$

where $b_{n+1} = \beta_{n+1}^2 \|v_n\|^2 - \|h_{n+1}\|^2$. On the other hand, $\lambda_1 = \|A\|$, $\|Av_n\|^2 \leqq \leqq \lambda_1 \langle Av_n, v_n\rangle$ imply that

$$b_{n+1} = \frac{1}{(\lambda_1 - r_n)^2} (\lambda_1^2 \|v_n\|^2 - \|Av_n\|^2) \geqq$$

$$\geqq \frac{\lambda_1}{(\lambda_1 - r_n)^2} \langle(\lambda_1 I - A) v_n, v_n\rangle = \frac{\lambda_1 r_n}{(\lambda_1 - r_n)^2}.$$

By our hypothesis λ_1 is an isolated point of $\sigma(A)$. Therefore the segment (M, λ_1) belongs to the resolvent set of A and thus the spectral family $\{E_\lambda\}$ is constant on (M, λ_1). Hence

$$r_n = \langle(\lambda_1 I - A) v_n, v_n\rangle = \int_m^{\lambda_1} (\lambda_1 - \lambda) \, \mathrm{d}\langle E_\lambda v_n, v_n\rangle =$$

$$= \int_m^M (\lambda_1 - \lambda) \, \mathrm{d}\langle E_\lambda v_n, v_n\rangle \geqq (\lambda_1 - M) \|v_n\|^2.$$

Furthermore, $\beta_{n+1}^2 - b_{n+1} \leqq \lambda_1(\lambda_1 - r_n)^{-1}$ and hence

$$(12) \qquad \frac{b_{n+1}}{\beta_{n+1}^2 - b_{n+1}} \geqq r_n \frac{1}{\lambda_1 - r_n} > \frac{r_n}{\lambda_1} \geqq \frac{\lambda_1 - M}{\lambda_1} \|v_n\|^2.$$

Hence according to (10), (11), (12) and

$$(13) \qquad \frac{\|z_{n+1}\|}{\|u_{n+1}\|} < \alpha_n \frac{\|z_n\|}{\|u_n\|},$$

$$\frac{\|z_{n+1}\|}{\|u_{n+1}\|} = \|w_{n+1} - \langle w_{n+1}, e_0\rangle e_0\|,$$

we obtain (9) with $\alpha_k = [1 - (a_k \|u_k\|^{-1})^2 (1 - M\lambda_1^{-1})]^{1/2}$ for each k $(k = 0, 1, 2, \ldots, n)$. Clearly, $0 < \alpha_k < 1$ for $\|z_k\| \leqq \|u_k\|$ and $M < \lambda_1$. We have that $\|z_{k+1}\| / \|u_{k+1}\| < \|z_k\| / \|u_k\|$ and moreover, $c_k^2 + \|v_k\|^2 = c_{k+1}^2 + \|v_{k+1}\|^2 = 1$ for each k. Hence $a_{k+1}^2 / \|u_{k+1}\|^2 > a_k^2 / \|u_k\|^2$ and therefore $\alpha_{k+1} < \alpha_k < 1$ for each k, for $a_k^2 = \|u_k\|^2 - \|z_k\|^2$.

We shall prove (8). Again, one can express each element u_n of (u_n) in the form $u_n = a_n e_0 + z_n$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $z_n \in \ker(A - \lambda_1 I)^{\perp}$ and $a_n > 0$. We have

$$(14)\qquad \lambda_1 - \mu_{n+1} = (\lambda_1 \|u_n\|^2 - \langle A u_n, u_n \rangle) \|u_n\|^{-2} = \\ = (\lambda_1 \|z_n\|^2 - \langle A z_n, z_n \rangle) \|u_n\|^{-2} = \langle (\lambda_1 I - A) z_n, z_n \rangle \|u_n\|^{-2} .$$

Now

$$(15)\qquad \langle (\lambda_1 I - A) z_n, z_n \rangle \geqq (\lambda_1 - M) \|z_n\|^2 .$$

Moreover, the orthogonal projection of $u_{n+1} = \mu_{n+1}^{-1} A u_n$ onto $\ker(A - \lambda_1 I)^{\perp}$ is equal to z_{n+1}, where $z_{n+1} = \mu_{n+1}^{-1} A z_n$. Then

$$(16)\qquad \|z_{n+1}\|^2 = \mu_{n+1}^{-2} \langle A^2 z_n, z_n \rangle = \\ = \mu_{n+1}^{-2} \int_m^{\lambda_1} \lambda^2 \, \mathrm{d}\langle E_\lambda z_n, z_n \rangle \geqq \left(\frac{m}{\mu_{n+1}}\right)^2 \int_m^{\lambda_1} \mathrm{d}\langle E_\lambda z_n, z_n \rangle = \left(\frac{m}{\mu_{n+1}}\right)^2 \|z_n\|^2 .$$

Now (14), (15), (16) give the first estimate in (8). Since $\sigma(\lambda_1 I - A)$ lies on the segment $[\lambda_1 - M, \lambda_1 - m]$ we have that

$$\lambda_1 - \mu_{n+1} \leqq (\lambda_1 - m) \|z_n\|^2 \, . \, \|u_n\|^{-2} .$$

Using (13) we obtain the other part of (8). The estimate of α_n follows at once from the expression for α_n and the inequality $a_n \geqq (1 - (M/\mu_n)^2) \|u_n\|^2$, which holds for sufficiently large n [3]. The theorem is proved.

Remark 4. The estimates (8), (9) show that the convergence of μ_n to λ_1 and the so called directional convergence of w_n to e_0 are better than the rate of convergence of the geometric sequence with quotient $\alpha_0 < 1$. Let us point out that under more general conditions on A and X, quite different estimates for (1) have been obtained by Marek [5] and Petryshyn [6].

Now assume that $A : X \to X$ is self-adjoint and positive definite. Put

$$u_n^{(\alpha)} = \int_m^{\lambda_1} \lambda^{-\alpha/2} \, \mathrm{d}E_\lambda u_n = A^{-\alpha/2} u_n$$

($\alpha = 0, \pm 1, \pm 2, \ldots$) and substitute $A^{\alpha/2} u_n^{(\alpha)}$ for u_n in (1). Then we obtain the procedures

$$(17)\qquad \mu_{n+1}^{(\alpha)} = \langle A^{\alpha+1} u_n^{(\alpha)}, u_u^{(\alpha)} \rangle \, . \, \|A^{\alpha/2} u_n^{(\alpha)}\|^{-2} , \\ u_{n+1}^{(\alpha)} = (\mu_{n+1}^{(\alpha)})^{-1} A u_n^{(\alpha)} , \\ (u_0^{(\alpha)} \neq 0, \; u_n^{(0)} = u_n, \; \mu_{n+1}^{(0)} = \mu_{n+1}) ,$$

where $n = 0, 1, 2, \ldots$; $\alpha = 0, \pm 1, \pm 2, \ldots$. For these procedures one can derive results similar to those of Theorems 1, 2, 3 [2], [1].

Put

$$u_n^{(\alpha)} = \frac{u_n^{(\alpha)}}{\|n_n^{(\alpha)}\|} ,$$

where $\alpha = 0, \pm 1, \pm 2, \ldots$, $n = 0, 1, 2, \ldots$, $u_n^{(0)} = u_n$, $w_n = w_n^{(0)}$, $u_n^{(\alpha)} = A^{-\alpha/2}u_n$ and (u_n) is defined by (1). Then

$$\text{(18)} \qquad \langle Aw_n^{(\alpha)}, w_n^{(\alpha)} \rangle = \frac{\langle A^{1-\alpha}u_n, u_n \rangle}{\langle A^{-\alpha}u_n, u_n \rangle},$$

$$(\alpha = 0, \pm 1, \pm 2, \ldots, \; n = 0, 1, 2, \ldots).$$

Theorem 9. *Let X be a real Hilbert space, $A : X \to X$ a linear positive definite and self-adjoint operator on X. Assume that λ_1 (not necessarily an isolated point of $\sigma(A)$ with finite multiplicity) is an eigenvalue of A and that the starting approximation $u_0^{(\alpha)}$ of (17) is not orthogonal to* $\ker(A - \lambda_1 I)$.

Then $\langle Aw_n^{(\alpha)}, w_n^{(\alpha)} \rangle \to \lambda_1$. If λ_1 is an isolated point of $\sigma(A)$, then $\|w_n^{(\alpha)} - e_0\| \to 0$ as $n \to \infty$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $\alpha = 0, \pm 1, \pm 2, \ldots$.

Proof. The first part of our theorem follows at once from (18) and Theorem 3 [2]. Furthermore, by Theorem 3 [2] we have that $\|u_n - Ne_0\| \to 0$ as $n \to \infty$, where $N = \sup_n \|u_n\| < +\infty$.

Since

$$A^{-\alpha/2}e_0 = \int_m^{\lambda_1} \lambda^{-\alpha/2} \, dE_\lambda e_0 = \lambda_1^{-\alpha/2} e_0$$

and $A^{-\alpha/2}$ is bounded, we obtain

$$\|u_n^{(\alpha)} - N\lambda_1^{-\alpha/2}e_0\| = \|A^{-\alpha/2}u_n - NA^{-\alpha/2}e_0\| \leqq$$
$$\leqq \|A^{-\alpha/2}\| \, \|u_n - Ne_0\| \to 0$$

as $n \to \infty$. By Lemma 1, 2 [1] the sequence $(\|u_n\|)_{n=1}^{\infty}$ is bounded monotone increasing with $u_0 \neq 0$. Hence $(\|A^{-\alpha/2}u_n\|)_{n=1}^{\infty}$ is bounded and $\|A^{-\alpha/2}u_n\| \geqq m^{-\alpha/2}\|u_n\| \geqq$ $\geqq m^{-\alpha/2}\|u_0\| > 0$ for each n. From $u_n^{(\alpha)} = A^{-\alpha/2}u_n \to NA^{-\alpha/2}e_0 = \lambda_1^{-\alpha/2}Ne_0$, $n \to \infty$ we get that $\|u_n^{(\alpha)}\| \to N\lambda_1^{-\alpha/2}$ and $\|u_n^{(\alpha)}\| \, e_0 \to \lambda_1^{-\alpha/2}Ne_0$ as $n \to \infty$.

Since

$$\|w_n^{(\alpha)} - e_0\| = \left\| \frac{u_n^{(\alpha)}}{\|u_n^{(\alpha)}\|} - e_0 \right\| = \frac{\|u_n^{(\alpha)} - \|u_n^{(\alpha)}\| e_0\|}{\|u_n^{(\alpha)}\|} \leqq$$
$$\leqq m^{\alpha/2}\|u_0\|^{-1}(\|u_n^{(\alpha)} - N\lambda_1^{-\alpha/2}e_0\| + \|N\lambda_1^{-\alpha/2}e_0 - \|u_n^{(\alpha)}\| \, e_0\|),$$

$\|w_n^{(\alpha)} - e_0\| \to 0$ as desired.

We shall show that the rate of convergence of the sequences $(\langle Aw_n^{(\alpha)}, w_n^{(\alpha)} \rangle)_{n=1}^{\infty}$ $(\alpha = -1, -2, \ldots)$ is not worse than the convergence of $(\langle Aw_n, w_n \rangle)_{n=1}^{\infty}$. Indeed, the generalized Schwarz inequality gives

$$\langle A^{-\alpha}u_n, u_n \rangle^2 = \langle AA^{-\alpha/2}u_n, A^{-(\alpha/2)-1}u_m \rangle^2 \leqq$$
$$\leqq \langle AA^{-\alpha/2}u_n, A^{-\alpha/2}u_n \rangle \langle AA^{-(\alpha/2)-1}u_n, A^{-\alpha/2-1}u_n \rangle =$$
$$= \langle A^{1-\alpha}u_n, u_n \rangle \langle A^{-\alpha-1}u_n, u_n \rangle .$$

Dividing this inequality by $\langle A^{-\alpha}u_n, u_n\rangle \langle A^{-\alpha-1}u_n, u_n\rangle$, we obtain that

$$\langle Aw_n^{(\alpha+1)}, w_n^{(\alpha+1)}\rangle \leqq \langle Aw_n^{(\alpha)}, w_n^{(\alpha)}\rangle$$

for each n and α ($\alpha = 0, \pm 1, \pm 2, \ldots$). Hence

$$\lambda_1 \geqq \ldots \geqq \langle Aw_n^{(-2)}, w_n^{(-2)}\rangle \geqq \langle Aw_n^{(-1)}, w_n^{(-1)}\rangle \geqq$$

$$\geqq \langle Aw_n, w_n\rangle \geqq \langle Aw_n^{(1)}, w_n^{(1)}\rangle \geqq \langle Aw_n^{(2)}, w_n^{(2)}\rangle \geqq \ldots .$$

Let us remark that the assumption of the positive definiteness of A in Theorem 8 is not essential. Indeed, if $A : X \to X$ is in general a self-adjoint operator on X, then $B = aI \pm A$, where a is a constant such that $a > \|A\|$, is positive definite and self-adjoint on X. Using the above results one can obtain the extreme value λ_1 of $\sigma(A)$ and the eigenvectors corresponding to λ_1 of course provided λ_1 is an eigenvalue of A). If in general A is only linear and bounded, then the derived theorems can be applied to the operator $T = A^*A$, which is self-adjoint and nonnegative, i.e. $T \geqq 0$.

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References

[1] *J. Kolomý:* Approximate determination of eigenvalues and eigenvectors of self-adjoint operators. Ann. Pol. Math. *38* (1980), 153—158.

[2] *J. Kolomý:* On determination of eigenvalues and eigenvectors of self-adjoint operators Apl. mat. *26* (1981), 161—170.

[3] *J. Kolomý:* Determination of eigenvalues and eigenvectors of self-adjoint operators. Mathematica *22* (1980), 53—58.

[4] *М. А. Красносельский* и другие: Приближенное решение операторных уравнений. Изд. Наука, Москва, 1969.

[5] *I. Marek:* Iterations of linear bounded operators in nonself-adjoint eigenvalue problems and Kellog's iteration process. Czech. Math. J. *12* (1962), 536—554.

[6] *W. V. Petryshyn:* On the eigenvalue problem $T(u) - \lambda S(u) = 0$ with unbounded and symmetric operators T and S. Phil. Trans. Royal Soc. London Ser. A, Math. Phys. Sci., No 1130, Vol. *262* (1968), 413—458.

[7] *V. Pták - J. Zemánek:* Continuité Lipschitzienne du spectre comme function d'un opérateur normal. Comment. Math. Univ. Carolinae *17* (1976), 507—512.

[8] *В. П. Пугачев:* О двух приёмах приближенного вычисления собственных значений и собственных векторов. Докл. акад. СССР, *110* (1956), 334—337.

[9] *Б. П. Пугачев:* Исследование одного метода приближенного вычисления собственных чисел и векторов. Труды сем. по функц. анал. Воронеж, Т. *4* (1960), 81—97.

[10] *F. Riesz, B. Sz.-Nagy:* Le▪ons d'analyse fonctionnelle, Ac. Sci. de Hongrif, Budapest, 1953.

[11] *Wang Jin-ru:* A gradient method for finding the eigenvalues and eigenvectors of a self-adjoint operator. Acta Math. Sinica *13* (1963), 23—28 (Chinese Math. Acta *4* (1963), 24—30).

[12] *K. Yosida:* Functional Analysis. Springer-Verlag, Berlin, 1965.

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