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ON A FUNDAMENTAL THEOREM OF THE LAPLACE TRANSFORM THEORY

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In this note we deal with an important theorem which plays a decisive role in Widder's theory of representability for the Laplace transform. This theorem was discovered by D. V. Widder and is to be found under the name "general representation theorem" in his classical book: The Laplace transform, 1946. Its main idea is, roughly speaking, that Laplace transforms of the so called Post-Widder approximations constructed for a given function and defined on the positive halfaxis, tend to this function if it has certain properties which are always fulfilled for Laplace images.

In the sequel we present a proof of a theorem of this type not only for numerically valued but also for vector-valued functions. Our assumptions are a little stronger than those of Widder but they are sufficient in all necessary cases.

The present proof is new, simple, elementary and, moreover, independent of the inversion theorem. Its technique differs considerably from that of Widder and is similar to that used by the author under more special assumptions in § 4 of his paper: Linear differential equations in Banach spaces, Rozpravy Československé akademie věd, řada mat. a přír. věd, *85* (1975), No 6.

We shall denote: (1) $\mathbb{R}$ – the real number field, (2) $(0, \infty)$ – the set of all positive numbers, (3) $M_1 \to M_2$ – the set of all mappings of the whole set M_1 into the set M_2.

Further, E will denote an arbitrary Banach space over $\mathbb{R}$ with the norm $\|\cdot\|$. We shall need only the most elementary properties of Banach spaces and of functions with values in a Banach space.

1. Lemma. $\frac{n^n}{n!} e^{-n} \leqq \frac{1}{\sqrt[4]{n}}$ *for every* $n \in 1\{, 2, \ldots\}$.

Proof. We have clearly

$$(1) \quad \frac{(n+1)^{n+1}}{(n+1)!} e^{-(n+1)} \left(\frac{n^n}{n!} e^{-n}\right)^{-1} = e^{-1}\left(1 + \frac{1}{n}\right)^n \text{ for every } n \in \{1, 2, \ldots\}.$$

Now we prove that

(2) $e^{-1}\left(1+\frac{1}{n}\right)^n \leqq e^{-1/4n}$ for every $n \in \{1, 2, \ldots\}$.

Indeed, the inequality (2) is equivalent to $e^{-(1/n)+(1/4n^2)}(1 + 1/n) \leqq 1$ for every $n \in \{1, 2, \ldots\}$, and to verify this last inequality it suffices to show that $e^{-\alpha+\alpha^2/4}(1 + \alpha) \leqq 1$ for every $0 \leqq \alpha \leqq 1$. But the function $e^{-\alpha+\alpha^2/4}(1 + \alpha)$ has the value 1 at the point 0 and is clearly nonincreasing for $0 \leqq \alpha \leqq 1$.

It follows from (1) and (2) that

(3) $\frac{(n+1)^{n+1}}{(n+1)!} e^{-(n+1)} \leqq e^{-1/4n} \frac{n^n}{n!} e^{-n}$ for every $n \in \{1, 2, \ldots\}$.

Further we prove

(4) $e^{-1/4n} \frac{1}{\sqrt[4]{n}} \leqq \frac{1}{\sqrt[4]{(n+1)}}$ for every $n \in \{1, 2, \ldots\}$.

Indeed, (4) is equivalent to $\sqrt[4]{[(n+1)/n]} \leqq e^{1/4n}$, i.e. to $1 + (1/n) \leqq e^{1/n}$ for every $n \in \{1, 2, \ldots\}$. But the last inequality is clearly true.

The desired inequality can now be proved by induction on n.

Indeed, it is true for $n = 1$. If it is true for a fixed $n \in \{1, 2, \ldots\}$, we obtain from (3) and (4) that

$$\frac{(n+1)^{n+1}}{(n+1)!} e^{-(n+1)} \leqq e^{-1/4n} \frac{n^n}{n!} e^{-n} \leqq e^{-1/4n} \frac{1}{\sqrt[4]{n}} \leqq \frac{1}{\sqrt[4]{(n+1)}}$$

which proves the induction step.

The proof is complete.

2. Proposition. *Let* $F \in (0, \infty) \to E$. *If*

(α) *the function* F *is twice continuously differentiable on* $(0, \infty)$,

(β) $\int_0^1 \xi^{j+1} \|F^{(j)}(\xi)\| \, d\xi < \infty, \quad \int_1^\infty \xi^{j-1} \|F^{(j)}(\xi)\| \, d\xi < \infty$ *for every* $j \in \{0, 1, 2\}$,

then

(a) *the function* $e^{-\lambda t} t^{q-1} F(1/t)$ *is integrable over* $(0, \infty)$ *for every* $\lambda > 0$ *and* $q \in \{0, 1, \ldots\}$,

(b) $\frac{((q+1)\lambda)^{q+1}}{q!} \int_0^\infty e^{-(q+1)\lambda\tau} \tau^{q-1} F\left(\frac{1}{\tau}\right) d\tau \to_{q\to\infty} \lambda F(\lambda)$ *for every* $\lambda > 0$.

Proof. Let us first observe that the assumption (β) is equivalent to

$$(1)\quad \int_0^{\mu} \frac{1}{\tau^{j+1}} \left\| F^{(j)}\left(\frac{1}{\tau}\right) \right\| \mathrm{d}\tau < \infty\,, \quad \int_{\mu}^{\infty} \frac{1}{\tau^{j+3}} \left\| F^{(j)}\left(\frac{1}{\tau}\right) \right\| \mathrm{d}\tau < \infty$$

for every $\mu > 0$ *and* $j \in \{0, 1, 2\}$.

We see easily that (1) implies (a).

Let us now fix a $\lambda > 0$.

For the sake of brevity we denote

$$(2)\quad K = 2\left(\int_0^{\lambda^{-1}} \frac{1}{\varrho} \left\| F\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho + \int_{\lambda^{-1}}^{\infty} \frac{1}{\varrho^3} \left\| F\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho\right) +$$
$$+ 4\left(\int_0^{\lambda^{-1}} \frac{1}{\varrho^2} \left\| F'\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho + \int_{\lambda^{-1}}^{\infty} \frac{1}{\varrho^4} \left\| F'\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho\right) +$$
$$+ \int_0^{\lambda^{-1}} \frac{1}{\varrho^3} \left\| F''\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho + \int_{\lambda^{-1}}^{\infty} \frac{1}{\varrho^5} \left\| F''\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho +$$
$$+ (1 + \lambda^2) \left\|F(\lambda)\right\| + (\lambda + \lambda^3) \left\|F'(\lambda)\right\|.$$

It follows from the assumption (α) and from (1) and (2) that

$$(3)\quad \left\| \frac{1}{\tau} F\left(\frac{1}{\tau}\right) - \lambda F(\lambda) \right\| = \left\| \frac{1}{\tau} F\left(\frac{1}{\tau}\right) - \frac{1}{\lambda^{-1}} F\left(\frac{1}{\lambda^{-1}}\right) \right\| =$$
$$= \left\| \int_{\lambda^{-1}}^{\tau} \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\sigma} F\left(\frac{1}{\sigma}\right)\right) \mathrm{d}\sigma \right\| = \left\| - \int_{\lambda^{-1}}^{\tau} \left(\frac{1}{\sigma^2} F\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma^3} F'\left(\frac{1}{\sigma}\right)\right) \mathrm{d}\sigma \right\| =$$
$$= \left\| - \int_{\lambda^{-1}}^{\tau} \left[\int_{\lambda^{-1}}^{\sigma} \frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\frac{1}{\varrho^2} F\left(\frac{1}{\varrho}\right) + \frac{1}{\varrho^3} F'\left(\frac{1}{\varrho}\right)\right) \mathrm{d}\varrho + \lambda^2 F(\lambda) + \lambda^3 F'(\lambda)\right] \mathrm{d}\sigma \right\| =$$
$$= \left\| - \int_{\lambda^{-1}}^{\tau} \left[- \int_{\lambda^{-1}}^{\sigma} \left(\frac{2}{\varrho^3} F\left(\frac{1}{\varrho}\right) + \frac{4}{\varrho^4} F'\left(\frac{1}{\varrho}\right) + \frac{1}{\varrho^5} F''\left(\frac{1}{\varrho}\right)\right) \mathrm{d}\varrho + \right.\right.$$
$$\left.\left. + \lambda^2 F(\lambda) + \lambda^3 F'(\lambda)\right] \mathrm{d}\sigma \right\| \leqq$$
$$\leqq (\tau - \lambda^{-1}) \left[2 \int_{\lambda^{-1}}^{\infty} \frac{1}{\varrho^3} \left\| F\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho + 4 \int_{\lambda^{-1}}^{\infty} \frac{1}{\varrho^4} \left\| F'\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho + \right.$$
$$\left. + \int_{\lambda^{-1}}^{\infty} \frac{1}{\varrho^5} \left\| F''\left(\frac{1}{\varrho}\right) \right\| \mathrm{d}\varrho + \lambda^2 \left\|F(\lambda)\right\| + \lambda^3 \left\|F'(\lambda)\right\| \right] \leqq$$
$$\leqq K\left(\tau - \frac{1}{\lambda}\right) \text{ for every } \tau > \frac{1}{\lambda}\,,$$

$$(4)\quad \left\|\frac{1}{\tau}F\left(\frac{1}{\tau}\right) - \lambda F(\lambda)\right\| = \left\|\frac{1}{\tau}F\left(\frac{1}{\tau}\right) - \frac{1}{\lambda^{-1}}F\left(\frac{1}{\lambda^{-1}}\right)\right\| =$$

$$= \left\|\int_\tau^{\lambda^{-1}} \frac{d}{d\sigma}\left(\frac{1}{\sigma}F\left(\frac{1}{\sigma}\right)\right) d\sigma\right\| = \left\|-\int_\tau^{\lambda^{-1}} \left(\frac{1}{\sigma^2}F\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma^3}F'\left(\frac{1}{\sigma}\right)\right) d\sigma\right\| =$$

$$= \left\|-\int_\tau^{\lambda^{-1}}\left[-\int_\sigma^{\lambda^{-1}} \frac{d}{d\varrho}\left(\frac{1}{\varrho^2}F\left(\frac{1}{\varrho}\right) + \frac{1}{\varrho^3}F'\left(\frac{1}{\varrho}\right)\right) d\varrho + \lambda^2 F(\lambda) + \lambda^3 F'(\lambda)\right] d\sigma\right\| =$$

$$= \left\|-\int_\tau^{\lambda^{-1}}\left[\int_\sigma^{\lambda^{-1}} \left(\frac{2}{\varrho^3}F\left(\frac{1}{\varrho}\right) + \frac{4}{\varrho^4}F'\left(\frac{1}{\varrho}\right) + \frac{1}{\varrho^5}F''\left(\frac{1}{\varrho}\right)\right) d\varrho +\right.\right.$$

$$\left.\left. + \lambda^2 F(\lambda) + \lambda^3 F'(\lambda)\right] d\sigma\right\| \leqq \int_\tau^{\lambda^{-1}}\left[\int_\sigma^{\lambda^{-1}} \left(\frac{2}{\varrho^3}\left\|F\left(\frac{1}{\varrho}\right)\right\| + \frac{4}{\varrho^4}\left\|F'\left(\frac{1}{\varrho}\right)\right\| +\right.\right.$$

$$\left.\left. + \frac{1}{\varrho^5}\left\|F''\left(\frac{1}{\varrho}\right)\right\|\right) d\varrho + \lambda^2\|F(\lambda)\| + \lambda^3\|F'(\lambda)\|\right] d\sigma \leqq$$

$$\leqq \int_\tau^{\lambda^{-1}} \frac{1}{\sigma^2}\left[\int_\sigma^{\lambda^{-1}} \left(\frac{2}{\varrho}\left\|F\left(\frac{1}{\varrho}\right)\right\| + \frac{4}{\varrho^2}\left\|F'\left(\frac{1}{\varrho}\right)\right\| + \frac{1}{\varrho^3}\left\|F''\left(\frac{1}{\varrho}\right)\right\| d\varrho +\right.\right.$$

$$\left.\left. + \sigma^2\lambda^2\|F(\lambda)\| + \sigma^2\lambda^3\|F'(\lambda)\|\right] d\sigma \leqq \int_\tau^{\lambda^{-1}} \frac{1}{\sigma^2}\left[2\int_0^{\lambda^{-1}} \frac{1}{\varrho}\left\|F\left(\frac{1}{\varrho}\right)\right\| d\varrho +\right.\right.$$

$$\left. + 4\int_0^{\lambda^{-1}} \frac{1}{\varrho^2}\left\|F'\left(\frac{1}{\varrho}\right)\right\| d\varrho + \int_0^{\lambda^{-1}} \frac{1}{\varrho^3}\left\|F''\left(\frac{1}{\varrho}\right)\right\| d\varrho + \|F(\lambda)\| + \|\lambda F'(\lambda)\|\right] d\sigma \leqq$$

$$\leqq K\int_\tau^{\lambda^{-1}} \frac{1}{\sigma^2}\, d\sigma = K\left(\frac{1}{\tau} - \lambda\right) \text{ for every } 0 < \tau < \frac{1}{\lambda}.$$

We need the following auxiliary identities which are well known or easy to prove:

$$(5)\quad \int_0^\infty e^{-\lambda\tau}\tau^q\, d\tau = \frac{q!}{\lambda^{q+1}} \text{ for every } q \in \{0, 1, \ldots\},$$

$$(6)\quad e^{-(q+1)\lambda\tau}\tau^q\left(\tau - \frac{1}{\lambda}\right) = -\frac{1}{(q+1)\lambda}\frac{d}{d\tau}\left[e^{-(q+1)\lambda\tau}\tau^{q+1}\right]$$

for every $\tau > 0$ and $q \in \{0, 1, \ldots\}$,

$$(7)\quad e^{-q\lambda\tau}\tau^q\left(\frac{1}{\tau} - \lambda\right) = \frac{1}{q}\frac{d}{d\tau}\left[e^{-q\lambda\tau}\tau^q\right] \text{ for every } q \in \{1, 2, \ldots\}.$$

Now, we obtain from (3), (4), (5), (6) and (7) that

$$(8)\quad \left\|\frac{((q+1)\lambda)^{q+1}}{q!}\int_0^\infty e^{-(q+1)\lambda\tau}\tau^{q-1} F\left(\frac{1}{\tau}\right) d\tau - \lambda F(\lambda)\right\| =$$

$$= \left\| \frac{((q+1)\lambda)^{q+1}}{q!} \int_0^\infty e^{-(q+1)\lambda\tau} \tau^q \left(\frac{1}{\tau} F\left(\frac{1}{\tau}\right) - \lambda F(\lambda) \right) d\tau \right\| \leqq$$

$$\leqq K \frac{((q+1)\lambda)^{q+1}}{q!} \left[\int_{\lambda^{-1}}^\infty e^{-(q+1)\lambda\tau} \tau^q \left(\tau - \frac{1}{\lambda} \right) d\tau + \int_0^{\lambda^{-1}} e^{-(q+1)\lambda\tau} \tau^q \left(\frac{1}{\tau} - \lambda \right) d\tau \right] \leqq$$

$$\leqq K \frac{((q+1)\lambda)^{q+1}}{q!} \left[\int_{\lambda^{-1}}^\infty e^{-(q+1)\lambda\tau} \tau^q \left(\tau - \frac{1}{\lambda} \right) d\tau + \int_0^{\lambda^{-1}} e^{-q\lambda\tau} \tau^q \left(\frac{1}{\tau} - \lambda \right) d\tau \right] =$$

$$= K \frac{((q+1)\lambda)^{q+1}}{q!} \left[- \frac{1}{(q+1)\lambda} \int_{\lambda^{-1}}^\infty \frac{d}{d\tau} \left(e^{-(q+1)\lambda\tau} \tau^{q+1} \right) d\tau + \right.$$

$$\left. + \frac{1}{q} \int_0^{\lambda^{-1}} \frac{d}{d\tau} \left(e^{-q\lambda\tau} \tau^q \right) d\tau \right] = K \frac{((q+1)\lambda)^{q+1}}{q!} \left[\frac{1}{(q+1)\lambda} \frac{e^{-(q+1)}}{\lambda^{q+1}} + \frac{1}{q} \frac{e^{-q}}{\lambda^q} \right] =$$

$$= K \frac{(q+1)^{q+1} e^{-(q+1)}}{(q+1)!} \left[\frac{1}{\lambda} + \frac{q+1}{q} \lambda e \right] \leqq$$

$$\leqq K \left[\frac{1}{\lambda} + 2\lambda e \right] \frac{(q+1)^{q+1} e^{-(q+1)}}{(q+1)!} \quad \text{for every } q \in \{1, 2, \ldots\}.$$

Using Lemma 1 we see easily that (8) implies (b).
The proof is complete.

3. Lemma. *Let* $F \in (0, \infty) \to E$. *If*

(α) *the function* F *is infinitely differentiable on* $(0, \infty)$,

(β) $\int_0^1 \xi^{p+1} \|F^{(p)}(\xi)\| \, d\xi < \infty$, $\int_1^\infty \xi^{p-1} \|F^{(p)}(\xi)\| \, d\xi < \infty$ *for every* $p \in \{0, 1, \ldots\}$.

then

(a) *the functions* $e^{-\mu t} t^{-(p+1)} F^{(p)}(1/t)$, $e^{-\mu t} t^{-p} F^{(p)}(1/t)$ *and* $e^{-\mu t} t^{-(p+2)} F^{(p+1)}(1/t)$ *are integrable over* $(0, \infty)$ *for every* $\mu > 0$ *and* $p \in \{0, 1, \ldots\}$,

(b) $$\int_0^\infty e^{-\mu\tau} \tau^{-(p+2)} F^{(p+1)}\left(\frac{1}{\tau}\right) d\tau = -\mu \int_0^\infty e^{-\mu\tau} \tau^{-p} F^{(p)}\left(\frac{1}{\tau}\right) d\tau -$$

$$- p \int_0^\infty e^{-\mu\tau} \tau^{-(p+1)} F^{(p)}\left(\frac{1}{\tau}\right) d\tau \text{ for every } \mu > 0 \text{ and } p \in \{0, 1, \ldots\}.$$

Proof. By a simple substitution we see from (α) and (β) that

(1) $$\int_0^1 \tau^{-(p+1)} \left\| F^{(p)}\left(\frac{1}{\tau}\right) \right\| d\tau < \infty \quad \text{for every } p \in \{0, 1, \ldots\},$$

(2) $$\int_1^\infty \tau^{-(p+3)} \left\| F^{(p)}\left(\frac{1}{\tau}\right) \right\| d\tau < \infty \quad \text{for every } p \in \{0, 1, \ldots\}.$$

Now the statement (a) is an easy consequence of (1) and (2).

To prove (b) we need some auxiliary observations.

First we prove that

(3) for every $p \in \{0, 1, \ldots\}$, there exists a sequence α_k, $k \in \{1, 2, \ldots\}$, such that $0 < \alpha_k < 1$ for every $k \in \{1, 2, \ldots\}$, $\alpha_k \to 0$ $(k \to \infty)$ and $\alpha_k^{-p} F^{(p)}(1/\alpha_k) \to 0$ $(k \to \infty)$.

Indeed, suppose that (3) does not hold. Consequently, there exist $0 < \delta \leqq 1$ and $\varepsilon > 0$ so that $\tau^{-p}\|F^{(p)}(1/\tau)\| \geqq \varepsilon$ for every $0 < \tau < \delta$. This implies that $\int_0^\delta \tau^{-(p+1)}\|F^{(p)}(1/\tau)\|\, \mathrm{d}\tau = \infty$ which contradicts (1).

Analogously we prove that

(4) for every $p \in \{0, 1, \ldots\}$, there exists a sequence β_k, $k \in \{1, 2, \ldots\}$, such that $\beta_k > 1$ for every $k \in \{1, 2, \ldots\}$, $\beta_k \to \infty$ $(k \to \infty)$ and $\beta_k^{-p-3} F^{(p)}(1/\beta_k) \to 0$ $(k \to \infty)$.

If the property (4) did not hold, then there would exist $\delta \geqq 1$ and $\varepsilon > 0$ so that $\tau^{-(p+3)}\|F^{(p)}(1/\tau)\| \geqq \varepsilon$ for every $\tau > \delta$. Consequently $\int_1^\infty \tau^{-(p+3)}\|F^{(p)}(1/\tau)\, \mathrm{d}\tau = \infty$, which would contradict (2).

As an immediate consequence of (3) and (4) we have

(5) for every $p \in \{0, 1, \ldots\}$, there exist two sequences α_k, β_k, $k \in \{1, 2, \ldots\}$ such that $0 < \alpha_k < 1$, $\beta_k > 1$ for every $k \in \{1, 2, \ldots\}$, $\alpha_k \to 0$, $\beta_k \to \infty$ $(k \to \infty)$ and $\mathrm{e}^{-\lambda\alpha_k}\alpha_k^{-p} F^{(p)}(1/\alpha_k) \to 0$ and $\mathrm{e}^{-\lambda\beta_k}\beta_k^{-p} F^{(p)}(1/\beta_k) \to 0$ $(k \to \infty)$ for every $\lambda > 0$.

On the other hand, we see immediately from (α) that

$$(6)\quad \frac{\mathrm{d}}{\mathrm{d}t} F^{(p)}\left(\frac{1}{t}\right) = -\frac{1}{t^2} F^{(p+1)}\left(\frac{1}{t}\right) \text{ for every } t > 0 \text{ and } p \in \{0, 1, \ldots\}.$$

Using (5) and (6) and integrating by parts we get for $\lambda > 0$ and $p \in \{0, 1, \ldots\}$

$$\int_0^\infty \mathrm{e}^{-\mu\tau}\tau^{-(p+2)} F^{(p+1)}\left(\frac{1}{\tau}\right) \mathrm{d}\tau = -\int_0^\infty \mathrm{e}^{-\mu\tau}\tau^{-p} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(F^{(p)}\left(\frac{1}{\tau}\right)\right) \mathrm{d}\tau =$$

$$= -\lim_{k\to\infty} \int_{\alpha_k}^{\beta_k} \mathrm{e}^{-\mu\tau}\tau^{-p} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} F^{(p)}\left(\frac{1}{\tau}\right)\right) \mathrm{d}\tau =$$

$$= -\lim_{k\to\infty}\left[\mathrm{e}^{-\mu\beta_k}\beta_k^{-p} F^{(p)}\left(\frac{1}{\beta_k}\right) - \mathrm{e}^{-\mu\alpha_k}\alpha_k^{-p} F^{(p)}\left(\frac{1}{\alpha_k}\right) - \int_{\alpha_k}^{\beta_k} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{-\mu\tau}\tau^{-p}\right) F^{(p)}\left(\frac{1}{\tau}\right) \mathrm{d}\tau\right] =$$

$$= -\lim_{k\to\infty}\left[\mu \int_{\alpha_k}^{\beta_k} \mathrm{e}^{-\mu\tau}\tau^{-p} F^{(p)}\left(\frac{1}{\tau}\right) \mathrm{d}\tau + p \int_{\alpha_k}^{\beta_k} \mathrm{e}^{-\mu\tau}\tau^{-(p+1)} F^{(p)}\left(\frac{1}{\tau}\right) \mathrm{d}\tau\right] =$$

$$= -\mu \int_0^\infty \mathrm{e}^{-\mu\tau}\tau^{-p} F^{(p)}\left(\frac{1}{\tau}\right) \mathrm{d}\tau - p \int_0^\infty \mathrm{e}^{-\mu\tau}\tau^{-(p+1)} F^{(p)}\left(\frac{1}{\tau}\right) \mathrm{d}\tau$$

which is the desired result (b).

4. Proposition. *Let $F \in (0, \infty) \to E$. If the assumptions* (α) *and* (β) *of Lemma* 1 *are fulfilled, then*

(a) *the functions* $e^{-\mu t} t^{-p-1} F^{(p)}(1/t)$ *and* $e^{-\mu t} t^{p-1} F(1/t)$ *are integrable over* $(0, \infty)$ *for every* $\mu > 0$ *and* $p \in \{0, 1, \ldots\}$,

(b) $$\int_0^\infty e^{-\mu\tau} \tau^{-(p+1)} F^{(p)}\left(\frac{1}{\tau}\right) d\tau = (-1)^p \mu^p \int_0^\infty e^{-\mu\tau} \tau^{p-1} F\left(\frac{1}{\tau}\right) d\tau$$

for every $\mu > 0$ *and* $p \in \{0, 1, \ldots\}$.

Proof. The statement (a) is an immediate consequence of the corresponding statement of Lemma 3.

To prove the statement (b), we conclude first from (a) that

(1) $$(-1)^{p+1} \int_0^\infty e^{-\mu\tau} \tau^p F\left(\frac{1}{\tau}\right) d\tau = (-1)^p \frac{d}{d\mu}\left[\int_0^\infty e^{-\mu\tau} \tau^{p-1} F\left(\frac{1}{\tau}\right) d\tau\right]$$

for every $\mu > 0$ and $p \in \{0, 1, \ldots\}$.

Now we proceed by induction on p.

The case $p = 0$ is evident.

Consequently, we now suppose that (b) holds for a fixed $p \in \{0, 1, \ldots\}$ and all $\mu > 0$ and we prove it for $p + 1$ and all $\mu > 0$.

Under these circumstances, we obtain from (b) with regard to (1) that

(2) $$(-1)^{p+1} \mu^{p+1} \int_0^\infty e^{-\mu\tau} \tau^p F\left(\frac{1}{\tau}\right) d\tau = (-1)^p \mu^{p+1} \frac{d}{d\mu}\left[\int_0^\infty e^{-\mu\tau} \tau^{p-1} F\left(\frac{1}{\tau}\right) d\tau\right] =$$

$$= \mu^{p+1} \frac{d}{d\mu}\left[\mu^{-p}(-1)^{-p} \mu^p \int_0^\infty e^{-\mu\tau} \tau^{p-1} F\left(\frac{1}{\tau}\right) d\tau\right] =$$

$$= \mu^{p+1} \frac{d}{d\mu}\left[\mu^{-p} \int_0^\infty e^{-\mu\tau} \tau^{-(p+1)} F^{(p)}\left(\frac{1}{\tau}\right) d\tau\right] =$$

$$= -\mu \int_0^\infty e^{-\mu\tau} \tau^{-p} F^{(p)}\left(\frac{1}{\tau}\right) d\tau - p \int_0^\infty e^{-\mu\tau} \tau^{-(p+1)} F^{(p)}\left(\frac{1}{\tau}\right) d\tau \text{ for every } \mu > 0.$$

Using Lemma 3 we get from (2) that

$$(-1)^{p+1} \mu^{p+1} \int_0^\infty e^{-\mu\tau} \tau^p F\left(\frac{1}{\tau}\right) d\tau = \int_0^\infty e^{-\mu\tau} \tau^{-(p+2)} F^{(p+1)}\left(\frac{1}{\tau}\right) d\tau$$

for every $\mu > 0$ which proves the induction step.

The proof is complete.

5. Theorem. *Let $F \in (0, \infty) \to E$. If*

(α) *the function F is infinitely differentiable on* $(0, \infty)$, (β) $\int_0^1 \xi^{p+1}\|F^{(p)}(\xi)\|\,\mathrm{d}\xi < \infty\,,\quad \int_1^\infty \xi^{p-1}\|F^{(p)}(\xi)\|\,\mathrm{d}\xi < \infty$ *for every* $p \in \{0, 1, \ldots\}$,

then

(a) *the functions* $\mathrm{e}^{-\lambda t}((q+1)/t)^{q+1}\,F^{(q)}((q+1)/t)$ *are integrable over* $(0, \infty)$ *for every* $\lambda > 0$ *and* $q \in \{0, 1, \ldots\}$,

(b) $\dfrac{(-1)^q}{q!}\displaystyle\int_0^\infty \mathrm{e}^{-\lambda\tau}\left(\frac{q+1}{\tau}\right)^{q+1} F^{(q)}\left(\frac{q+1}{\tau}\right)\mathrm{d}\tau \to_{q\to\infty} F(\lambda)$ *for every* $\lambda > 0$.

Proof. The first statement (a) is an immediate consequence of the corresponding statement of Proposition 4.

By the statement (a) we have in particular that

(1) the function $\mathrm{e}^{-\lambda t}(1/t)\,F(1/t)$ is integrable over $(0, \infty)$.

Further, it follows from (α) that

(2) the function $(1/t)\,F(1/t)$ is continuous on $(0, \infty)$.

Now we get by Proposition 4 with $p = q$ and $\mu = \lambda(q+1)$ after a simple substitution that

$$(3)\quad \frac{(-1)^q}{q!}\int_0^\infty \mathrm{e}^{-\lambda\tau}\left(\frac{q+1}{\tau}\right)^{q+1} F^{(q)}\left(\frac{q+1}{\tau}\right)\mathrm{d}\tau =$$

$$= \frac{(-1)^q}{q!}(q+1)\int_0^\infty \mathrm{e}^{-\lambda(q+1)\tau}\tau^{-(q+1)}F^{(q)}\left(\frac{1}{\tau}\right)\mathrm{d}\tau =$$

$$= \frac{\lambda^q(q+1)^{q+1}}{q!}\int_0^\infty \mathrm{e}^{-\lambda(q+1)\tau}\tau^{q-1}F\left(\frac{1}{\tau}\right)\mathrm{d}\tau =$$

$$= \lambda^{-1}\frac{((q+1)\,\lambda)^{q+1}}{q!}\int_0^\infty \mathrm{e}^{-(q+1)\lambda\tau}\tau^{q-1}F\left(\frac{1}{\tau}\right)\mathrm{d}\tau$$

for every $\lambda > 0$ and $q \in \{0, 1, \ldots\}$.

The statement (b) is now an immediate consequence of (3) by virtue of Proposition 2.

The proof is complete.

6. Lemma. *For every* $t > 0$ *and* $\mu > 0$, $|(\mathrm{e}^{-\mu t} - 1)/\mu + t| \leqq \frac{1}{2}\mu t^2$.

Proof.

$$\left|\frac{\mathrm{e}^{-\mu t} - 1}{\mu} + t\right| = \left|t - \int_0^t \mathrm{e}^{-\mu\tau}\,\mathrm{d}\tau\right| = \left|\int_0^t (1 - \mathrm{e}^{\mu\tau})\,\mathrm{d}\tau\right| = \mu\int_0^t\int_0^\tau \mathrm{e}^{-\mu\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau \leqq \frac{\mu t^2}{2}.$$

7. Theorem. *Let $F \in (0, \infty) \to E$. If the assumptions* (α) *and* (β) *of Theorem* 5 *are fulfilled, then*

(a) *the functions* $e^{-\lambda t} t^p ((q+1)/t)^{q+1} F^{(q)}((q+1)/t)$ *are integrable over* $(0, \infty)$ *for every* $\lambda > 0$ *and* $p, q \in \{0, 1, \ldots\}$,

(b) $$\frac{(-1)^{p+q}}{q!} \int_0^\infty e^{-\lambda\tau} \tau^p \left(\frac{q+1}{\tau}\right)^{q+1} F^{(q)}\left(\frac{q+1}{\tau}\right) d\tau \to_{q\to\infty} F^{(p)}(\lambda)$$

for every $\lambda > 0$ *and* $p \in \{0, 1, \ldots\}$.

Proof. The statement (a) follows immediately from (a) of Theorem 5.

To prove the statement (b), we proceed by induction on p.

For the sake of simplicity we shall occasionally write

$$f_q(t) = \frac{(-1)^q}{q!} \left(\frac{q+1}{t}\right)^{q+1} F^{(q)}\left(\frac{q+1}{t}\right) \quad \text{for} \quad t > 0 \quad \text{and} \quad q \in \{0, 1, \ldots\}.$$

The case $p = 0$ follows from Theorem 5.

We now prove the induction step, i.e., we suppose that for a fixed $p \in \{0, 1, \ldots\}$,

(1) $$(-1)^p \int_0^\infty e^{-\lambda\tau} \tau^p f_q(\tau) \, d\tau \to_{q\to\infty} F^{(p)}(\lambda) \text{ for every } \lambda > 0,$$

and we prove that (1) implies

(2) $$(-1)^{p+1} \int_0^\infty e^{-\lambda\tau} \tau^{p+1} f_q(\tau) \, d\tau \to_{q\to\infty} F^{(p+1)}(\lambda) \text{ for every } \lambda > 0.$$

To this aim we need some auxiliary observations. First we recall that clearly

(3) $$\frac{F^{(p)}(\lambda+\mu) - F^{(p)}(\lambda)}{\mu} \to_{\mu\to 0+} F^{(p+1)}(\lambda) \text{ for every } \lambda > 0.$$

Further we get by Lemma 6 that

(4) $$\left\| \int_0^\infty \left[\frac{e^{-(\lambda+\mu)} - e^{-\lambda\tau}}{\mu} + e^{-\lambda\tau}\tau \right] \tau^p f_q(\tau) \, d\tau \right\| \leqq \int_0^\infty \left| \frac{e^{-\mu\tau} - 1}{\mu} + \tau \right| e^{-\lambda\tau} \tau^p \|f_q(\tau)\| d\tau \leqq$$

$$\leqq \frac{\mu}{2} \int_0^\infty e^{-\lambda\tau} \tau^{p+2} \|f_q(\tau)\| \, d\tau \text{ for every } \lambda > 0, \ \mu > 0 \text{ and } q \in \{0, 1, \ldots\}.$$

Finally it follows from (1) which is supposed to be valid that

(5) $$(-1)^p \int_0^\infty \frac{e^{-(\lambda+\mu)} - e^{-\mu\tau}}{\mu} \tau^p f_q(\tau) \, d\tau -$$

$$- \frac{F^{(p)}(\lambda+\mu) - F^{(p)}(\lambda)}{\mu} \to_{q\to\infty} 0 \text{ for every } \lambda > 0 \text{ and } \mu > 0.$$

On the other hand, a routine verification shows that we can write

$$(6)\quad (-1)^{p+1}\int_0^\infty e^{-\lambda\tau}\tau^{p+1}f_q(\tau)\,d\tau - F^{(p+1)}(\lambda) =$$

$$= \left[(-1)^p\int_0^\infty \frac{e^{-(\lambda+\mu)} - e^{-\lambda\tau}}{\mu}\tau^p f_q(\tau)\,d\tau - \frac{F^{(p)}(\lambda+\mu) - F^{(p)}(\lambda)}{\mu}\right] +$$

$$+ \left[(-1)^{p+1}\int_0^\infty \left(\frac{e^{-(\lambda+\mu)} - e^{-\lambda\tau}}{\mu} + e^{-\lambda\tau}\tau\right)\tau^p f_q(\tau)\,d\tau\right] +$$

$$+ \left[\frac{F^{(p)}(\lambda+\mu) - F^{(p)}(\lambda)}{\mu} - F^{(p+1)}(\lambda)\right] \text{for every } \lambda > 0, \mu > 0 \text{ and } q \in \{0, 1, \ldots\}.$$

Let now $\varepsilon > 0$ and $\lambda > 0$ be fixed.

By (3) and (4) we can find a fixed $\mu_0 > 0$ such that

$$(7)\quad \left\|\frac{F^{(p)}(\lambda+\mu_0) - F^{(p)}(\lambda)}{\mu_0} - F^{(p+1)}(\lambda)\right\| \leqq \frac{\varepsilon}{3},$$

$$(8)\quad \left\|(-1)^{p+1}\int_0^\infty \left(\frac{e^{-(\lambda+\mu_0)} - e^{-\lambda\tau}}{\mu_0} + e^{-\lambda\tau}\tau\right)\tau^p f_q(\tau)\,d\tau\right\| \leqq \frac{\varepsilon}{3}$$

for every $q \in \{0, 1, \ldots\}$.

Further it follows from (5) that there exists a $q_0 \in \{0, 1, \ldots\}$ such that

$$(9)\quad \left\|(-1)^p\int_0^\infty \frac{e^{-(\lambda+\mu_0)} - e^{-\lambda\tau}}{\mu_0}\tau^p f_q(\tau)\,d\tau - \frac{F^{(p)}(\lambda+\mu_0) - F^{(p)}(\lambda)}{\mu_0}\right\| \leqq \frac{\varepsilon}{3}$$

for every $q \geqq q_0$.

Summing up (6)–(9) we see that

$$\left\|(-1)^{p+1}\int_0^\infty e^{-\lambda\tau}\tau^{p+1}f_q(\tau)\,d\tau - F^{(p+1)}(\lambda)\right\| \leqq \varepsilon$$

for every $q \geqq q_0$ which proves (2).

This verifies the induction step and therefore the statement (b) is valid.

The proof is complete.

8. Proposition. *Let $F \in (0, \infty) \to E$. If*

(α) *the function F is infinitely differentiable on $(0, \infty)$,* (β) *the functions* $\lambda^{p+1} F^{(p)}(\lambda)$ *are bounded on* $(0, \infty)$ *for every* $p \in \{0, 1, \ldots\}$,

then

$$\int_0^1 \xi^{p+1} \|F^{(p)}(\xi)\| \, d\xi < \infty \text{ and } \int_1^\infty \xi^{p-1} \|F^{(p)}(\xi)\| \, d\xi < \infty \text{ for every } p \in \{0, 1, \ldots\}.$$

Proof is easy.

9. Proposition. *Let* $F \in (0, \infty) \to E$. *If*

(α) *the function* F *is infinitely differentiable on* $(0, \infty)$,

(β) $F(\lambda) \to 0$ $(\lambda \to \infty)$,

(γ) *there exists a* $\vartheta > 1$ *such that* $\int_0^\infty \mu^{\vartheta p + \vartheta - 2} \|F^{(p)}(\mu)\|^\vartheta \, d\mu < \infty$ *for every* $p \in \{1, 2, \ldots\}$,

then

$$\int_0^1 \xi^{p+1} \|F^{(p)}(\xi)\| \, d\xi < \infty, \quad \int_1^\infty \xi^{p-1} \|F^{(p)}(\xi)\| \, d\xi < \infty \text{ for every } p \in \{0, 1, \ldots\}.$$

Proof. Let us first consider $p \in \{1, 2, \ldots\}$.
By virtue of Hölder's inequality we get from (γ) that

$$\int_0^1 \xi^{p+1} \|F^{(p)}(\xi)\| \, d\xi = \int_0^1 \xi^{2/\vartheta} [\xi^{p+1-(2/\vartheta)} \|F^{(p)}(\xi)\|] \, d\xi \leqq$$

$$\leqq \left[\int_0^1 \xi^{2/(\vartheta-1)} \right]^{(\vartheta-1)/\vartheta} \left[\int_0^1 \xi^{\vartheta p + \vartheta - 2} \|F^{(p)}(\xi)\|^\vartheta \, d\xi \right]^{1/\vartheta} \leqq$$

$$\leqq \left[\int_0^1 \xi^{2/(\vartheta-1)} \, d\xi \right]^{(\vartheta-1)/\vartheta} \left[\int_0^\infty \mu^{\vartheta p + \vartheta - 2} \|F^{(p)}(\mu)\|^\vartheta \, d\mu \right]^{1/\vartheta} < \infty,$$

$$\int_1^\infty \xi^{p-1} \|F^{(p)}(\xi)\| \, d\xi = \int_1^\infty \xi^{-2(\vartheta-1)/\vartheta} [\xi^{p+1-(2/\vartheta)} \|F^{(p)}(\xi)\|] \, d\xi \leqq$$

$$\leqq \left[\int_1^\infty \xi^{-2} \, d\xi \right]^{(\vartheta-1)\vartheta} \left[\int_1^\infty \xi^{\vartheta p + \vartheta - 2} \|F^{(p)}(\xi)\|^\vartheta \, d\xi \right]^{1/\vartheta} \leqq$$

$$\leqq \left[\int_0^\infty \mu^{\vartheta p + \vartheta - 2} \|F^{(p)}(\mu)\|^\vartheta \, d\mu \right]^{1/\vartheta} < \infty.$$

It remains to deal with the case $p = 0$.
To this aim we first deduce from (β) that $F(\xi) = \int_\xi^\infty F'(\mu) \, d\mu$ for every $\xi > 0$.

Now we get by Hölder's inequality for $\xi > 0$ that

$$\|F(\xi)\| \leqq \int_\xi^\infty \|F'(\mu)\| \, d\mu = \int_\xi^\infty \mu^{-2(\vartheta-1)/\vartheta}[\mu^{2(\vartheta-1)/\vartheta}\|F'(\mu)\|] \, d\mu \leqq$$

$$\leqq \left[\int_\xi^\infty \mu^{-2} \, d\mu\right]^{(\vartheta-1)/\vartheta} \left[\int_\xi^\infty \mu^{2\vartheta-2}\|F'(\mu)\|^\vartheta \, d\mu\right]^{1/\vartheta} \leqq \xi^{-(\vartheta-1)/\vartheta}\left[\int_0^\infty \mu^{2\vartheta-2} F'(\mu)^\vartheta \, d\mu\right]^{1/\vartheta}.$$

The desired inequalities with $p = 0$ follow immediately from this estimate and from (γ) with $p = 1$.

10. Proposition. *Let* $F \in (0, \infty) \to E$. *If*

(α) *the function* F *is infinitely differentiable on* $(0, \infty)$,

(β) $F(\lambda) \to 0$ $(\lambda \to \infty)$,

(γ) $\int_0^\infty \mu^{p-1}\|F^{(p)}(\mu)\| \, d\mu < \infty$ *for every* $p \in \{1, 2, \ldots\}$,

then

$$\int_0^1 \xi^{p+1}\|F^{(p)}(\xi)\| \, d\xi < \infty, \quad \int_1^\infty \xi^{p-1}\|F^{(p)}(\xi)\| \, d\xi < \infty \quad \textit{for every} \quad p \in \{0, 1, \ldots\}.$$

Proof. The case $p = \{1, 2, \ldots\}$ is immediately clear from (γ).

We have only to deal with the case $p = 0$.

First, by (β) we get $F(\xi) = \int_\xi^\infty F'(\mu) \, d\mu$ for every $\xi > 0$.

This identity implies $\|F(\xi)\| \leqq \int_\xi^\infty \|F'(\mu)\| \, d\mu \leqq \int_0^\infty \|F'(\mu)\| \, d\mu < \infty$ which at once gives $\int_0^1 \xi^{p+1}\|F(\xi)\| \, d\xi < \infty$.

On the other hand, it follows from $\int_0^\infty \|F'(\mu)\| \, d\mu < \infty$ that there exists a sequence $\lambda_k > 0$, $k \in 1, 2, \ldots$, such that $\lambda_k \to \infty$ $(k \to \infty)$ and $F'(\lambda_k) \to 0$ $(k \to \infty)$. Since $\int_0^\infty \mu\|F''(\mu)\| \, d\mu < \infty$ (cf. (γ)) implies that $\int_\xi^\infty \|F''(\mu)\| \, d\mu < \infty$ for every $\xi > 0$, we obtain that $F'(\xi) = \int_\xi^\infty F''(\mu) \, d\mu$ for every $\xi > 0$. Consequently we can write $\|\xi F'(\xi)\| = \|\xi \int_\xi^\infty (1/\mu) \, \mu F''(\mu) \, d\mu\| \leqq \int_\xi^\infty \mu\|F''(\mu)\| \, d\mu$ for every $\xi > 0$. By virtue of (γ) this implies $\xi F'(\xi) \to 0$. Using this fact and integrating by parts we obtain easily $\|\xi F'(\xi)\| = \|\int_\xi^\infty F'(\mu) \, d\mu - \int_\xi^\infty \mu F''(\mu) \, d\mu\| \leqq \int_0^\infty \|F'(\mu)\| \, d\mu + \int_0^\infty \mu\|F''(\mu)\| \, . \, d\mu$ for every $\xi > 0$. The above results now give $\|F(\xi)\| = \|\int_\xi^\infty F'(\mu) \, d\mu\| = \|\int_\xi^\infty (1/\mu) \, \mu F'(\mu) \, d\mu\| \leqq \int_\xi^\infty (1/\mu) \, \|\mu F'(\mu)\| \, d\mu \leqq (1/\xi) \, [\int_0^\infty \|F'(\mu)\| \, d\mu + \int_0^\infty \mu\|F''(\mu)\| \, d\mu]$ for every $\xi > 0$. Consequently $\int_1^\infty \xi^{-1}\|F(\xi)\| \, d\xi \leqq (\int_1^\infty \xi^{-2} \, d\xi) \, . \, [\int_0^\infty \|F'(\mu)\| \, d\mu + \int_0^\infty \mu\|F''(\mu)\| \, d\mu] = \int_0^\infty \|F'(\mu)\| \, d\mu + \int_0^\infty \mu\|F''(\mu)\| \, d\mu$ which proves the last desired inequality, by virtue of (γ).

The proof is complete.

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