Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 106 (1981), No. 2, 186--196

Persistent URL: `http://dml.cz/dmlcz/118088`

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ON INCREMENT OF THE TANGENT ARGUMENT ALONG A CURVE

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(Received May 23, 1979)

In this paper we introduce an increment of the tangent argument along a plane curve (not necessarily smooth) as the limit of increments of secant argument. This number will be characterized by means of a decomposition into Jordan curves and by means of indices of points with respect to the given curve. It will easily follow that for every smooth positively oriented Jordan curve the increment of the tangent argument along it equals 2π. This special case is known as a consequence of a deep theorem of Lindelöf ([3]).

I am thankful for the advice I have received from Professor Ilja Černý who read the manuscript.

1. INTRODUCTION AND NOTATION

By a *curve* we shall understand a continuous mapping φ of a compact interval $\langle\alpha, \beta\rangle$ into the open complex plane $\mathbf{C}$. The image $\varphi(\langle\alpha, \beta\rangle)$ will be denoted by $\langle\varphi\rangle$. The curve $\dot{-}\varphi : \langle -\beta, -\alpha\rangle \to \mathbf{C}$ is defined by $\dot{-}\varphi(t) = \varphi(-t)$. Let $\varphi : \langle\alpha, \beta\rangle \to \mathbf{C}$ and $\psi : \langle\gamma, \delta\rangle \to \mathbf{C}$ be curves, $\varphi(\beta) = \psi(\gamma)$. Define a curve $\varphi \dot{+} \psi : \langle\alpha, \beta - \gamma + \delta\rangle \to \mathbf{C}$ by

$$(\varphi \dot{+} \psi)(t) = \varphi(t) \quad (t \in \langle\alpha, \beta\rangle),$$

$$(\varphi \dot{+} \psi)(t) = \psi(t + \gamma - \beta) \quad (t \in \langle\beta, \beta - \gamma + \delta\rangle).$$

We shall write "$\varphi \dot{-} \psi$" instead of "$\varphi \dot{+} (\dot{-}\psi)$".

A curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{C}$ is termed *closed*, if $\varphi(\alpha) = \varphi(\beta)$; if φ is closed and $\varphi(t) \neq \varphi(s)$ whenever $\alpha \leqq t < s < \beta$, then φ is called a *Jordan curve.*

For a closed curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{C}$, φ^* denotes the $(\beta - \alpha)$-periodic extension of φ.

The *index of a point* z with respect to a closed curve φ is denoted by $\operatorname{ind}_\varphi z$. The function $\operatorname{ind}_\varphi$ is constant on every connected subset $\mathbf{G}$ of $\mathbf{C} - \langle\varphi\rangle$; the common value of $\operatorname{ind}_\varphi z$ $(z \in \mathbf{G})$ is denoted by $\operatorname{ind}_\varphi \mathbf{G}$.

According to the Jordan theorem, for any Jordan curve φ the set $\mathbf{C} - \langle\varphi\rangle$ has precisely two components: the *interior* $\operatorname{Int} \varphi$ (bounded), and the *exterior* $\operatorname{Ext} \varphi$

(unbounded). The *orientation* of a Jordan curve φ is defined by or $\varphi = \operatorname{ind}_\varphi \operatorname{Int} \varphi$; it equals 1 or -1. Note that $\operatorname{ind}_\varphi \operatorname{Ext}\varphi = 0$.

By a *cycle* we shall understand a finite sequence of closed curves. We shall not distinguish between a closed curve φ and the cycle (φ). If $\Gamma = (\varphi_1, \ldots, \varphi_N)$ is a cycle, denote $\langle \Gamma \rangle = \bigcup_{n=1}^{N} \langle \varphi_n \rangle$. The index of a point or a set with respect to a cycle is defined as the sum of indices with respect to the curves of the cycle.

We shall say that a cycle $\Gamma = (\psi_1, \ldots, \psi_N)$ is a *decomposition* of a curve φ, if there are curves

$$\varphi_j : \langle \alpha_j, \beta_j \rangle \to \mathbf{C}\,, \quad \tilde{\varphi}_j : \langle \hat{\alpha}_j, \hat{\beta}_j \rangle \to \mathbf{C} \quad (j = 1, \ldots, J)\,,$$

a permutation f of the set $\{1, \ldots, J\}$ and integers $1 \leqq p_1 < p_2 < \ldots < p_{N-1} < J$ so that

(i) $\varphi = \varphi_1 \dotplus \ldots \dotplus \varphi_J$,

(ii) $\psi_1 = \tilde{\varphi}_{f(1)} \dotplus \ldots \dotplus \tilde{\varphi}_{f(p_1)}$,

$\psi_2 = \tilde{\varphi}_{f(p_1+1)} \dotplus \ldots \dotplus \tilde{\varphi}_{f(p_2)}$,

$\ldots$

$\psi_N = \tilde{\varphi}_{f(p_{N-1}+1)} \dotplus \ldots \dotplus \tilde{\varphi}_{f(J)}$,

(iii) $\hat{\beta}_j - \hat{\alpha}_j = \beta_j - \alpha_j$, $\tilde{\varphi}_j(t - \alpha_j + \hat{\alpha}_j) = \varphi_j(t)$
$(j = 1, \ldots, J;\ t \in \langle \alpha_j, \beta_j \rangle)$.

2. SIMPLE INTERSECTION POINTS

Let $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ be a closed curve. A point $w \in \mathbf{C}$ is called an *intersection point* of φ, if φ maps at least two points from $\langle \alpha, \beta)$ to w. If w is the image of precisely two points from $\langle \alpha, \beta)$, then the intersection point w is termed simple.

Lemma 1. *Let $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ be a closed curve. Let $w = \varphi(t_1) = \varphi(t_2)$ $(t_1 < t_2 < \beta)$ be a simple isolated intersection point (i.e. a simple intersection point, which is an isolated point of the set of all intersection points of the curve φ). Choose points $\tau_1 \in (t_1, t_2)$ and $\tau_2 \in (t_2, \beta)$. Fix $r > 0$ so small that*

(1) *$\{z : |z - w| \leqq r\}$ contains neither an intersection point $w' \neq w$, nor $\varphi(\tau_1)$, nor $\varphi(\tau_2)$.*

Let $\langle \alpha_j, \beta_j \rangle$ $(j = 1, 2)$ be the largest intervals containing t_j and satisfying $\varphi^*((\alpha_j, \beta_j)) \subset \{z : |z - w| < r\}$. Assume that

(2) *the set $\varphi^*(\{\alpha_1, \beta_1\})$ separates the circle $\mathbf{K} = \{z : |z - w| = r\}$ between the points $\varphi^*(\alpha_2)$ and $\varphi^*(\beta_2)$.*

Then there exist a $\varrho_0 > 0$ and an integer ι such that

$$\operatorname{ind}_\varphi(\mathbf{U} - \langle\varphi\rangle) = \{\iota, \iota - 1, \iota + 1\}$$

for all ϱ-neighborhoods $\mathbf{U}$ *of* w *with* $0 < \varrho < \varrho_0$.

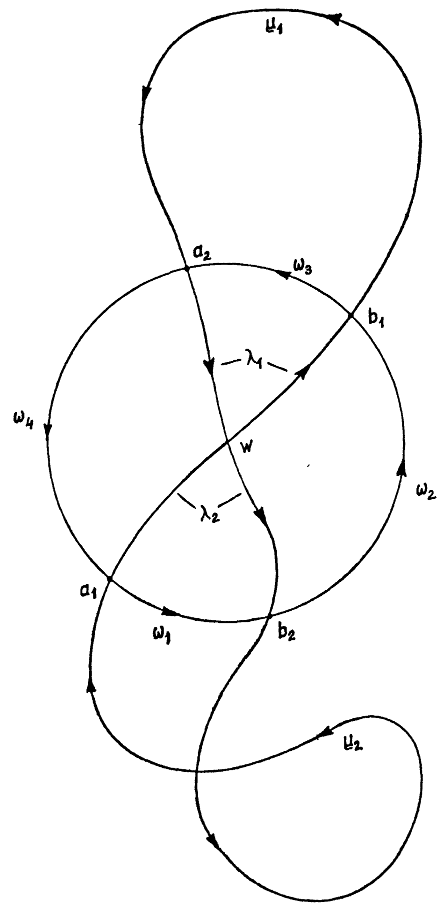

Fig. 1

Proof. (See Fig. 1.) Denote

$$\begin{aligned}
\lambda_1 &= \varphi^*|\langle\alpha_2, t_2\rangle + \varphi^*|\langle t_1, \beta_1\rangle\,,\\
\lambda_2 &= \varphi^*|\langle\alpha_1, t_1\rangle + \varphi^*|\langle t_2, \beta_2\rangle\,,\\
\mu_1 &= \varphi^*|\langle\beta_1, \alpha_2\rangle\,,\\
\mu_2 &= \varphi^*|\langle\beta_2, \alpha_1 + \beta - \alpha\rangle\,,\\
a_j &= \varphi^*(\alpha_j)\,, \quad b_j = \varphi^*(\beta_j) \quad (j = 1, 2)\,,\\
\omega^*(t) &= w + re^{it} \quad (t \in (-\infty, +\infty))\,.
\end{aligned}$$

Choose $s_0 < s_2 < s_4$ such that $s_4 - s_0 = 2\pi$, $\omega^*(s_0) = \omega^*(s_4) = a_1$, $\omega^*(s_2) = b_1$. By (2), one of the points a_2, b_2 belongs to $\omega^*((s_0, s_2))$ while the other one belongs to $\omega^*((s_2, s_4))$. For the symmetry reason we may suppose that $b_2 \in \omega^*((s_0, s_2))$. Find $s_1 \in (s_0, s_2)$ and $s_3 \in (s_2, s_4)$ with $\omega^*(s_1) = b_2$, $\omega^*(s_3) = a_2$. Denote $\omega_j = \omega^* \mid \langle s_{j-1}, s_j \rangle$ $(j = 1, \dots, 4)$, $\omega = \omega_1 \dotplus \dots \dotplus \omega_4$. Then ω is a Jordan curve with or $\omega = 1$.

Put $\varrho_0 = \operatorname{dist}(w, \langle \mu_1 \rangle \cup \langle \mu_2 \rangle)$. Let $\boldsymbol{U}$ be an open ϱ-neighborhood of w, $0 < \varrho < \varrho_0$.

Choose $z_1 \in \boldsymbol{U} \cap \operatorname{Int}(\lambda_1 \dotplus \omega_3)$, $z_2 \in \boldsymbol{U} \cap \operatorname{Int}(\lambda_2 \dot{-} \omega_1)$ and $z_3 \in \boldsymbol{U} - (\operatorname{Int}(\lambda_1 \dotplus \omega_3) \cup \operatorname{Int}(\lambda_2 \dot{-} \omega_1) \cup \langle \varphi \rangle)$ arbitrarily. (It is easy to see that these sets are not empty and their union is equal to $\boldsymbol{U} - \langle \varphi \rangle$.)

For two cycles Γ_1, Γ_2 write $\Gamma_1 \sim_k \Gamma_2$ if $\operatorname{ind}_{\Gamma_1} z_k = \operatorname{ind}_{\Gamma_2} z_k$ $(k = 1, 2, 3)$. We obtain

$$\varphi \sim_1 (\lambda_1 \dotplus \mu_1, \lambda_2 \dotplus \mu_2) \sim_1 (\lambda_1 \dot{-} \omega_2 \dot{-} \omega_1 \dot{-} \omega_4,$$
$$\lambda_2 \dot{-} \omega_1, \mu_1 \dot{-} \omega_3, \mu_2 \dotplus \omega_1, \omega).$$

Since z_1 lies in the unbounded component of both $\lambda_1 \dotplus \omega_2 \dot{-} \omega_1 \dot{-} \omega_4$ and $\lambda_2 \dot{-} \omega_1$, we have

$$\varphi \sim_1 (\mu_1 \dot{-} \omega_3, \mu_2 + \omega_1, \omega).$$

Similarly we can prove

$$\varphi \sim_2 (\mu_1 \dot{-} \omega_3, \mu_2 \dotplus \omega_1, \dot{-}\omega),$$
$$\varphi \sim_3 (\mu_1 \dot{-} \omega_3, \mu_2 \dotplus \omega_1).$$

Hence ϱ_0 and $\iota = \operatorname{ind}_\Gamma \boldsymbol{U}$, where $\Gamma = (\mu_1 \dot{-} \omega_3, \mu_2 \dotplus \omega_1)$, have the required properties.

Definition. Let $\varphi : \langle \alpha, \beta \rangle \to \boldsymbol{C}$ be a closed curve. An intersection point w of φ will be termed *essential* if w is simple, isolated and (1) implies (2).

The integer ι from Lemma 1 which corresponds to an essential intersection point w will be denoted by $\iota_\varphi w$.

3. DECOMPOSITION INTO JORDAN CURVES

Lemma 2. *Let φ: $\langle \alpha, \beta \rangle \to \boldsymbol{C}$ be a closed curve. Assume that*

(3) *the set $\boldsymbol{T} = \{t \in \langle \alpha, \beta \rangle : \varphi(t') = \varphi(t)$ for some $t' \in \langle \alpha, \beta \rangle - \{t\}\}$ is finite.*

Then there exists a Jordan curve ψ such that

$$\operatorname{Int} \psi \cap \langle \varphi \rangle = \emptyset \tag{4}$$

and

(5) *there are $\alpha_j \in \boldsymbol{T}$, $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_J = \beta$, a permutation f of the set*

$\{1, \ldots, J\}$ *and a positive integer* $K \leqq J$ *with*

$$\psi = \varphi \mid \langle \alpha_{f(1)-1}, \alpha_{f(1)} + \ldots + \varphi \mid \langle \alpha_{f(p)-1}, \ldots, \alpha_{f(p)} \rangle \, .$$

Proof. Denote by Ψ the set of all Jordan curves satisfying (5). Obviously Ψ is nonempty. Further, Ψ contains a minimal element ψ_0 in the following sense: if $\psi \in \Psi$ and $\operatorname{Int} \psi \subset \operatorname{Int} \psi_0$, then $\operatorname{Int} \psi = \operatorname{Int} \psi_0$. Indeed, the assumption of existence of an infinite sequence $\{\psi_j\}_{j=1}^{\infty} \subset \Psi$ with

$$\operatorname{Int} \psi_1 \underset{\neq}{\supset} \operatorname{Int} \psi_2 \underset{\neq}{\supset} \ldots$$

leads to a contradiction with the finite cardinality of $\boldsymbol{T}$.

Suppose $\operatorname{Int} \psi_0 \cap \langle \varphi \rangle \neq \emptyset$. Then $\varphi(t_0) \in \operatorname{Int} \psi_0$ for some $t \in \langle \alpha, \beta \rangle$. Find the smallest interval $\langle \xi, \eta \rangle$ containing t_0 with $\varphi^*((\xi, \eta)) \subset \operatorname{Int} \psi_0$. Since $\partial(\operatorname{Int} \psi_0) = \psi_0$ by the Jordan theorem, we have $\psi_0(s) = \varphi^*(\eta)$ for some s. Find the smallest $t > s$ $\psi_0^*(t) = \varphi(\xi)$.

If $\varphi^* \mid \langle \xi, \eta \rangle$ is one-one, then $\varphi^* \mid \langle \xi, \eta \rangle + \psi_0^* \mid \langle s, t \rangle$ is a Jordan curve, the interior of which is a proper subset of $\operatorname{Int} \psi_0$. If $\varphi^* \mid \langle \xi, \eta \rangle$ is not one-one, then, using the fact that $\boldsymbol{T}$ is finite, we can find an interval $\langle \xi', \eta' \rangle \subset \langle \xi, \eta \rangle$ such that $\varphi^* \mid \langle \xi', \eta' \rangle$ is a Jordan curve with *Int* $\varphi^* \mid \langle \xi', \eta' \rangle \subseteq$ *Int* ψ_0. In both the situations ψ_0 is not a minimal element of Ψ, which is a contradiction.

Lemma 3. *Let be a closed curve satisfying* (3). *Then there exists a decomposition* $(\psi_1, \ldots, \psi_N)$ *of* φ *such that*

(6) ψ_n *are Jordan curves* $(n = 1, \ldots, N)$,

(7) *for* $j < k$ *either* $\operatorname{Int} \psi_j \subset \operatorname{Int} \psi_k$ *or* $\operatorname{Int} \psi_j \cap \operatorname{Int} \psi_k = \emptyset$.

Proof. It follows by using repeatedly Lemma 2; another proof is due to I. Černý ([1]).

Lemma 4. *Let* $\boldsymbol{M}$ *be a compact connected subset of* $\boldsymbol{C}$, $\boldsymbol{G}$ *a component of* $\boldsymbol{C} - \boldsymbol{M}$. *Let* $\psi : \langle \alpha, \beta \rangle \to \boldsymbol{C}$ *be a Jordan curve with* $\operatorname{Int} \psi \subset \boldsymbol{G}$. *Assume that there are precisely* J *points* $t_j \in \langle \alpha, \beta \rangle$ *such that* $\varphi(t_j) \in \boldsymbol{M}$. *Then* $\boldsymbol{G} - \langle \psi \rangle$ *has exactly* $J + 1$ *components, one of them being* $\operatorname{Int} \psi$.

Proof. We shall prove the following assertion, which implies Lemma 4 by induction:

Let $\varphi : \langle \alpha, \beta \rangle \to \boldsymbol{C}$ be a curve, either Jordan or one-one. Let $\varphi(\{\alpha, \beta\}) \subset \partial \boldsymbol{G}$, $\varphi((\alpha, \beta)) \subset \boldsymbol{G}$. Then $\boldsymbol{G} - \langle \varphi \rangle$ has exactly two components.

The Riemann theorem on conformal mapping implies that $\boldsymbol{G}$ is homeomorphic to $\boldsymbol{C}$ (if $\boldsymbol{G}$ is bounded) or to $\boldsymbol{C} - \{0\}$ (if $\boldsymbol{G}$ is unbounded). We give the proof for the former case, the latter being similar.

Let $\boldsymbol{G}^* = \boldsymbol{G} \cup \{\infty_{\boldsymbol{G}}\}$, $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$ be the one-point compactifications of $\boldsymbol{G}$ and $\boldsymbol{C}$, respectively. Clearly, any homeomorphism $f : \boldsymbol{G} \to \boldsymbol{C}$ may be extended to a homeomorphism $f^* : \mathbf{G}^* \to \mathbf{C}^*$ if we define $f^*(\infty_{\mathbf{G}}) = \infty$. Then $f^* \circ \varphi$ is a Jordan curve in $\mathbf{C}^*$ and, by the Jordan theorem, $\mathbf{C}^* - \langle f^* \circ \varphi \rangle = \mathbf{C} - \langle f \circ \varphi \rangle$ has exactly two components. Applying f^*_{-1} we get the assertion.

Lemma 5. *Let* $\Gamma = (\psi_1, \ldots, \psi_N)$ *be a cycle satisfying* (6), (7) *and suppose that*

(8) $\langle \Gamma \rangle$ *is connected,*

(9) $\mathbf{W} = \bigcup_{n \neq n'} \langle \psi_n \rangle \cap \langle \psi_{n'} \rangle$ *is finite,*

(10) $\bigcup_{n \neq n' \neq n'' \neq n} \langle \psi_n \rangle \cap \langle \psi_{n'} \rangle \cap \langle \psi_{n''} \rangle = \emptyset$,

(11) *for every* $w \in \mathbf{W}$ *there is such an integer* ι *that* $\operatorname{ind}_\Gamma (U - \langle \Gamma \rangle) = \{\iota, \iota - 1, \iota + 1\}$ *for each sufficiently small neighborhood* $\mathbf{U}$ *of* w.

Denote by $\mathbf{G}_1, \ldots, \mathbf{G}_P$ *the components of* $\mathbf{C} - \langle \Gamma \rangle$, *by* $w_1, \ldots, w_Q$ *the elements of* $\mathbf{W}$ *and by* $\iota_1, \ldots, \iota_Q$ *the integers corresponding to them by* (11).
Then

$$\sum_{n=1}^{N} \operatorname{or} \psi_n = \sum_{p=1}^{P} \operatorname{ind}_\Gamma G_p - \sum_{q=1}^{Q} \iota_q . \tag{12}$$

Proof. For $N = 1$ the assertion holds. Let it hold whenever the cycle Γ contains less than N members. By (7), $\operatorname{Int} \psi_1 \cap \langle \Gamma \rangle = \emptyset$. Denote $\Gamma_0 = (\psi_2, \ldots, \psi_N)$. Let $\mathbf{G}$ be the component of $\mathbf{C} - \langle \Gamma_0 \rangle$ including $\operatorname{Int} \psi_1$. By Lemma 4, $\mathbf{G} - (\langle \psi_1 \rangle \cup \operatorname{Int} \psi_1)$ has exactly K components where K is the number of points of $\mathbf{W} \cap \langle \psi_1 \rangle$. After a convenient reordering of $\{\mathbf{G}_1, \ldots, \mathbf{G}_P\}$ we may denote them by $\{\mathbf{G}_1, \ldots, \mathbf{G}_K\}$ while $\operatorname{Int} \psi_1$ may be denoted by $\mathbf{G}_P$.

Let $w \in \mathbf{W} \cap \langle \psi_1 \rangle$. By (10), there is exactly one curve ψ_M $(M > 1)$ satisfying $w \in \langle \psi_M \rangle$. Denote by Γ_1 the cycle $(\psi_2, \ldots, \psi_{M-1}, \psi_{M+1}, \ldots, \psi_N)$. Let $\mathbf{U}$ be so small a neighborhood of w that $\langle \Gamma_1 \rangle \cap \mathbf{U} = \emptyset$. Then two situations are possible:

I) $\operatorname{Int} \psi_1 \subset \operatorname{Int} \psi_M$; then $\operatorname{ind}_\Gamma (\mathbf{U} - \langle \Gamma \rangle) = \{\operatorname{ind}_{\Gamma_1} \mathbf{U}, \operatorname{ind}_{\Gamma_1} \mathbf{U} + \operatorname{or} \psi_M, \operatorname{ind}_{\Gamma_1} \mathbf{U} + \operatorname{or} \psi_M + \operatorname{or} \psi_1\}$. By (11) obviously $\operatorname{or} \psi_1 = \operatorname{or} \psi_M$ and consequently $\iota = \operatorname{ind}_{\Gamma_1} \mathbf{U} + \operatorname{or} \psi_M = \operatorname{ind}_{\Gamma_0} \mathbf{G}$.

II) $\operatorname{Int} \psi_1 \subset \operatorname{Ext} \psi_M$; then $\operatorname{ind}_\Gamma (\mathbf{U} - \langle \Gamma \rangle) = \{\operatorname{ind}_{\Gamma_1} \mathbf{U}, \operatorname{ind}_{\Gamma_1} \mathbf{U} + \operatorname{or} \psi_1, \operatorname{ind}_{\Gamma_1} \mathbf{U} + \operatorname{or} \psi_M\}$; by (11) $\operatorname{or} \psi_1 = -\operatorname{or} \psi_M$ and $\iota = \operatorname{ind}_{\Gamma_1} \mathbf{U} = \operatorname{ind}_{\Gamma_0} \mathbf{G}$.

We have proved $\iota = \operatorname{ind}_{\Gamma_0} \mathbf{G}$ for every $w \in \mathbf{W} \cap \langle \psi_1 \rangle$. Further, $\operatorname{ind}_{\Gamma_0} \mathbf{G} = \operatorname{ind}_\Gamma \mathbf{G}_p$ for $p = 1, \ldots, K$, $\operatorname{ind}_\Gamma \operatorname{Int} \psi_1 = \operatorname{ind}_{\Gamma_0} \mathbf{G} + \operatorname{or} \psi_1$ and thus

$$\sum_{p=1}^{K} \operatorname{ind}_\Gamma \mathbf{G}_p + \operatorname{ind}_\Gamma \mathbf{G}_P - \sum_{w_q \in \langle \psi_1 \rangle} \iota_q = \operatorname{ind}_{\Gamma_0} \mathbf{G} + \operatorname{or} \psi_1 .$$

Obviously Γ_0 has the properties (6), (7), ..., (11) and

$$\operatorname{ind}_{\Gamma_0} \mathbf{G}_p = \operatorname{ind}_\Gamma \mathbf{G}_p \quad \text{for} \quad p = K + 1, \ldots, P - 1 .$$

Hence the induction hypothesis yields

$$\sum_{n=1}^{N} \operatorname{or} \psi_n = \sum_{p=K+1}^{P-1} \operatorname{ind}_\Gamma \mathbf{G}_p + \operatorname{ind}_{\Gamma_0} \mathbf{G} - \sum_{w_q \notin \langle \psi_1 \rangle} \iota_q .$$

Consequently

$$\sum_{p=1}^{P} \operatorname{ind}_\Gamma \mathbf{G}_p - \sum_{q=1}^{Q} \iota_q = \sum_{p=1}^{K} \operatorname{ind}_\Gamma \mathbf{G}_p + \operatorname{ind}_\Gamma \mathbf{G}_P -$$

$$- \sum_{w_q \in \langle \psi_1 \rangle} \iota_q + \sum_{p=K+1}^{P-1} \operatorname{ind}_\Gamma \mathbf{G}_p - \sum_{w_q \notin \langle \psi_1 \rangle} \iota_q = \sum_{n=2}^{N} \operatorname{or} \psi_n + \operatorname{or} \psi_1 ,$$

which proves the assertion.

Theorem 1. *Let a closed curve φ have only a finite number of intersection points $w_1, \ldots, w_Q$, each of them being essential. Then there exists a decomposition $\Gamma = (\psi_1, \ldots, \psi_N)$* of φ satisfying (6), ..., (11).

If we denote by $\mathbf{G}_1, \ldots, \mathbf{G}_P$ the components of $\mathbf{C} - \langle \varphi \rangle$, then

$$\sum_{n=1}^{N} \operatorname{or} \psi_n = \sum_{p=1}^{P} \operatorname{ind}_\Gamma \mathbf{G}_p - \sum_{q=1}^{Q} \iota_\varphi(w_q) . \tag{13}$$

Proof. It is sufficient to take any decomposition Γ of φ satisfying (6) and (7) (its existence follows from Lemma 3). The properties (8), (9), (10) are evident, the property (11) follows from Lemma 1. Using Lemma 5 we obtain (13).

4. INCREMENT OF THE TANGENT ARGUMENT

Let $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ be a closed curve. We say that φ is *smooth*, if there is a closed nowhere vanishing curve $\psi : \langle \alpha, \beta \rangle \to \mathbf{C}$ with $\psi(t) = \varphi'(t)$ for $t \in (\alpha, \beta)$. We shall denote this curve ψ by $D\varphi$.

The number $2\pi \operatorname{ind}_{D\varphi} 0$ has the geometric interpretation of increment of the tangent argument along the curve φ. The restriction to smooth curves is not convenient for our purpose. Therefore we shall define the increment of the tangent argument in another way.

By a *homotopy* we shall understand a continuous mapping $H : \langle \alpha, \beta \rangle \times \langle 0, 1 \rangle \to \mathbf{C}$ such that $0 \notin H(\langle \alpha, \beta \rangle \times \langle 0, 1 \rangle)$ and $H(\cdot, s)$ is a closed curve for each $s \in \langle 0,1 \rangle$.

It is well known that under these assumptions

$$\operatorname{ind}_{H(\cdot,0)} 0 = \operatorname{ind}_{H(\cdot,1)} 0 . \tag{14}$$

Given a closed curve $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ and $h > 0$, we shall denote by φ_h the closed curve defined on $\langle \alpha, \beta \rangle$ by

$$\varphi_h(t) = \varphi^*(t + h) - \varphi(t) .$$

Assuming

(15) there is a $\Delta > 0$ with $\varphi^*(t + h) \neq \varphi(t)$ whenever $t \in \langle \alpha, \beta \rangle$ and $0 < h \leqq \Delta$,

φ_h is nowhere vanishing for $h \in (0, \Delta\rangle$. Hence the mapping H defined by

$$H(t, s) = \varphi_{h+s(\Delta-h)}(t)$$

is a homotopy and, by (14), $\operatorname{ind}\varphi_h 0 = \operatorname{ind}_{\varphi_\Delta} 0$ for all $h \in (0, \Delta)$. This allows us to introduce the following definition:

Let $\varphi : \langle, \beta\rangle \to \mathbf{C}$ be a closed curve satisfying (15). Then the number $\lim_{h\to 0+} 2\pi \operatorname{ind}_{\varphi_h} 0$ $(= 2\pi \operatorname{ind}_{\varphi_\Delta} 0)$ will be called the *increment of the tangent argument* along the curve φ and denoted by $\mathscr{T}(\varphi)$.

Theorem 2. *Let $\varphi : \langle\alpha, \beta\rangle \to \mathbf{C}$ be a smooth closed curve. Then $\mathscr{T}(\varphi) = 2\pi \operatorname{ind}_{D\varphi} 0$.*

Proof. Obviously φ satisfies (15). Define a mapping $H : \langle\alpha, \beta\rangle \times \langle 0, 1\rangle \to \mathbf{C}$ by

$$H(t, s) = \begin{cases} \varphi_{s\Delta}(t)/(s\Delta) & (s \in (0, 1\rangle), \\ D\,\varphi(t) & (s = 0). \end{cases}$$

It follows from (15) that H is nowhere vanishing. Clearly $H(\cdot, s)$ is a closed curve for every $s \in \langle 0, 1\rangle$ and H is continuous on $\langle\alpha, \beta\rangle \times (0, 1\rangle$.

Given any $t_0 \in (\alpha, \beta)$ and $\varepsilon > 0$, find $\delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow |\varphi'(t) - \varphi'(t_0)| < \varepsilon .$$

Let $|t - t_0| < \frac{1}{2}\delta$ and $0 < s < \delta/(2\Delta)$. Then

$$\left|\frac{\varphi^*(t + s\Delta) - \varphi(t)}{s\Delta}\right| \leqq \int_0^1 |\varphi(t + s\,\Delta\sigma) - \varphi(t_0)|\,\mathrm{d}\sigma \leqq \varepsilon .$$

This provəs the continuity of H at the points $[t_0, 0]$, $t_0 \in (\alpha, \beta)$. The continuity of H at the points $[\alpha, 0]$ and $[\beta, 0]$ can be proved analogously.

We have verified that H is a homotopy and thus, by (14),

$$\operatorname{ind}_{D\varphi} 0 = \operatorname{ind}_{\varphi_\Delta/\Delta} 0' = \operatorname{ind}_{\varphi_\Delta} 0 = (2\pi)^{-1}\,\mathscr{T}(\varphi) .$$

Lemma 6. *Let ψ_1 and $\psi_2 : \langle\alpha, \beta\rangle \to \mathbf{C}$ be Jordan curves. Then there is a homeomorphism Φ of* Int $\psi_1 \cup \langle\psi_1\rangle$ *onto* Int $\psi_2 \cup \langle\psi_2\rangle$ *such that*

$$\Phi(\psi_1(t)) = \psi_2(t) \quad \textit{for all} \quad t \in \langle\alpha, \beta\rangle .$$

Proof. We may suppose that $\langle\alpha, \beta\rangle = \langle 0, 2\pi\rangle$. Denote by ω the curve defined on $\langle 0, 2\pi\rangle$ by $\omega(t) = \mathrm{e}^{\mathrm{i}t}$.

A well known theorem from the plane topology (see [2]) says that every homeomorphism of the unit circle into $\mathbf{C}$ can be extended to a homeomorphism of $\mathbf{C}$ onto $\mathbf{C}$.

Let Ψ_1, Ψ_2 be such extensions of the mappings defined by $e^{it} \mapsto \psi_1(t)$, $e^{it} \mapsto \psi_2(t)$, respectively. Obviously, the superposition $\Psi_2 \circ \Psi_1^{-1}$, restricted to $\operatorname{Int} \psi_1 \cup \langle \psi_1 \rangle$, has the desired prOperties.

Theorem 3. *Let $\varphi : \langle \alpha, \beta \rangle \to \mathbf{C}$ be a Jordan curve. Then* $\mathscr{T}(\varphi) = 2\pi \operatorname{or} \varphi$.

Proof. We may suppose that $\langle \alpha, \beta \rangle = \langle 0, 2\pi \rangle$.

By Lemma 6 there is a homeomorphism Φ of $\{z : |z| \leqq 1\}$ onto $\langle \varphi \rangle \cup \operatorname{Int} \varphi$ with

$$\Phi(e^{it}) = \varphi(t) \quad (t \in \langle 0, 2\pi \rangle) . \tag{16}$$

Fix an $h \in (0, 2\pi)$. Put

$$H(t, s) = \Phi(e^{i(t+h)}) - \Phi(se^{it})$$

for $t \in \langle 0, 2\pi \rangle$, $s \in \langle 0, 1 \rangle$. Since Φ is continuous and one-one, H is a homotopy. Using (14) we obtain

$$\operatorname{or} \varphi = \operatorname{ind}_\varphi \Phi(0) = \operatorname{ind}_{H(\cdot, 0)} 0 = \operatorname{ind} \varphi_h \, 0 ,$$

and hence

$$2\pi \operatorname{or} \varphi = \mathscr{T}(\varphi) .$$

Lemma 7. *Under the hypotheses and notation from Lemma* 1, *consider* $h > 0$ *such that*

$$\varphi^*(\langle t_1 - h, t_1 + h \rangle \cup \langle t_2 - h; t_2 + h \rangle) \subset \{z : |z - w| < r\} .$$

Let

$$\begin{aligned}
\chi_1(t) &= \varphi^*(t_1 + t) - \varphi^*(t_1 - h + t) , \\
\chi_2(t) &= \varphi^*(t_2 + t) - \varphi^*(t_2 - h + t) , \\
\chi_3(t) &= \varphi^*(t_1 + t) - \varphi^*(t_2 - h + t) , \\
\chi_4(t) &= \varphi^*(t_2 + t) - \varphi^*(t_1 - h + t) , \quad t \in \langle 0, h \rangle .
\end{aligned}$$

Then

$$\int_{\chi_1} z^{-1} \, dz + \int_{\chi_2} z^{-1} \, dz = \int_{\chi_3} z^{-1} \, dz + \int_{\chi_4} z^{-1} \, dz .$$

Proof. Put

$$\begin{aligned}
\gamma &= s_2 - s_1 + \beta_1 - t_1 + \beta_2 - t_2 - 2h , \\
\phi_j(t) &= (-1)^j h^{-1} . (t - t_j) \quad (t \in \langle t_j, t_j + h \rangle, \ j = 1, 2) , \\
\hat{\omega}(t) &= e^{i\pi t/\gamma} \quad (t \in \langle 0, \gamma \rangle) , \\
\psi_1 &= \phi_1 \dotplus \hat{\omega} \dotplus \phi_2 , \\
\psi_2 &= \varphi^* | \langle t_1, \beta_1 \rangle \dotplus \omega_2 \dotplus \varphi^* | \langle t_2, \beta_2 \rangle .
\end{aligned}$$

It is easy to see that ψ_2 is a Jordan curve. By Lemma 6 there is a homeomorphism Φ of $\operatorname{Int}\psi_1 \cup \langle\psi_1\rangle$ onto $\operatorname{Int}\psi_2 \cup \langle\psi_2\rangle$ such that $\Phi(\psi_1(t)) = \psi_2(t)$, which implies $\Phi(\varphi^*(t)) = \hat{\varphi}_j(t)$ $(t \in \langle t_j, t_j + h\rangle,\ j = 1, 2)$. Put $\tilde{H}(t, s) = \Phi(th^{-1}e^{\pi is})$ $(t \in \langle 0, h\rangle$, $s \in \langle 0, 1\rangle)$. The mapping $\tilde{H}$ is continuous and has these properties:

$$\tilde{H}(\langle 0, h\rangle \times \langle 0, 1\rangle \subset \langle\psi_2\rangle \cup \operatorname{Int}\psi_2\,, \tag{17}$$

$$\tilde{H}(t, s) = w \Leftrightarrow t = 0\,, \tag{18}$$

$$\tilde{H}(t, 0) = \varphi^*(t_2 + t),\ \tilde{H}(t, 1) = \varphi^*(t_1 + t)\,. \tag{19}$$

We define mappings H_1, H_2 on $\langle 0, h\rangle \times \langle 0, 1\rangle$ by $H_j(t, s) = \tilde{H}(t, s) - \varphi(t_j - h + t)$. (The mappings H_1 and H_2 vanish nowhere because of (17), (18).) Consider a homotopy $H : (t, s) \mapsto \varphi^s(t)$, where

$$\varphi^s = \chi_1 \dot{-} \chi_3 \dot{+} H_2(\cdot, s) \dot{-} H_1(\cdot, s)\,.$$

Using (14) we obtain

$$\operatorname{ind}_{\chi_1 \dot{-} \chi_3 \dot{+} \chi_2 \dot{-} \chi_4} 0 = \operatorname{ind}_{\chi_1 \dot{-} \chi_3 \dot{+} \chi_3 \dot{-} \chi_1} 0 = 0\,,$$

and hence

$$\int_{\chi_1 \dot{-} \chi_3 \dot{+} \chi_2 \dot{-} \chi_4} z^{-1}\,dz = 0\,,$$

which proves the assertion.

Theorem 4. *Let φ be a closed curve with only a finite number of intersection points, each of them being essential. Then $\mathscr{T}(\varphi)$ is defined and*

$$\mathscr{T}(\varphi) = 2\pi \sum_{n=1}^{N} \operatorname{or} \varphi^n\,,$$

where $(\varphi^1, \ldots, \varphi^N)$ is an arbitrary decomposition of φ into Jordan curves.

Proof. Clearly φ satisfies (15). Assume $0 < h < \Delta$ and

$$\varphi^*(\langle t_1 - h,\ t_1 + h\rangle \cup \langle t_2 - h,\ t_2 + h\rangle) \subset \{z : |z - w| < r\}$$

whenever $w = \varphi(t_1) = \varphi(t_2)$ $(t_1 < t_2)$ and r satisfies (1). It follows without difficulties from Lemma 8 that

$$\int_{\varphi_h} z^{-1}\,dz = \sum_{n=1}^{N} \int_{\varphi_h{}^n} z^{-1}\,dz\,,$$

which means

$$\operatorname{ind}_{\varphi_h} 0 = \sum_{n=1}^{N} \operatorname{ind}_{\varphi_h{}^n} 0\,.$$

Using Theorem 3 and letting $h \to 0$ we obtain

$$\mathscr{T}(\varphi) = \sum_{n=1}^{N} \mathscr{T}(\varphi^n) = 2\pi \sum_{n=1}^{N} \operatorname{or} \varphi^n .$$

Theorem 5. *Let φ: $\langle \alpha, \beta \rangle \to \mathbf{C}$ be a closed curve with only a finite number of intersection points, each of them being essential. Let $(\varphi_1, \ldots, \varphi_N)$ be a decomposition of φ into Jordan curves. Denote by $\mathbf{G}_1, \ldots, \mathbf{G}_P$ the components of $\mathbf{C} - \langle \varphi \rangle$ and by $w_1, \ldots, w_Q$ the intersection points of φ. Then $\mathscr{T}(\varphi)$ is defined and*

$$(2\pi)^{-1} \mathscr{T}(\varphi) = \sum_{n=1}^{N} \operatorname{or} \varphi_n = \sum_{p=1}^{P} \operatorname{ind}_\varphi \mathbf{G}_p - \sum_{q=1}^{Q} \iota_\varphi w_q .$$

Proof. This is only a summary of the results of Theorem 1 and Theorem 4.

References

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