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FINITE SPHERICAL GEOMETRIES

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The concept of a finite projective geometry is well-known (see e.g. [1]). A projective geometry is an ordered triple (P, L, α), where P and L are two disjoint sets and α is a binary relation which is a subset of $P \times L$ such that the following conditions are satisfied:

(i) to any two elements a, b of P there exists exactly one element $c \in L$ such that $(a, c) \in \alpha$, $(b, c) \in \alpha$;

(ii) to any two elements x, y of L there exists exactly one element $z \in P$ such that $(z, x) \in \alpha$, $(z, y) \in \alpha$;

(iii) there exist four elements a, b, c of P such that no three of them are in the relation α with the same element of L.

The elements of the set P are called *points*, the elements of the set L are called *lines*, the relation α is called *the relation of incidence.*

If the sets P, L are finite, the projective geometry (P, L, α) is called *finite*. It is well-known that for a finite projective geometry there exists a positive integer n such that each point is incident with exactly $n + 1$ lines and each line is incident with exactly $n + 1$ points. Such a projective geometry will be denoted by $P\,G(n)$. The total number of points of $P\,G(n)$ is equal to $n^2 + n + 1$ and the total number of lines is also equal to this number.

It is advantageous to consider $P\,G(n)$ as a bipartite graph on the sets P and L in which a vertex $x \in P$ is adjacent to a vertex $y \in L$ if and only if $(x, y) \in \alpha$.

Choose a vertex $l \in L$ and denote the set of all vertices x of P such that $(x, l) \in \alpha$ by $\Gamma(l)$. By $A\,G(n)$ denote the subgraph of $P\,G(n)$ induced by the set $(P \setminus \Gamma(l)) \cup \cup (L \setminus \{l\})$. The graph $A\,G(n)$ will be called *an affine geometry.* The construction of $A\,G(n)$ corresponds to the interrelation between the projective geometry and the affine one in the plane. The vertex (line) l is an improper line, the vertices (points) from $\Gamma(l)$ are improper points, the vertices (points and lines) of $A\,G(n)$ are proper points and proper lines. In the affine geometry $A\,G(n)$ each point is incident with exactly $n + 1$ lines and each line is incident with exactly n vertices. The total number of points is equal to n^2, the total number of lines is equal to $n^2 + n$.

In this paper we shall introduce a new type of finite geometries, namely, finite spherical geometries. First we shall mention the characteristic properties of the geometry on a sphere. The fundamental concepts of this geometry are points and circles. To any three points there exists exactly one circle which contains all of them. Now choose a point s on a sphere. (We denote it by a lowercase letter, because we do so also for all other points considered in this paper.) Consider a stereographical projection of the sphere with s deleted onto a plane (not containing s) with the centre of projection s. Such a projection gives a one-to-one correspondence between the set of all circles on the sphere which contain s and the set of all lines in the plane. The incidence of points and circles on the sphere corresponds to the incidence of their images in the projection. Hence we can say that the set of all points of the sphere except s and the set of all circles on the sphere which contain s form an affine geometry isomorphic to the affine geometry in the Euclidean plane.

This leads us to a general definition of a spherical geometry. For the sake of simplicity we shall express this definition in the terminology of the graph theory.

A spherical geometry is a bipartite graph on the sets P, C such that the following conditions are satisfied:

(1) to any three distinct vertices of P there exists exactly one vertex of C which is adjacent to all of them;

(2) for each vertex $s \in P$, the subgraph induced by the set $(P \setminus \{s\}) \cup \Gamma(s)$, where $\Gamma(s)$ is the set of vertices which are adjacent to s, is isomorphic to an affine geometry, while the elements of $P \setminus \{s\}$ correspond to the points and the elements of $\Gamma(s)$ correspond to the lines of this affine geometry. The vertices of P are called *points*, the vertices of C are called *circles of the spherical geometry*.

If a spherical geometry is a finite graph, we call it a *finite spherical geometry*. If p is the number of points of a finite spherical geometry, then the set $P \setminus \{s\}$ for each $s \in P$ has $p - 1$ elements. This number must be equal to the number of points of a finite affine geometry $A\,G(n)$, hence $p = n^2 + 1$ for some n. Then $\Gamma(s)$ has $n^2 + n$ elements and thus the degree of each vertex of P is equal to $n^2 + n$. We shall denote a finite spherical geometry by $S\,G(n)$, where n is the number introduced above.

Hence each point is adjacent to $n^2 + n$ circles. Each circle is adjacent to $n + 1$ points, because in the affine geometry induced by the set $(P \setminus \{s\}) \cup \Gamma(s)$ in which it is contained it is adjacent to n vertices and moreover it is adjacent to s. We shall compute the total number of circles in $S\,G(n)$. As each point has the degree $n^2 + n$ and the number of points is $n^2 + 1$, the number of edges of $S\,G(n)$ is $(n^2 + 1)(n^2 + n)$. As each circle has the degree $n + 1$, the total number of circles is equal to $(n^2 + 1) \,.\, (n^2 + n)/(n + 1) = n^3 + n$.

It is evident that $S\,G(n)$ exists only for such n for which $A\,G(n)$ and also $P\,G(n)$ exists. Now let n be a number for which $A\,G(n)$ exists. We shall construct the spherical geometry $S\,G(n)$. Take the point set $P = \{p_1, \ldots, p_q\}$, where $q = n^2 + 1$. Construct an affine geometry $H_1' \cong A\,G(n)$ with the set of points $p_2, \ldots, p_q$; its lines will be the circles of the required $S\,G(n)$ which are adjacent to p_1. Add p_1 to H'_1 and join p_1 with all the lines of H'_1; denote the graph thus obtained by H_1. Now construct an affine geometry H'_2 with the points set $\{p_1, p_3, \ldots, p_q\}$ such that the subgraph of H_1 induced by the union of $P \setminus \{p_2\}$ and the set of all lines adjacent to both p_1 and p_2 is its subgraph and the other lines of H'_2 are not contained in H_1. By H_2 denote the graph obtained from H'_2 by adding p_2 and joining it with all the lines of H'_2. Construct an affine geometry H'_3 with the point set $\{p_1, p_2, p_4, \ldots, p_q\}$ such that the subgraph of $H_1 \cup H_2$ induced by the union of $P \setminus \{p_3\}$ and the set of all lines adjacent to p_3 and simultaneously either to p_1 or p_2 is its subgraph and the other lines of H'_3 are not contained in $H_1 \cup H_2$. By H_3 denote the graph obtained from H'_3 by adding p_3 and joining it with all lines of H'_3. We continue in the same way; at each step we have some bundles of lines of an affine geometry prescribed and we add new lines to complete this geometry, which is always possible. In the end we obtain the required finite spherical geometry $S\,G(n)$.

We have proved the following theorem:

Theorem. *The finite spherical geometry $S\,G(n)$ exists if and only if the projective geometry $P\,G(n)$ exists. Each circle of $S\,G(n)$ is adjacent to $n + 1$ points, each point is adjacent to $n^2 + n$ circles. The total number of points of $S\,G(n)$ is $n^2 + 1$, the total number of circles is $n^3 + n$.*

A spherical geometry is isomorphic to the so-called conform geometry in the plane (see eg. [2]). In the conform geometry one improper point is added to the plane and all lines are considered as circles containing this improper point. Hence if we delete one point from $S\,G(n)$ (considered as corresponding to the improper point), we obtain a geometry in the plane whose elements are points, lines and circles, i.e. an affine geometry enriched by circles.

The simplest finite spherical geometry is $S\,G(2)$. We obtain it by taking five points and assigning a circle to each triple of these points in a one-to-one manner.

References

[1] *Birkhoff, G.:* Lattice Theory. New York 1948.
[2] *Бушманова, Г. В. Норден А. П.:* Элементы конформной геометрии, Казань 1972.

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