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COMPLEMENTS OF CONGRUENCES IN AN Ω-GROUP

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1. In the present note we are concerned with the problem of the existence of complements of congruences in an Ω-group. The notion of a congruence in a universal algebra was introduced in [1] I, where the reader may find a basic information on the object (also see [4–6]). A *congruence* in an algebra G is a stable symmetric and transitive binary relation in G. Symmetric and transitive binary relations in the set G ($\equiv$ *partitions in* G) form a complete lattice, denoted by $P(G)$, with respect to the inclusion; congruences in the algebra G also form a complete lattice $\mathscr{K}(G)$, which is a closed $\wedge$-subsemilattice of $P(G)$. We shall deal with congruences in an Ω-group G which are relative complements of a congruence $C \in \mathscr{K}(G)$ in a given interval $[A, B]$, $A \leqq C \leqq B$ being congruences in G. We shall consider a complement in the lattice $P(G)$ – the socalled *P-complement*, as well as in the lattice $\mathscr{K}(G)$, called the *$\mathscr{K}$-complement* (Definition 1.1). In an analogous way we distinguish a *Dedekind P-complement* and a *Dedekind $\mathscr{K}$-komplement* (Definition 2.1). Criteria for the existence of a relative P-complement are given in 1.5, 1.6 and 1.7. In Theorem 2.7 we show that no congruence is a Dedeking P-complement of a congruence C in $[A, B]$, $A < C < B$.

Let us recall the notation and some results that are needed. Let A be a symmetric and transitive binary relation (ST-relation) in a set G. For $x \in G$ let $A(x) = \{y \in G : yAx\}$ and $\bigcup A = \bigcup\{x \in G : A(x)\}$. If $A(x) \neq \emptyset$ then $A(x)$ is said to be a *block* of A and $\bigcup A$ its *domain*. The set of all blocks of an ST-relation in G is called a *partition in G*. We use the same notation for both the ST-relation and this partition, because there is a $1-1$ correspondence between the set of all ST-relations in G and the set of all partitions in G, as is well known. We shall also find it useful to consider, if need be, the partitions in G as ST-relations and vice versa. If G is an Ω-group then $\bigcup A$ is an Ω-subgroup of G and $A(0)$ is an ideal of $\bigcup A$. If $\{A_\alpha\} \subseteq \mathscr{K}(G)$ and $B = \bigvee_{\alpha \in K} A_\alpha$, then $\bigcup B$ is the Ω-subgroup $\langle \bigcup_\alpha(\bigcup A_\alpha)\rangle$ generated in G by the set $\bigcup_\alpha(\bigcup A_\alpha)$ and $B(0) = \langle\langle \bigcup_\alpha A_\alpha(0)\rangle\rangle_{\cup B}$, the ideal generated in $\bigcup B$ by the set $\bigcup_\alpha A_\alpha(0)$ and $A = \bigcup A/A(0)$ (see 1.4 and 1.6 [1]).

In what follows G means an Ω-group.

1.1 Definition. Let $A \leqq C \leqq B$ be congruences in G. $D \in \mathscr{K}(G)$ is said to be a *relative P-complement* or a *relative $\mathscr{K}$-complement* of C in the interval $[A, B]$, when D is a relative complement of C in $[A, B]$ with respect to the lattice $P(G)$ (i.e. to the lattice operations $\vee_P$, $\wedge_P$) or to the lattice $\mathscr{K}(G)$ i(e. to $\vee_{\mathscr{K}}$, $\wedge_{\mathscr{K}}$), respectively.

1.2 Lemma. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Ω-subgroups of G. If $\mathfrak{A} \cup \mathfrak{B} = \mathfrak{C}$ then the sets $\mathfrak{A}$ and $\mathfrak{B}$ are comparable by inclusion.*

See [2] Lemma 2.2.

1.3 Let $A \leqq B$ be congruences in G. The partition A has evidently a unique relative P-complement in $[A, B]$, namely B. Analogously for B. Hence by studying the relative complementarity we may suppose, without loss of generality, $A < C < B$.

1.4 Lemma. (see [1] 2.8.2). *Let $A < C < B$ be congruences in G and let $D \in \mathscr{K}(G)$ be a relative P-complement of C in $[A, B]$. Then D is a relative $\mathscr{K}$-complement of C in $[A, B]$ and it holds*

(1) $B(0) = C(0) + D(0) = D(0) + C(0)$, $A(0) = C(0) \cap D(0)$

(2) $C(0) + \bigcup A = \bigcup B$, $\bigcup C = \bigcup B$, $\bigcup D = \bigcup A$ *or*

$D(0) + \bigcup A = \bigcup B$, $\bigcup C = \bigcup A$, $\bigcup D = \bigcup B$

(3) $C(0) = A(0) \Leftrightarrow D(0) = B(0) \Leftrightarrow C = \bigcup B/A(0) \Leftrightarrow D = \bigcup A/B(0)$

(4) $C(0) = B(0) \Leftrightarrow D(0) = A(0) \Leftrightarrow C = \bigcup A/B(0) \Leftrightarrow D = \bigcup B/A(0)$.

Proof. D is a relative $\mathscr{K}$-complement of C in $[A, B]$ because

(*) $\quad C \wedge_{\mathscr{K}} D = C \wedge_P D = A \quad$ and $\quad B \geqq C \vee_{\mathscr{K}} D \geqq C \wedge_P D = B$.

(1) By [1] 2.8.2 we have

(**) $\quad \bigcup C = \bigcup A \quad$ and $\quad \bigcup D = \bigcup B \quad$ or $\quad \bigcup C = \bigcup B \quad$ and $\quad \bigcup D = \bigcup A$.

Suppose the first case of (**) occurs. By [2] 1.3 it holds

$$C \vee_P D(0) = [C(0) + \bigcup C \cap D(0)] \cup [\bigcup D \cap C(0) + D(0)] =$$
$$= [C(0) + \bigcup A \cap D(0)] \cup [\bigcup B \cap C(0) + D(0)] = C(0) + D(0).$$

The order of summands in the square brackets may be changed. It follows

$$B(0) = C \vee_P D(0) = C(0) + D(0) = D(0) + C(0).$$

The equality $A(0) = C(0) \cap D(0)$ is evident.

(2) By [2] 1.6 and (**) we have

$$x \in \bigcup D \setminus (D(0) + \bigcup C \cap \bigcup D) = \bigcup B \setminus (D(0) + \bigcup A) \Rightarrow C \vee_P D(x) = \\ = x + D(0) \Rightarrow B(x) = x + B(0) = x + D(0).$$

If $D(0) + \bigcup A \neq \bigcup B$ then $B(0) = D(0)$, hence $C(0) \subseteq D(0)$. By (1) $C(0) = C(0) \cap \cap D(0) = A(0)$, hence $C = A$, a contradiction. Supposing the other case of (**) occurs we obtain the first condition of (2).
(3) Suppose $C(0) = A(0)$. If the first possibility of (**) holds then $C = A$, a contradiction. If the second possibility is true then $C = \bigcup B/A(0)$. By (1) $B(0) = C(0) + + D(0) = A(0) + D(0) = D(0)$, hence by (**) $D = \bigcup A/B(0)$. Now, by (1) $C(0) = = C(0) \cap B(0) = C(0) \cap D(0) = A(0)$.
We get (4) from (3) by interchanging C and D.

1.5 Theorem. *Let $A < C < B$, $D \in [A, B]$ be congruences in G. Then D is a relative P-complement of C in $[A, B]$ iff the following conditions (1) and (2) hold:*

(1) $\bigcup C = \bigcup A$ *and* $\bigcup D = \bigcup B$ *or* $\bigcup C = \bigcup B$ *and* $\bigcup D = \bigcup A$

(2) $C(0) + \bigcup A = \bigcup C$, $D(0) + \bigcup A = \bigcup D$, $C(0) \cap D(0) = A(0)$ *and*

$C(0) + D(0) = B(0)$.

Proof. Let
(α) C have $D \in \mathscr{K}(G)$ as its relative P-complement in $[A, B]$.
By [1] 2.8 it holds

(a) $C(0) + \bigcup A = \bigcup C$ or (b) $B(0) = C(0)$.

Because D has $C \in \mathscr{K}(G)$ for its relative P-complement in $[A, B]$ we have

(a′) $D(0) + \bigcup A = \bigcup D$ or (b′) $B(0) = D(0)$.

By 1.4(2) we have

(a″) $C(0) + \bigcup A = \bigcup B$, $\bigcup C = \bigcup A$ and $\bigcup D = \bigcup B$.

(b)″ $D(0) + \bigcup A = \bigcup B$, $\bigcup C = \bigcup A$ and $\bigcup D = \bigcup B$.

The same Lemma implies

(c) $C(0) \cap D(0) = A(0)$ and $C(0) + D(0) = B(0) = D(0) + C(0)$.

It follows that one of the 8 possibilities a ∧ a′ ∧ a″ to b ∧ b′ ∧ b″ is true. We investigate each of them as follows.
(a ∧ a′ ∧ a″) ∧ c ⇒ (2) and the second condition of (1) (a ∧ b′ ∧ a″) ∧ c. From (b′) and (c) it follows that $C(0) = A(0)$ and hence by (a) $C = (C(0) + \bigcup A)/A(0) = = (A(0) + \bigcup A)/A(0) = A$, a contradiction b ∧ a′ ∧ a″ implies $C = \bigcup B/B(0) = B$, a contradiction, b ∧ b′ ∧ a″ implies $C = \bigcup B/B(0) = B$, a contradiction.

The remaining 4 possibilities are obtained from the above by interchanging C and D. Thus it is proved that (α) $\Rightarrow$ (1) and (2).

Conversely, let (1) and (2) be true. Suppose that the first condition of (1) holds. We shall show $C \vee_{\mathscr{K}} D = C \vee_P D$. Using Lemma 1.6 [2] we obtain (in virtue of the fact that $\bigcup C \cap \bigcup D = \bigcup A$ by (1))

$$\bigcup_{x \in \bigcup A} C \vee_P D(x) = \bigcup_{x \in \bigcup A} [C \vee_P D(0) + x] \supseteq \bigcup_{x \in \bigcup A} [C(0) + x] = C(0) + \bigcup A = \bigcup C\,.$$

Similarly

$$\bigcup_{x \in \bigcup A} C \vee_P D(x) \supseteq D(0) + \bigcup A = \bigcup D\,.$$

Thus the blocks $C \vee_P D(x)$ for $x \in \bigcup A$ cover the set $\bigcup C \cup \bigcup D = \bigcup B \cup \bigcup A = \bigcup B$. So they exhaust all blocks of the partition $C \vee_P D$. Further

$$B(0) \supseteq C \vee_{\mathscr{K}} D(0) = \langle\!\langle C(0), D(0) \rangle\!\rangle_{\langle \cup C, \cup D \rangle} \supseteq C(0) + D(0) = B(0)\,,$$

thus

$$C \vee_{\mathscr{K}} D(0) = C(0) + D(0) = B(0)\,.$$

Finally

$$C \vee_{\mathscr{K}} D(0) \supseteq C \vee_P D(0) \supseteq \bigcup_{x \in D(0)} [C(0) + x] = C(0) + D(0) = C \vee_{\mathscr{K}} D(0)\,,$$

so $C \vee_{\mathscr{K}} D(0) = C \vee_P D(0)$. By [2] 1.3, if $x \in \bigcup C \cap \bigcup D = \bigcup A$ then

$$C \vee_P D(x) = C \vee_P D(0) + x = C \vee_{\mathscr{K}} D(0) + x = C \vee_{\mathscr{K}} D(x)\,,$$

so $C \vee_P D = C \vee_{\mathscr{K}} D$. From the above it is also clear that $C \vee_P D = C \vee_{\mathscr{K}} D = \bigcup B / B(0) = B$. Further, it holds evidently

$$C \wedge_P D = C \wedge_{\mathscr{K}} D = \bigcup C \cap \bigcup D / C(0) \cap D(0) = \bigcup A / A(0) = A\,,$$

which completes the proof of Theorem.

1.6 Theorem. *Let $A < C < B$ be congruences in G and let $D \in \mathscr{K}(G)$ be a relative $\mathscr{K}$-complement of C in $[A, B]$. Then D is a relative P-complement of C in $[A, B]$ iff*

(1) $C(0) + \bigcup A = \bigcup C$ *and/or* $D(0) + \bigcup A = \bigcup D$.

Proof. Let D be a relative $\mathscr{K}$-complement of C in $[A, B]$ and let (1) be true. If we prove $C \vee_P D = C \vee_{\mathscr{K}} D$ then D will be a relative P-complement of C in $[A, B]$ (because $C \wedge_P D = C \wedge_{\mathscr{K}} D$). But this follows from [2] 2.5 since $A \nparallel C \nparallel B$ implies $C \parallel D$.

Now, we give a proof of the stronger version of the converse implication (with "and" in (1)). Let $D \in \mathscr{K}(G)$ be a relative P-complement (and hence also a relative

$\mathscr{K}$-complement) of C in $[A, B]$. Then $\bigcup C \cap \bigcup D = \bigcup A$ and $\bigcup C \cup \bigcup D = \bigcup B$. It follows either $\bigcup A = \bigcup C \cap \bigcup D = \bigcup D$ (hence $\bigcup C = \bigcup B$) or $\bigcup A = \bigcup C \cap \bigcup D = \bigcup C$ (and hence $\bigcup D = \bigcup B$). Now, let $C(0) + \bigcup A \neq \bigcup C$. By [2] 1.6, if $x \in \bigcup C \setminus (C(0) + \bigcup A) = \bigcup C \setminus (C(0) + \bigcup C \cap \bigcup D)$ then $x + C(0) = C \vee_P D(x) = C \vee_{\mathscr{K}} D(x) = x + B(0)$, hence $C(0) = B(0)$. If $\bigcup A = \bigcup C$ then $C(0) + \bigcup A = C(0) + \bigcup C = \bigcup C$, a contradiction. If $\bigcup B = \bigcup C$ then $C = \bigcup B / B(0) = B$, a contradiction. The case $D(0) + \bigcup A \neq \bigcup D$ is symmetric. Hence the stronger version of (1) follows.

(The weaker version of the converse implication (with "or" in (1)) follows immediately from 1.4.)

1.7 Theorem. *Let $A < C < B$ and $D \in [A, B]$ be congruences in G. Then D is a relative P-complement of C in $[A, B]$ iff the following identities hold*

(1) $C \wedge D = A$,

(2) $C(0) + \bigcup A = \bigcup B$ *or* $D(0) + \bigcup A = \bigcup B$,

(3) $C(0) + \bigcup C \cap D(0) = B(0)$ *or* $D(0) + \bigcup D \cap C(0) = B(0)$.

Proof. Necessity. (1) is evident and (2) follows immediately from 1.5.
(3) By 1.4, D is a relative $\mathscr{K}$-complement of C in $[A, B]$, hence by [2] 1.3

$$B(0) = C \vee_{\mathscr{K}} D(0) = C \vee_P D(0) = [C(0) + \bigcup C \cap D(0)] \cup [D(0) + \bigcup D \cap C(0)] .$$

Both members on the right are Ω-subgroups, thus by 1.2 one of them is a subset of the other, i.e.

$$\text{either} \quad B(0) = C(0) + \bigcup C \cap D(0) \quad \text{or} \quad B(0) = D(0) + \bigcup D \cap C(0) .$$

Sufficiency will be proved similarly as that of 1.5. Let the conditions (1), (2) and (3) be fulfilled. We shall prove $C \vee_{\mathscr{K}} D = C \vee_P D = B$.

In virtue of $\bigcup C \cap \bigcup D = \bigcup A$ it follows from [2] 1.6 that

$$\bigcup_{x \in \bigcup A} C \vee_P D(x) = \bigcup_{x \in \bigcup A} [C \vee_P D(0) + x] \supseteq \bigcup_{x \in \bigcup A} [C(0) + x] = C(0) + \bigcup A .$$

Similarly

$$\bigcup_{x \in \bigcup A} C \vee_P D(x) \supseteq D(0) + \bigcup A .$$

One of these sets is equal to $\bigcup B$. Therefore the blocks $C \vee_P D(x)$ for $x \in \bigcup A$ cover the set $\bigcup B$ and thus exhaust all blocks of the partition $C \vee_P D$. Finally, [2] 1.3 implies for $x \in \bigcup C \cap \bigcup D = \bigcup A$

$$B(0) + x \supseteq C \vee_{\mathscr{K}} D(x) \supseteq C \vee_P D(x) =$$
$$= [C(0) + \bigcup C \cap D(0)] \cup [D(0) + \bigcup D \cap C(0)] \supseteq B(0) + x ,$$

thus $B(0) + x = C \vee_{\mathscr{K}} D(x) = C \vee_P D(x)$, i.e. $C \vee_P D = C \vee_{\mathscr{K}} D = \bigcup B/B(0) = B$, which completes the proof of Theorem.

2.1 Definition. Let $A \leqq C \leqq B$ be elements of a lattice S. An element $D \in [A, B]$ is called a *Dedekind complement* of C in $[A, B]$ if (2a) $E = C \vee (D \wedge E)$ for every $C \leqq E \leqq B$, and (2b) $F = D \wedge (C \vee F)$ for every $A \leqq F \leqq D$.

A Dedekind complement of C in $[A, B]$ is a relative complement of C in $[A, B]$ since (2a) for $E = B$ implies $B = C \vee (D \wedge B) = C \vee D$, and (2b) for $F = A$ implies $A = D \wedge (C \vee A) = D \wedge C$.

Note that $C = A$ or $C = B$ has exactly one Dedekind complement D in $[A, B]$, namely $D = B$ or $D = A$, respectively.

2.2 Let A, B, C, D be congruences in an algebra G. There are two types of the Dedekind complement.

D is called a *Dedekind P-complement* of C in $[A, B]$ or a *Dedekind $\mathscr{K}$-complement* of C in $[A, B]$ if D is a Dedekind complement of C in $[A, B]$ referred to the lattice $S = P(G)$ or $S = \mathscr{K}(G)$, respectively.

2.3 Definition. Let C and D be elements of a lattice S. We say that (C, D) is a *modular pair* (in S) and we write (C, D) M, when

$$D \wedge (C \vee F) = (D \wedge C) \vee F \quad \text{for every} \quad F \leqq D\,.$$

Dually, we say that (C, D) is a *dual modular pair* (in S) and we write (C, D) M*, when

$$D \vee (C \wedge E) = (D \vee C) \wedge E \quad \text{for every} \quad D \leqq E\,.$$

See [3] Def. 1.1.

By [3] 1.4, Definition 2.3 can be reformulated as follows:

2.4 Lemma. *(C, D) M iff $D \wedge (C \vee F) = F$ for every $C \wedge D \leqq F \leqq D$ (which means (C, D) M in the lattice $[C \wedge D, C \vee D]$);*

$$(C, D)\,\mathrm{M}^* \quad \textit{iff} \quad D \vee (C \wedge E) = E \quad \textit{for every} \quad D \leqq E \leqq C \vee D$$

(which means (C, D) M in the lattice $[C \wedge D, C \vee D]$).*

2.5 Lemma. *Let $A \leqq C \leqq B$ be elements of a lattice S. An element $D \in [A, B]$ is a Dedekind complement of C in $[A, B]$ iff*

(2a′) *(D, C) M*, i.e. $C \vee (D \wedge E) = (C \vee D) \wedge E$ for every $C \leqq E$,*

(2b′) *(C, D) M, i.e. $D \wedge (C \vee F) = (D \wedge C) \vee F$ for every $F \leqq D$.*

Proof follows from 2.4. The condition (2a), Def. 1.1, is equivalent to (2a′) and (2b) is equivalent to (2b′).

2.6 Note. From 2.6 it follows that the relation "to be a Dedekind complement in $[A, B]$" is symmetric, i.e.

if D is a Dedekind complement of C in $[A, B]$ then
C is a Dedekind complement of D in $[A, B]$.

2.7 Theorem. *Let $A < C < D$ be congruences in G. Then no congruence in G is a Dedekind P-complement of C in $[A, B]$.*

Proof. Let $D \in \mathscr{K}(G)$ be a Dedekind P-complement of C in $[A, B]$. Then D is a relative P-complement of C $[A, B]$ and thus by 1.4

(Q) (1) $\bigcup C = \bigcup A$ and $\bigcup D = \bigcup B$ or (2) $\bigcup C = \bigcup B$ and $\bigcup D = \bigcup A$

and simultaneously

(3) $C(0) \cap D(0) = A(0)$ and $C(0) + D(0) = B(0)$.

By 2.5, D is a Dedekind P-complement of C in $[A, B]$ iff (2a′) and (2b′) are fulfilled, which is equivalent by [1] 2.2 and 2.3.1 to the simultaneous validity of the following conditions (R) and (S):

(R) (a) $D(0) \subseteq \bigcup C$ or (b) $C(0) \subseteq D(0)$,

(S) (α) $D(0) \cap \bigcup C \subseteq C(0) \cap \bigcup D$ or (β) $D(0) \cap \bigcup C \supseteq C(0) \cap \bigcup D$.

The statement $R \wedge S$ is equivalent to one of the following four statements $a \wedge \alpha$ to $b \wedge \beta$:

($a \wedge \alpha \equiv$) $D(0) \subseteq C(0)$,

($a \wedge \beta \equiv$) $\bigcup C \supseteq D(0) \supseteq C(0) \cap \bigcup D$,

($b \wedge \alpha \equiv$) $D(0) \cap \bigcup C \subseteq C(0) \subseteq D(0)$,

($b \wedge \beta \equiv$) $D(0) \supseteq C(0)$.

If we use either the condition (1) or (2) of (Q) we obtain $a \wedge \beta \wedge (1 \vee 2) \Rightarrow$ either $\bigcup A \supseteq D(0) \supseteq C(0)$ or $D(0) \supseteq C(0) \cap \bigcup A$,
$b \wedge \alpha \wedge (1 \vee 2) \Rightarrow$ either $D(0) \cap \bigcup A \subseteq C(0) \subseteq D(0)$ or $D(0) = C(0)$. If we use in addition the condition (3) of (Q) we obtain $a \wedge \alpha \wedge 3 \Rightarrow C(0) = B(0)$ since $C(0) = D(0) + C(0) = B(0)$. It follows that $A(0) = C(0) \cap D(0) = D(0)$, so either $C = \bigcup A/B(0)$ and $D = \bigcup B/A(0)$ (Q1), or $C = \bigcup B/B(0) = B$ by (Q2), a contradiction.
$a \wedge \beta \wedge 1 \wedge 3 \Rightarrow C(0) = A(0)$ for $A(0) = D(0) \cap C(0) = C(0)$. It follows by (Q1) that $C = \bigcup A/A(0) = A$, a contradiction.
$a \wedge \beta \wedge 2 \wedge 3 \Rightarrow A(0) = C(0) \cap D(0) \supseteq C(0) \cap \bigcup A$ $(\supseteq A(0))$. It follows that $A(0) = C(0) \cap \bigcup A$. Moreover, we have $\bigcup C = \bigcup B$ and $\bigcup D = \bigcup A$.
$b \wedge \alpha \wedge 1 \wedge 3 \Rightarrow C(0) = A(0)$ since $A(0) = D(0) \cap C(0) = C(0)$. It follows by (Q1) that $C = \bigcup A/A(0) = A$, a contradiction.

b $\wedge$ α $\wedge$ 2 $\wedge$ 3 $\Rightarrow B(0) = C(0) + D(0) = C(0)$. It follows that $C = \bigcup B/B(0) = B$, a contradiction.

b $\wedge$ β $\wedge$ 3 $\Rightarrow A(0) = C(0)$ because of $A(0) = C(0) \cap D(0) = C(0)$. Hence either $C = A$ by (Q1), a contradiction, or $C = \bigcup B/A(0)$ and $D = \bigcup A/B(0)$ by (Q2) and 1.4.

Let us review the conclusions obtained up to now. We have proved that either $C = \bigcup A/B(0)$ and $D = \bigcup B/A(0)$ or $C = \bigcup B/A(0)$ and $D = \bigcup A/B(0)$ or (a $\wedge$ β $\wedge$ 2 $\wedge$ 3) $A(0) = C(0) \cap \bigcup A$, $\bigcup C = \bigcup B$ and $\bigcup D = \bigcup A$.

We can easily verify that $D = \bigcup B/A(0)$ is a Dedekind P-complement of $C = \bigcup A/B(0)$ only if either $\bigcup A = \bigcup B$ or $A(0) = B(0)$. If $\bigcup A \neq \bigcup B$ and $A(0) \neq B(0)$ then the set $\bigcup B \setminus \bigcup A$ contains two different blocks of the partition $D = \bigcup B/A(0)$. We choose E such that some of its blocks meets these blocks of D. Then (2a) cannot be fulfilled. It follows either $\bigcup A = \bigcup B$ or $A(0) = B(0)$.

By symmetry, the same result is obtained if $C = \bigcup B/A(0)$ and $D = \bigcup A/B(0)$.

Now, if $\bigcup A = \bigcup B$ then $C = B$ and if $A(0) = B(0)$ then $C = A$, a contradiction in both cases.

The remaining case is a $\wedge$ β $\wedge$ 2 $\wedge$ 3,

$$(*) \qquad A(0) = C(0) \cap \bigcup A\,, \quad \bigcup C = \bigcup B \quad \text{and} \quad \bigcup D = \bigcup A\,.$$

(This condition implies a $\wedge$ β, so $(*)$ is a necessary and sufficient condition for some $D \in \mathcal{K}(G)$ to be a Dedekind P-complement of C in $[A, B]$. But we shall show that the condition $(*)$ also leads to a contradiction.)

The congruence D is uniquely determined. In fact, since any Dedekind P-complement $D \in \mathcal{K}(G)$ of C in $[A, B]$ is a relative P-complement of C in $[A, B]$, we have by 1.4 that $B(0) = C(0) + D(0)$. Further, it holds

$$(**) \qquad [C(0) + D(0)] \cap \bigcup A = C(0) \cap \bigcup A + D(0) \cap \bigcup A\,.$$

The inclusion $\supseteq$ is evident. Let us prove the converse inclusion. For an element a on the left it holds $a = c + d \in \bigcup A$ for a suitable $c \in C(0)$ and $d \in D(0)$. Then $c \in \bigcup A - d \subseteq \bigcup A + \bigcup D = \bigcup D = \bigcup A$ (by $(*)$), thus $c \in C(0) \cap \bigcup A$ and hence $a = c + d \in C(0) \cap \bigcup A + D(0) \cap \bigcup A$. Hence the inclusion follows. By $(*)$ and $(**)$ we obtain the null-block $D(0)$ of the partition D as follows: $B(0) \cap \bigcup A = [C(0) + D(0)] \cap \bigcup A = C(0) \cap \bigcup A + D(0) \cap \bigcup A = A(0) + D(0) \cap \bigcup D = A(0) + D(0) = D(0)$. Thus $D = \bigcup A/B(0) \cap \bigcup A$.

By 2.6, the congruence C is a Dedekind P-complement of D in $[A, B]$. As proved above, C is uniquely determined and equal to $\bigcup A/B(0) \cap \bigcup A$. Then $A = C \wedge D$ implies $B(0) \cap \bigcup A = A(0)$. Hence $C = \bigcup A/A(0) = A$, a contradiction. This completes the proof of of Theorem.

References

[1] *T. D. Mai:* Partitions and congruences in algebras. Archivum Math. (Brno) *10* (1974), I 111—122; II 159—172; III 173—187; IV 231—253.

[2] *F. Šik:* Joins of congruences in Ω-groups (to appear).

[3] *F. Maeda* and *S. Maeda:* Theory of symmetric lattices. Berlin—Heidelberg 1970.
[4] *H. Draškovičová:* The lattice of partitions in a set. Acta Fac. Rer. Nat. Univ. Comen. Math. *24* (1970), 37—65.
[5] *O. Borůvka:* The theory of partitions in a set. (In Czech) Publ. Fac. Sci. Univ. Brno, No *278* (1946), 1—37.
[6] *O. Borůvka:* Foundations of the theory of groupoids and groups. Berlin 1974 (in German: Berlin 1960, in Czech: Praha 1962).

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