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SOME REMARKS ON THE NEVANLINNA THEORY OF HOLOMORPHIC MAPPINGS OF RIEMANN SURFACES

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Introduction. This paper contains several remarks to H. Wu's results published in [1] and [2]. For this reason, the notation from [1] will be used here without further comments. For the readers' convenience we mention that the definitions of the fundamental quantities are found also in the Russian translation of [2]: Х. Ву, Теория равнораспределения для голоморфных кривых, Издательство "Мир", Москва 1973, pp. 35–80.

Throughout this paper it is assumed that **V** is an open Riemann surface, **M** is a closed Riemann surface and $f : \mathbf{V} \to \mathbf{M}$ is a holomorphic mapping.

1.

1A. H. WU in [1], p. 508 gave a simple proof, in the case of a parabolic Riemann surface, that $f(\mathbf{V})$ is open dense in **M**. A stronger result is found in [2], p. 47: $\delta(a) = 0$ for almost every $a \in \mathbf{M}$. We extend the latter result to Riemann surfaces admitting *finite harmonic exhaustion.*

1B. Theorem. *Let $f : \mathbf{V} \to \mathbf{M}$ be a holomorphic mapping of an open Riemann surface **V**, admitting finite harmonic exhaustion, with unbounded characteristic function $T(r)$* (i.e. $\lim_{r\to\infty} T(r) = \infty$). *Then*

$$\delta(a) = 0$$

for almost every $a \in \mathbf{M}$.

Proof. Let us denote (here $u_a(z)$ is the *proximity function*, see [1], p. 483)

$$m(r, a) = \frac{1}{2\pi}\int_{\partial V[r]} f^*u_a * \mathrm{d}\tau .$$

It is known (see [1], p. 508) that

$$(1) \qquad \int_M m(r, a)\, \Omega = \text{const.}$$

for every $r \geqq r(\tau)$. Equation (1) and the Fatou lemma yield

$$\liminf_{r \to s} m(r, a) < \infty \tag{2}$$

for a.e. $a \in M$.

Thus for every $a \in \boldsymbol{M} \setminus N$ (where $\int_N \Omega = 0$) there exists a sequence $\{r_i^a\}_{i=1}^{\infty}$ with the following properties:

$$\lim_{i \to \infty} r_i^a = s\,, \quad \lim_{i \to \infty} m(r_i^a, a) = \sigma(a) < \infty\,. \tag{3}$$

If the mapping $f : \boldsymbol{V} \to \boldsymbol{M}$ has unbounded characteristic function $T(r)$, the defect $\delta(a)$ can be defined by

$$\delta(a) = \liminf_{r \to s} \frac{m(r, a)}{T(r)}$$

as is easy to see from First Main Theorem.

From (3) we obtain

$$0 \leqq \delta(a) = \liminf_{r \to s} \frac{m(r, a)}{T(r)} \leqq \liminf_{i \to \infty} \frac{m(r_i^a, a)}{T(r)} = 0\,,$$

and therefore

$$\delta(a) = 0\,,$$

QED.

2. In this paragraph let $\boldsymbol{V}$ denote an open Riemann surface admitting *infinite* harmonic exhaustion, i.e. a parabolic Riemann surface.

2A. H. Wu calls the mapping $f : \boldsymbol{V} \to \boldsymbol{M}$ *transcendental* iff

$$\lim_{r \to \infty} \frac{r}{T(r)} = 0\,. \tag{4}$$

The following interpretation of condition (4) is given in [1], Lemma 8.3, p. 516.

If $\boldsymbol{V}$ is obtained from a compact Riemann surface $\boldsymbol{M}'$ by deleting a finite number of points $m_1, \ldots, m_k$, then $f : \boldsymbol{V} \to \boldsymbol{M}$ is transcendental iff f is not a restriction of a holomorphic mapping $\tilde{f} : \boldsymbol{M}' \to \boldsymbol{M}$.

2B. Another interpretation of the transcendental mapping is possible with the help of the *Weierstrass property.*

Definition. A holomorphic mapping $f : \boldsymbol{V} \to \boldsymbol{M}$ is said to have the *Weierstrass property at the ideal boundary* β of $\boldsymbol{V}$ if the *global cluster set*

$$C_{\boldsymbol{V}}(f) = \bigcap_{r \geqq r_0} \overline{f(\boldsymbol{V} \setminus V[r])}$$

at β is *total,* i.e.

$$C_{\boldsymbol{V}}(f) = \boldsymbol{M}\,.$$

This definition originates from [3], p. 117.

Theorem. *A mapping $f : \mathbf{V} \to \mathbf{M}$ is transcendetal iff f has the Weierstrass property at the ideal boundary β of $\mathbf{V}$.*

Proof. 1. Let f be transcendental. Then $\delta(a) = 0$ almost everywhere on $\mathbf{M}$ by 1A, hence $f(\mathbf{V} \setminus V[r])$ is dense in $\mathbf{M}$ for every $r \geqq r(\tau)$.

2. Conversely, let us assume that f has the Weierstrass property at the ideal boundary β of $\mathbf{V}$. Then there exists a point $a \in \mathbf{M}$ such that $\lim_{r \to \infty} n(r, a) = \infty$ and also $\lim_{r \to \infty} N(r, a) = \infty$. Thus as a consequence of First Main Theorem and because of $m(r, a) \geqq 0$ we have

$$T(r) + \text{const.} \geqq N(r, a)\,. \tag{5}$$

If both sides of Inequality (5) are divided by r, we obtain

$$\frac{T(r) + \text{const.}}{r} \geqq \frac{N(r, a)}{r}\,. \tag{6}$$

Furthermore, l'Hospital's rule yields

$$\lim_{r \to \infty} \frac{T(r)}{r} = \lim_{r \to \infty} \frac{T(r) + \text{const.}}{r} \geqq \lim_{r \to \infty} \frac{N(r, a)}{r} = \lim_{r \to \infty} n(r, a) = \infty\,,$$

(or we can proceed without using l'Hospital's rule, see [4];

$$\lim_{r \to \infty} \frac{N(r, a)}{r} = \lim_{r \to \infty} \frac{\int_{r_0}^{r} n(t, a)\, dt}{r} \geqq \lim_{r \to \infty} \frac{\int_{r/2}^{r} n(t, a)\, dt}{r} \geqq \lim_{r \to \infty} \frac{r/2}{r}\, n(r/2, a) = \infty)$$

QED.

3. In this paragraph, let $\mathbf{V}$ denote an open Riemann surface admitting *finite* harmonic exhaustion.

Theorem. *Let $f : \mathbf{V} \to \mathbf{M}$ be a holomorphic mapping with unbounded characteristic function $T(r)$. Then f has the Weierstrass property at the ideal boundary β of $\mathbf{V}$.*

Proof is an easy consequence of Theorem 1B.

4. In view of Theorem 3 we introduce the following definition.

Definition. *Let $f : \mathbf{V} \to \mathbf{M}$ be a holomorphic mapping from an open Riemann surface $\mathbf{V}$ having finite or infinite harmonic exhaustion, into $\mathbf{M}$. The mapping f is called transcendental iff*

$$\lim_{r \to s} \frac{T(r)}{r} = \infty\,. \tag{7}$$

Remark. For the case $s = \infty$, condition (7) is equivalent with condition (4). For $s < \infty$, condition (7) is equivalent with $\lim_{r \to s} T(r) = \infty$.

Thus, if $f : \mathbf{V} \to \mathbf{M}$ is transcendental in the sense of our definition, then f has the Weierstrass property at the ideal boundary β of $\mathbf{V}$. Hence the boundary β of $\mathbf{V}$ behaves as an essential singularity of the mapping f.

5. In this paragraph only open Riemann surfaces with *finite Euler characteristic* $\chi(\mathbf{V})$ are considered.

5A. If $f : \mathbf{V} \to \mathbf{M}$ is a transcendental mapping from a parabolic Riemann surface into $\mathbf{M}$, then the right hand side of the defect relation

$$\sum_{a \in \mathbf{M}} \delta(a) \leqq \chi(\mathbf{M}) + \chi, \tag{8}$$

is finite, i.e. the set of deficient values is at most countable.

In the case of a Riemann surface with *finite* harmonic exhaustion, the condition of transcendency does not ensure the finiteness of the right hand side of the defect relation

$$\sum_{a \in \mathbf{M}} \delta(a) \leqq \chi(\mathbf{M}) + \chi + \varepsilon. \tag{9}$$

The finiteness of the right hand side of this relation is ensured by the following condition:

$$\lim_{r \to s} \frac{\log \dfrac{1}{s - r}}{T(r)} = 0. \tag{10}$$

5B. In the following, an interpretation of condition (10) is proposed.

Theorem. *If $f : \mathbf{V} \to \mathbf{M}$ is a holomorphic mapping of an open Riemann surface, admitting finite harmonic exhaustion, into $\mathbf{M}$, for which condition (10) is valid, then the covering surface $(\mathbf{M})_f^{\mathbf{V}}$ is regularly exhaustible.*

Proof. If the generalized L'Hospital's rule (see Lemma 8.7 in [1]) is applied to equation (10), we obtain

$$\liminf_{r \to s} \frac{1}{(s - r)\, v(r)} = 0. \tag{11}$$

Equation (11) proves our Theorem, see [3], p. 170, 18D. QED.

6. In [3], the following theorem has been proved (see [3], p. 118):

6A. Theorem. *Let $\mathbf{V}$ be a parabolic Riemann surface. Every meromorphic function on $\mathbf{V}$ with the Weierstrass property assumes every value infinitely many times in $\mathbf{V}$ except perhaps for a countable union of compact sets of capacity zero.*

6B. It is possible to generalize this theorem to the case of a holomorphic mapping from an open Riemann surface admitting *finite or infinite* harmonic exhaustion, into an *arbitrary closed* Riemann surface $\mathbf{M}$.

Theorem. *Let* $\mathbf{V}$ *be an open Riemann surface admitting finite or infinite harmonic exhaustion, and* $\mathbf{M}$ *a compact Riemann surface. Every transcendental holomorphic mapping* $f : \mathbf{V} \to \mathbf{M}$ *assumes every value infinitely many times in* $\mathbf{V}$ *except perhaps for a countable union of compact sets of capacity zero.*

Proof. If K_n,

$$K_n = \{a \in \mathbf{M};\ n(r, a) \leqq n,\ r \in (r_0, s)\},$$

is of positive capacity then there exists a compact set $K \subset K_n$ such that K is of positive capacity and contained in an open set U_0. The set U_0 is determined by the following conditions: $U_0 \subset U$, where $\{U, \varphi\}$ is a chart for which

$$\varphi(\overline{U}) = \{z \in \mathbf{C},\ |z| \leqq 1\},\quad \varphi(U_0) = \{z \in \mathbf{C},\ |z| < \tfrac{1}{2}\}.$$

Let $g(z, a)$ denote *Green's function* of the region U with a pole at $a \in U$. For $a \in U$, $z \in \mathbf{M} \setminus U$ we put $g(z, a) \equiv 0$.

First we prove the following assertion: *For* $(z, a) \in \mathbf{M} \times U_0$,

$$u_a(z) \leqq g(z, a) + \text{const.} \tag{16}$$

holds.

The function $u_a(z)$ is, as a function of two variables (z, a), continuous on the compact set $(\mathbf{M} \setminus U) \times \overline{U}_0$ (see Theorems 2.1 and 2.8 in [1]). Thus for $(z, a) \in (\mathbf{M} \setminus U) \times \overline{U}_0$ $u_a(z)$ is bounded, i.e. $u_a(z) \leqq$ const.. For $(z, a) \in \overline{U} \times \overline{U}_0$ it is

$$u_a(z) = \log \frac{1}{|z - z(a)|} + \phi_a(z), \tag{17}$$

where $\phi_a(z)$ is a continuous function of two variables (z, a) on the compact set $\overline{U} \times \overline{U}_0$. Thus for $(z, a) \in \overline{U} \times \overline{U}_0$ we have $\phi_a(z) \leqq$ const.

Because $g(z, a)$ is expressed in U as

$$g(z, a) = \log \frac{1}{|z - z(a)|} + v(z, a), \tag{18}$$

where $v(z, a)$ is a harmonic function in a neighborhood of the point a, the validity of inequality (16) is evident.

Let μ be the *equilibrium measure* on K (for definition see C. Constantinescu, and A. Cornea: Ideale Ränder Riemannscher Flächen, p. 48). Then

$$\int_K g(z, a)\, d\mu(a) \leqq 1 \quad \text{for} \quad z \in U.$$

Thus (16) implies

$$\int_K u_a(z)\, d\mu(a) \leqq \text{const.} \quad \text{for} \quad z \in \mathbf{M}.$$

Hence

$$\int_K m(r, a)\, d\mu(a) = \int_K \left[\int_{\partial V[r]} f^* u_a * d\tau\right] d\mu(a) = \int_{\partial V[r]} \left[\int_K u_a \circ f\, d\mu(a)\right] * d\tau = O(1)\,.$$

Furthermore,

$$\int_K N(r, a)\, d\mu(a) = \int_K \left[\int_{r_0}^{r} n(t, a)\, dt\right] d\mu(a) =$$

$$= \int_{r_0}^{r} \left[\int_K n(t, a)\, d\mu(a)\right] dt \leqq \text{const.}\, r = O(r)\,.$$

Evidently

$$\int_K T(r)\, d\mu(a) = T(r)\, \mu(K)\,.$$

From First Main Theorem we obtain

$$T(r) = O(1) + O(r)\,,$$

which contradicts the assumption of f being transcendental. Therefore the set K is of capacity zero. QED.

References

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