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Časopis pro pěstování matematiky, Vol. 105 (1980), No. 3, 266--277

Časopis pro pěstování matematiky, roč. 105 (1980), Praha

CHARACTERIZATIONS OF THE SPHERE IN E^4 BY MEANS OF THE PSEUDOPARALLEL MEAN CURVATURE VECTOR FIELD

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(Received December 12, 1977)

In [2], we have characterized the 2-dimensional sphere in E^4 using the notion of the parallelness of the mean curvature normal field. In the present paper, we are going to introduce the concept of the pseudoparallelness of the mean curvature vector field and to apply it to some characterizations of the sphere in E^4.

1. Let M be a surface in the 4-dimensional Euclidean space E^4 and ∂M its boundary. Let $\{U_\alpha\}$ be an open covering of M such that in each domain U_α there is an orthonormal frame $\{M; v_1, v_2, v_3, v_4\}$ with $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$ where $T(M)$,$N(M)$ denote the tangent and the normal bundle of M, respectively. Then we have

$$\begin{aligned}(1)\qquad \mathrm{d}M &= \omega^1 v_1 + \omega^2 v_2\,,\\ \mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4\,,\\ \mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4\,,\\ \mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4\,,\\ \mathrm{d}v_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3\,;\end{aligned}$$

$$\begin{aligned}(2)\qquad &\mathrm{d}\omega^i = \omega^k \wedge \omega_k^i\,, \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j\,,\\ &\omega_i^j + \omega_j^i = 0\,, \quad \omega^3 = \omega^4 = 0 \quad (i, j, k = 1, 2, 3, 4)\,.\end{aligned}$$

The well-known prolongation procedure implies further the existence of real-valued functions a_i, b_i, c_i $(i = 1, 2)$, $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 1, 2)$ and $A_i, B_i, \ldots, E_i$ $(i = 1, 2)$ in each U_α such that

$$\begin{aligned}(3)\qquad &\omega_1^3 = a_1\omega^1 + b_1\omega^2\,, \quad \omega_2^3 = b_1\omega^1 + c_1\omega^2\,,\\ &\omega_1^4 = a_2\omega^1 + b_2\omega^2\,, \quad \omega_2^4 = b_2\omega^1 + c_2\omega^2\,;\end{aligned}$$

$$\begin{aligned}(4)\qquad &\mathrm{d}a_1 - 2b_1\omega_1^2 - a_2\omega_3^4 = \alpha_1\omega^1 + \beta_1\omega^2\,,\\ &\mathrm{d}b_1 + (a_1 - c_1)\,\omega_1^2 - b_2\omega_3^4 = \beta_1\omega^1 + \gamma_1\omega^2\,,\end{aligned}$$

$$dc_1 + 2b_1\omega_1^2 - c_2\omega_3^4 = \gamma_1\omega^1 + \delta_1\omega^2 ,$$
$$da_2 - 2b_2\omega_1^2 + a_1\omega_3^4 = \alpha_2\omega^1 + \beta_2\omega^2 ,$$
$$db_2 + (a_2 - c_2)\,\omega_1^2 + b_1\omega_3^4 = \beta_2\omega^1 + \gamma_2\omega^2 ,$$
$$dc_2 + 2b_2\omega_1^2 + c_1\omega_3^4 = \gamma_2\omega^1 + \delta_2\omega^2 ;$$

$$\begin{aligned}(5)\qquad & d\alpha_1 - 3\beta_1\omega_1^2 - \alpha_2\omega_3^4 = A_1\omega^1 + (B_1 - b_1K - \tfrac{1}{2}a_2k)\,\omega^2 ,\\
& d\beta_1 + (\alpha_1 - 2\gamma_1)\,\omega_1^2 - \beta_2\omega_3^4 = (B_1 + b_1K + \tfrac{1}{2}a_2k)\,\omega^1 + \\
& \qquad + (C_1 + a_1K - \tfrac{1}{2}b_2k)\,\omega^2 ,\\
& d\gamma_1 + (2\beta_1 - \delta_1)\,\omega_1^2 - \gamma_2\omega_3^4 = (C_1 + c_1K + \tfrac{1}{2}b_2k)\,\omega^1 + \\
& \qquad + (D_1 + b_1K - \tfrac{1}{2}c_2k)\,\omega^2 ,\\
& d\delta_1 + 3\gamma_1\omega_1^2 - \delta_2\omega_3^4 = (D_1 - b_1K + \tfrac{1}{2}c_2k)\,\omega^1 + E_1\omega^2 ,\\
& d\alpha_2 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 = A_2\omega^1 + (B_2 - b_2K + \tfrac{1}{2}a_1k)\,\omega^2 ,\\
& d\beta_2 + (\alpha_2 - 2\gamma_2)\,\omega_1^2 + \beta_1\omega_3^4 = (B_2 + b_2K - \tfrac{1}{2}a_1k)\,\omega^1 + \\
& \qquad + (C_2 + a_2K + \tfrac{1}{2}b_1k)\,\omega^2 ,\\
& d\gamma_2 + (2\beta_2 - \delta_2)\,\omega_1^2 + \gamma_1\omega_3^4 = (C_2 + c_2K - \tfrac{1}{2}b_1k)\,\omega^1 + \\
& \qquad + (D_2 + b_2K + \tfrac{1}{2}c_1k)\,\omega^2 ,\\
& d\delta_2 + 3\gamma_2\omega_1^2 + \delta_1\omega_3^4 = (D_2 - b_2K - \tfrac{1}{2}c_1k)\,\omega^1 + E_2\omega^2\end{aligned}$$

where

$$(6)\qquad K = a_1c_1 - b_1^2 + a_2c_2 - b_2^2 , \quad k = (a_1 - c_1)\,b_2 - (a_2 - c_2)\,b_1 ,$$

the function K being the Gauss curvature of M.

Denote by

$$(7)\qquad H = (a_1 + c_1)^2 + (a_2 + c_2)^2$$

the mean curvature of M and by

$$(8)\qquad \xi = (a_1 + c_1)\,v_3 + (a_2 + c_2)\,v_4$$

the mean curvature vector field in $N(M)$. In the following suppose that $\xi \neq 0$, and thus $H \neq 0$ on M.

Let $P(M)$ be the vector bundle on M such that $P_m(M)$ is the union of $T_m(M)$ and ξ_m for each point $m \in M$. The vector field ξ is said to be *pseudoparallel* in $P(M)$ if $t\xi \in P(M)$ for each vector field $t \in T(M)$.

From (8) we see that

$$d\xi = [(\alpha_1 + \gamma_1)\,\omega^1 + (\beta_1 + \delta_1)\,\omega^2]\,v_3 + [(\alpha_2 + \gamma_2)\,\omega^1 + (\beta_2 + \delta_2)\,\omega^2]\,v_4$$
$$(\text{mod } v_1, v_2)$$

and thus ξ is pseudoparallel if and only if

$$(a_1 + c_1)(\alpha_2 + \gamma_2) - (a_2 + c_2)(\alpha_1 + \gamma_1) = 0\,, \tag{9}$$
$$(a_1 + c_1)(\beta_2 + \delta_2) - (a_2 + c_2)(\beta_1 + \delta_1) = 0\,.$$

Let us remark that, according to (9),

$$(\alpha_1 + \gamma_1)(\beta_2 + \delta_2) - (\beta_1 + \delta_1)(\alpha_2 + \gamma_2) = 0\,. \tag{10}$$

Now, we have the following

Lemma 1. *Let the mean curvature vector field ξ be pseudoparallel in $P(M)$. Then*

$$k = 0$$

on M.

Proof. Differentiating (9) and using (4), (5) and (10), we obtain

$$\begin{aligned}(11)\qquad & (a_1 + c_1)(A_2 + C_2 + c_2K - \tfrac{1}{2}b_1k) - \\ & - (a_2 + c_2)(A_1 + C_1 + c_1K + \tfrac{1}{2}b_2k) = 0\,, \\ & (a_1 + c_1)[B_2 + D_2 + \tfrac{1}{2}(a_1 + c_1)k] - \\ & - (a_2 + c_2)[B_1 + D_1 - \tfrac{1}{2}(a_2 + c_2)k] = 0\,, \\ & (a_1 + c_1)[B_2 + D_2 - \tfrac{1}{2}(a_1 + c_1)k] - \\ & - (a_2 + c_2)[B_1 + D_1 + \tfrac{1}{2}(a_2 + c_2)k] = 0\,, \\ & (a_1 + c_1)(C_2 + E_2 + a_2K + \tfrac{1}{2}b_1k) - \\ & - (a_2 + c_2)(C_1 + E_1 + a_1K - \tfrac{1}{2}b_2k) = 0\,,\end{aligned}$$

These equations yield immediately the assertion.

2. In the theorems proved in this paper, we use the 1-form

$$\begin{aligned}(12)\qquad \tau &= \tau_1\omega^1 + \tau_2\omega^2 = \\ &= [(a_1 - c_1)\beta_1 + (a_2 - c_2)\beta_2 - b_1(\alpha_1 - \gamma_1) - b_2(\alpha_2 - \gamma_2)]\,\omega^1 + \\ &+ [(a_1 - c_1)\gamma_1 + (a_2 - c_2)\gamma_2 - b_1(\beta_1 - \delta_1) - b_2(\beta_2 - \delta_2)]\,\omega^2\,.\end{aligned}$$

By exterior differentiation of τ we get

$$d\tau = -[2J + (H - 4K)K - 2k^2]\,\omega^1 \wedge \omega^2 \tag{13}$$

where

$$J = \beta_1(\beta_1 - \delta_1) + \gamma_1(\gamma_1 - \alpha_1) + \beta_2(\beta_2 - \delta_2) + \gamma_2(\gamma_2 - \alpha_2)\,. \tag{14}$$

In this section we examine the dependence of the form τ and the function J on the choice of tangent frames of M. For this reason, consider another field of frames $\{M; \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ given, in each U_α, by

$$(15) \qquad \begin{aligned} v_1 &= \varepsilon_1 \cos \varrho \,.\, \bar{v}_1 - \sin \varrho \,.\, \bar{v}_2 \,, \quad v_3 = \varepsilon_2 \cos \sigma \,.\, \bar{v}_3 - \sin \sigma \,.\, \bar{v}_4 \,, \\ v_2 &= \varepsilon_1 \sin \varrho \,.\, \bar{v}_1 + \cos \varrho \,.\, \bar{v}_2 \,, \quad v_4 = \varepsilon_2 \sin \sigma \,.\, \bar{v}_3 + \cos \sigma \,.\, \bar{v}_4 \,; \end{aligned}$$

$$(\varepsilon_1^2 = \varepsilon_2^2 = 1)$$

and denote by a bar all functions and formulas related to $\{M; \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$.

Lemma 2. *On M, it is*

$$\bar{\tau} = \varepsilon_1 \tau \,, \quad \bar{J} = J \,.$$

Proof. From (1) and $\mathrm{d}M = \bar{\omega}^i \bar{v}_i$, $\mathrm{d}\bar{v}_i = \bar{\omega}_i^j \bar{v}_j$ we obtain

$$(16) \qquad \begin{aligned} \bar{\omega}^1 &= \varepsilon_1(\cos \varrho \,.\, \omega^1 + \sin \varrho \,.\, \omega^2) \,, \\ \bar{\omega}^2 &= -\sin \varrho \,.\, \omega^1 + \cos \varrho \,.\, \omega^2 \,; \end{aligned}$$

$$(17) \qquad \begin{aligned} \bar{\omega}_1^2 &= \varepsilon_1(\mathrm{d}\varrho + \omega_1^2) \,, \\ \bar{\omega}_3^4 &= \varepsilon_2(\mathrm{d}\sigma + \omega_3^4) \,; \end{aligned}$$

$$(18) \qquad \begin{aligned} \bar{\omega}_1^3 &= \varepsilon_1\varepsilon_2(\cos \varrho \cos \sigma \,.\, \omega_1^3 + \sin \varrho \cos \sigma \,.\, \omega_2^3 + \\ &\quad + \cos \varrho \sin \sigma \,.\, \omega_1^4 + \sin \varrho \sin \sigma \,.\, \omega_2^4) \,, \\ \bar{\omega}_2^3 &= \varepsilon_2(-\sin \varrho \cos \sigma \,.\, \omega_1^3 + \cos \varrho \cos \sigma \,.\, \omega_2^3 - \\ &\quad - \sin \varrho \sin \sigma \,.\, \omega_1^4 + \cos \varrho \sin \sigma \,.\, \omega_2^4) \,, \\ \bar{\omega}_1^4 &= \varepsilon_1(-\cos \varrho \sin \sigma. \, \omega_1^3 - \sin \varrho \sin \sigma \,.\, \omega_2^3 + \\ &\quad + \cos \varrho \cos \sigma \,.\, \omega_1^4 + \sin \varrho \cos \sigma \,.\, \omega_2^4) \,, \\ \bar{\omega}_2^4 &= \sin \varrho \sin \sigma \,.\, \omega_1^3 - \cos \varrho \sin \sigma \,.\, \omega_2^3 - \\ &\quad - \sin \varrho \cos \sigma \,.\, \omega_1^4 + \cos \varrho \cos \sigma \,.\, \omega_2^4 \,. \end{aligned}$$

Thus from (16), (18) using (3) and ($\bar{3}$) we get

$$(19) \qquad \begin{aligned} \bar{a}_1 &= \varepsilon_2(R_1 \cos \sigma + R_2 \sin \sigma) \,, \\ \bar{b}_1 &= \varepsilon_1\varepsilon_2(S_1 \cos \sigma + S_2 \sin \sigma) \,, \\ \bar{c}_1 &= \varepsilon_2(T_1 \cos \sigma + T_2 \sin \sigma) \,, \\ \bar{a}_2 &= -(R_1 \sin \sigma - R_2 \cos \sigma) \,, \\ \bar{b}_2 &= \varepsilon_1(S_1 \sin \sigma - S_2 \cos \sigma) \,, \\ \bar{c}_2 &= -(T_1 \sin \sigma - T_2 \cos \sigma) \end{aligned}$$

where

$$(20)\quad \begin{aligned} R_1 &= R_1(a_1, b_1, c_1) = a_1 \cos^2 \varrho + 2b_1 \sin \varrho \cos \varrho + c_1 \sin^2 \varrho\,, \\ S_1 &= S_1(a_1, b_1, c_1) = a_1 \sin \varrho \cos \varrho + b_1(\sin^2 \varrho - \cos^2 \varrho) - c_1 \sin \varrho \cos \varrho\,, \\ T_1 &= T_1(a_1, b_1, c_1) = a_1 \sin^2 \varrho - 2b_1 \sin \varrho \cos \varrho + c_1 \cos^2 \varrho\,, \end{aligned}$$

and $R_2 = R_2(a_2, b_2, c_2)$, $S_2 = S_2(a_2, b_2, c_2)$, $T_2 = T_2(a_2, b_2, c_2)$ have the same meaning. Further, from (17), (19) according to (5) and ($\bar{5}$) we have

$$(21)\quad \begin{aligned} \bar{\alpha}_1 &= \quad \varepsilon_1\varepsilon_2 \cos \sigma(R_1^\alpha \cos \varrho + R_1^\beta \sin \varrho) + \varepsilon_1\varepsilon_2 \sin \sigma(R_2^\alpha \cos \varrho + R_2^\beta \sin \varrho)\,, \\ \bar{\beta}_1 &= -\varepsilon_2 \cos \sigma(R_1^\alpha \sin \varrho - R_1^\beta \cos \varrho) - \varepsilon_2 \sin \sigma(R_2^\alpha \sin \varrho - R_2^\beta \cos \varrho) = \\ &= -\varepsilon_2 \cos \sigma(S_1^\alpha \cos \varrho + S_1^\beta \sin \varrho) - \varepsilon_2 \sin \sigma(S_2^\alpha \cos \varrho + S_2^\beta \sin \varrho)\,, \\ \bar{\gamma}_1 &= \quad \varepsilon_1\varepsilon_2 \cos \sigma(S_1^\alpha \sin \varrho - S_1^\beta \cos \varrho) + \varepsilon_1\varepsilon_2 \sin \sigma(S_2^\alpha \sin \varrho - S_2^\beta \cos \varrho) = \\ &= \quad \varepsilon_1\varepsilon_2 \cos \sigma(T_1^\alpha \cos \varrho + T_1^\beta \sin \varrho) + \varepsilon_1\varepsilon_2 \sin \sigma(T_2^\alpha \cos \varrho + T_2^\beta \sin \varrho)\,, \\ \bar{\delta}_1 &= -\varepsilon_2 \cos \sigma(T_1^\alpha \sin \varrho - T_1^\beta \cos \varrho) - \varepsilon_2 \sin \sigma(T_2^\alpha \sin \varrho - T_2^\beta \cos \varrho)\,, \\ \bar{\alpha}_2 &= -\varepsilon_1 \sin \sigma(R_1^\alpha \cos \varrho + R_1^\beta \sin \varrho) + \varepsilon_1 \cos \sigma(R_2^\alpha \cos \varrho + R_2^\beta \sin \varrho)\,, \\ \bar{\beta}_2 &= \quad \sin \sigma(R_1^\alpha \sin \varrho - R_1^\beta \cos \varrho) - \cos \sigma(R_2^\alpha \sin \varrho - R_2^\beta \cos \varrho) = \\ &= \quad \sin \sigma(S_1^\alpha \cos \varrho + S_1^\beta \sin \varrho) - \cos \sigma(S_2^\alpha \cos \varrho + S_2^\beta \sin \varrho)\,, \\ \bar{\gamma}_2 &= -\varepsilon_1 \sin \sigma(S_1^\alpha \sin \varrho - S_1^\beta \cos \varrho) + \varepsilon_1 \cos \sigma(S_2^\alpha \sin \varrho - S_2^\beta \cos \varrho) = \\ &= -\varepsilon_1 \sin \sigma(T_1^\alpha \cos \varrho + T_1^\beta \sin \varrho) + \varepsilon_1 \cos \sigma(T_2^\alpha \cos \varrho + T_2^\beta \sin \varrho)\,, \\ \bar{\delta}_2 &= \quad \sin \sigma(T_1^\alpha \sin \varrho - T_1^\beta \cos \varrho) - \cos \sigma(T_2^\alpha \sin \varrho - T_2^\beta \cos \varrho) \end{aligned}$$

where

$$(22)\quad \begin{aligned} R_i^\alpha &= R_i(\alpha_i, \beta_i, \gamma_i)\,, \quad S_i^\alpha = S_i(\alpha_i, \beta_i, \gamma_i)\,, \quad T_i^\alpha = T_i(\alpha_i, \beta_i, \gamma_i)\,, \\ R_i^\beta &= R_i(\beta_i, \gamma_i, \delta_i)\,, \quad S_i^\beta = S_i(\beta_i, \gamma_i, \delta_i)\,, \quad T_i^\beta = T_i(\beta_i, \gamma_i, \delta_i) \quad (i = 1, 2)\,. \end{aligned}$$

In virtue of (19) and (21) we get

$$(23)\quad \begin{aligned} \bar{a}_1 - \bar{c}_1 &= \quad \varepsilon_2(R_1 - T_1) \cos \sigma + \varepsilon_2(R_2 - T_2) \sin \sigma\,, \\ \bar{a}_2 - \bar{c}_2 &= -(R_1 - T_1) \sin \sigma + (R_2 - T_2) \cos \sigma\,, \\ \bar{\alpha}_1 - \bar{\gamma}_1 &= \quad \varepsilon_1\varepsilon_2 \cos \sigma[(R_1^\alpha - T_1^\alpha) \cos \varrho + (R_1^\beta - T_1^\beta) \sin \varrho] + \\ &\qquad + \varepsilon_1\varepsilon_2 \sin \sigma[(R_2^\alpha - T_2^\alpha) \cos \varrho + (R_2^\beta - T_2^\beta) \sin \varrho]\,, \\ \bar{\beta}_1 - \bar{\delta}_1 &= -\varepsilon_2 \cos \sigma[(R_1^\alpha - T_1^\alpha) \sin \varrho - (R_1^\beta - T_1^\beta) \cos \varrho] - \\ &\qquad - \varepsilon_2 \sin \sigma[(R_2^\alpha - T_2^\alpha) \sin \varrho - (R_2^\beta - T_2^\beta) \cos \varrho]\,, \end{aligned}$$

$$\bar{\alpha}_2 - \bar{\gamma}_2 = -\varepsilon_1 \cos\sigma[(R_1^\alpha - T_1^\alpha)\cos\varrho + (R_1^\beta - T_1^\beta)\sin\varrho] + \\ + \varepsilon_1 \sin\sigma[(R_2^\alpha - T_2^\alpha)\cos\varrho + (R_2^\beta - T_2^\beta)\sin\varrho],$$

$$\bar{\beta}_2 - \bar{\delta}_2 = \sin\sigma[(R_1^\alpha - T_1^\alpha)\sin\varrho - (R_1^\beta - T_1^\beta)\cos\varrho] - \\ - \cos\sigma[(R_2^\alpha - T_2^\alpha)\sin\varrho - (R_2^\beta - T_2^\beta)\cos\varrho].$$

Using (19), (21) and (23), we obtain

$$\bar{\tau}_1 = (U_1^\alpha + U_2^\alpha)\cos\varrho + (U_1^\beta + U_2^\beta)\sin\varrho,$$
$$\bar{\tau}_2 = -\varepsilon_1(U_1^\alpha + U_2^\alpha)\sin\varrho + \varepsilon_1(U_1^\beta + U_2^\beta)\cos\varrho$$

where

$$U_i^\alpha = S_i(R_i^\alpha - T_i^\alpha) - S_i^\alpha(R_i - T_i),$$
$$U_i^\beta = S_i(R_i^\beta - T_i^\beta) - S_i^\beta(R_i - T_i) \quad (i = 1, 2).$$

However, direct calculation yields

$$U_1^\alpha = (a_1 - c_1)\beta_1 - b_1(\alpha_1 - \gamma_1),$$
$$U_1^\beta = (a_1 - c_1)\gamma_1 - b_1(\beta_1 - \delta_1)$$

and analogous relations for U_2^α, U_2^β. Thus

$$\bar{\tau}_1 = \tau_1\cos\varrho + \tau_2\sin\varrho,$$
$$\bar{\tau}_2 = -\varepsilon_1\tau_1\sin\varrho + \varepsilon_1\tau_2\cos\varrho$$

which proves, together with (16), the first identity of our lemma.

To prove the other one, introduce the symbols V_1, V_2 by the relation

$$V_i = -S_i^\alpha(R_i^\beta - T_i^\beta) + S_i^\beta(R_i^\alpha - T_i^\alpha) \quad (i = 1, 2).$$

Using (21), (23), we get

$$\bar{J} = V_1 + V_2.$$

On the other hand, according to (20) and (22) we have

$$V_1 = \beta_1(\beta_1 - \delta_1) + \gamma_1(\gamma_1 - \alpha_1)$$

and analogously for V_2. This and the equation (14) complete the proof.

3. The main tool used in the proofs of the theorems contained in this paper is the Stokes theorem asserting

$$\int_{\partial M} \tau = \int_M \mathrm{d}\tau \tag{24}$$

for any 1-form τ on M. Assuming that ∂M consists of umbilical points ($a_1 - c_1 = 0$,

$a_2 - c_2 = 0, b_1 = 0, b_2 = 0$), we have $\tau = 0$ on ∂M and, according to (24) and (13),

$$\int_M [2(J - k^2) + (H - 4K) K] \omega^1 \wedge \omega^2 = 0 . \tag{25}$$

Now, we are going to prove the first theorem characterizing the sphere among surfaces in E^4:

Theorem 1. *Let M be a surface in E^4 and ∂M its boundary. Let*

(i) *$K > 0$ on M;*

(ii) *ξ be pseudoparallel in $P(M)$;*

(iii) *$H =$ const. $\neq 0$ on M;*

(iv) *∂M consist of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. From (4) and (7) we have

$$\begin{aligned} dH = {} & 2[(a_1 + c_1)(\alpha_1 + \gamma_1) + (a_2 + c_2)(\alpha_2 + \gamma_2)] \omega^1 + \\ & + 2[(a_1 + c_1)(\beta_1 + \delta_1) + (a_2 + c_2)(\beta_2 + \delta_2)] \omega^2 \end{aligned}$$

and according to (ii), (iii) we obtain (9) and

$$\begin{aligned} (a_1 + c_1)(\alpha_1 + \gamma_1) + (a_2 + c_2)(\alpha_2 + \gamma_2) &= 0 , \\ (a_1 + c_1)(\beta_1 + \delta_1) + (a_2 + c_2)(\beta_2 + \delta_2) &= 0 . \end{aligned}$$

As $H \neq 0$, this system of equations has the only solution

$$\begin{aligned} \alpha_1 + \gamma_1 = 0 , \quad & \beta_1 + \delta_1 = 0 , \\ \alpha_2 + \gamma_2 = 0 , \quad & \beta_2 + \delta_2 = 0 . \end{aligned} \tag{26}$$

Then the relation (14) has the form

$$J = 2(\beta_1^2 + \gamma_1^2 + \beta_2^2 + \gamma_2^2) .$$

Thus we have, according to Lemma 1, $k = 0$ and $J \geqq 0$ on M.

Further, from (6), (7) we get

$$H - 4K = (a_1 - c_1)^2 + (a_2 - c_2)^2 + 4b_1^2 + 4b_2^2$$

and hence $H - 4K \geqq 0$ on M. Thus, according to (i),

$$2J + (H - 4K) K \geqq 0 .$$

on M. By the Stokes theorem we have

$$\int_M [2J + (H - 4K) K] \omega^1 \wedge \omega^2 = 0$$

which implies $H - 4K = 0$ on M, so that each point of M is umbilical.

Remark. Using (4), the equations (26) imply

$$d(a_1 + c_1) - (a_2 + c_2)\,\omega_3^4 = 0\,,$$
$$d(a_2 + c_2) + (a_1 + c_1)\,\omega_3^4 = 0\,.$$

Thus we have also proved that under the assumptions (ii), (iii) of Theorem 1 the mean curvature vector field ξ is parallel.

In the preceding theorem, we have given a modification of the H-theorem concerning ovaloids in E^3 to the case of surfaces in E^4. Now, we are going to prove a generalization of this result.

Consider the mean curvature H of M and its covariant derivatives defined, according to [1], p. 16, by

$$dH = H_1\omega^1 + H_2\omega^2\,,$$
$$dH_1 - H_2\omega_1^2 = H_{11}\omega^1 + H_{12}\omega^2\,,\quad dH_2 + H_1\omega_1^2 = H_{12}\omega^1 + H_{22}\omega^2.$$

Using (4), (5), (7), we have

(27) $$\tfrac{1}{2}H_1 = (a_1 + c_1)(\alpha_1 + \gamma_1) + (a_2 + c_2)(\alpha_2 + \gamma_2)\,,$$
$$\tfrac{1}{2}H_2 = (a_1 + c_1)(\beta_1 + \delta_1) + (a_2 + c_2)(\beta_2 + \delta_2)\,;$$

(28) $$\tfrac{1}{2}H_{11} = (a_1 + c_1)(A_1 + C_1 + c_1K + \tfrac{1}{2}b_2k) + (a_2 + c_2)(A_2 + C_2 + c_2K - \tfrac{1}{2}b_1k) + (\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2,$$

$$\tfrac{1}{2}H_{12} = (a_1 + c_1)(B_1 + D_1) + (a_2 + c_2)(B_2 + D_2) + (\alpha_1 + \gamma_1)(\beta_1 + \delta_1) + (\alpha_2 + \gamma_2)(\beta_2 + \delta_2)\,,$$

$$\tfrac{1}{2}H_{22} = (a_1 + c_1)(C_1 + E_1 + a_1K - \tfrac{1}{2}b_2k) + (a_2 + c_2)(C_2 + E_2 + a_2K + \tfrac{1}{2}b_1k) + (\beta_1 + \delta_1)^2 + (\beta_2 + \delta_2)^2.$$

It is easy to prove that

(29) $$\bar{H}_1^2 + \bar{H}_2^2 = H_1^2 + H_2^2\,.$$

Now, we prove

Theorem 2. *Let M be a surface in E^4 and ∂M its boundary. Let*

(i) $K > 0$ *on* M;

(ii) ξ *be pseudoparallel in* $P(M)$;

(iii) $16(H - 4K)\,HK \geqq H_1^2 + H_2^2 > 0$ *on* M;

(iv) ∂M *consist of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. We use the integral formula

$$\int_M I\omega^1 \wedge \omega^2 = 0$$

and, first of all we prove that $I = 0$ where

$$I = 2(J - k^2) + (H - 4K)\,K \tag{30}$$

and J is the invariant introduced by (14).

The assumption (ii) is expressed by the system of equations (9) and, according to Lemma 1, implies $k = 0$ on M. As $H \neq 0$, the system (9), (27) has the only solution

$$\begin{aligned} &\alpha_1 + \gamma_1 = \tfrac{1}{2}(a_1 + c_1)\,H^{-1}H_1\,, \qquad \beta_1 + \delta_1 = \tfrac{1}{2}(a_1 + c_1)\,H^{-1}H_2\,, \\ &\alpha_2 + \gamma_2 = \tfrac{1}{2}(a_2 + c_2)\,H^{-1}H_1\,, \qquad \beta_2 + \delta_2 = \tfrac{1}{2}(a_2 + c_2)\,H^{-1}H_2\,. \end{aligned} \tag{31}$$

Hence in virtue of (31), the equation (30) has the form

$$\begin{aligned} I = &-(a_1 + c_1)\,H^{-1}(H_2\beta_1 + H_1\gamma_1) - (a_2 + c_2)\,H^{-1}(H_2\beta_2 + H_1\gamma_2) + \\ &+ 4(\beta_1^2 + \gamma_1^2 + \beta_2^2 + \gamma_2^2) + (H - 4K)\,K \end{aligned}$$

and consequently

$$\begin{aligned} I = &[2\beta_1 - \tfrac{1}{4}(a_1 + c_1)\,H^{-1}H_2]^2 + [2\gamma_1 - \tfrac{1}{4}(a_1 + c_1)\,H^{-1}H_1]^2 + \\ &+ [2\beta_2 - \tfrac{1}{4}(a_2 + c_2)\,H^{-1}H_2]^2 + [2\gamma_2 - \tfrac{1}{4}(a_2 + c_2)\,H^{-1}H_1]^2 - \\ &- \tfrac{1}{16}H^{-1}(H_1^2 + H_2^2) + (H - 4K)\,K\,. \end{aligned}$$

The condition (iii) thus yields $I \geqq 0$ so that $I = 0$ on M. This implies

$$\begin{aligned} &\beta_1 = \tfrac{1}{8}(a_1 + c_1)\,H^{-1}H_2\,, \qquad \gamma_1 = \tfrac{1}{8}(a_1 + c_1)\,H^{-1}H_1\,, \\ &\beta_2 = \tfrac{1}{8}(a_2 + c_2)\,H^{-1}H_2\,, \qquad \gamma_2 = \tfrac{1}{8}(a_2 + c_2)\,H^{-1}H_1 \end{aligned}$$

and

$$16(H - 4K)\,HK = H_1^2 + H_2^2\,. \tag{32}$$

According to (31), we have further

$$\begin{aligned} &\alpha_1 = \tfrac{3}{8}(a_1 + c_1)\,H^{-1}H_1\,, \qquad \delta_1 = \tfrac{3}{8}(a_1 + c_1)\,H^{-1}H_2\,, \\ &\alpha_2 = \tfrac{3}{8}(a_2 + c_2)\,H^{-1}H_1\,, \qquad \delta_2 = \tfrac{3}{8}(a_2 + c_2)\,H^{-1}H_2 \end{aligned}$$

and hence

$$\alpha_1 = 3\gamma_1\,, \quad \delta_1 = 3\beta_1\,, \quad \alpha_2 = 3\gamma_2\,, \quad \delta_2 = 3\beta_2\,. \tag{33}$$

We prove further that a surface M satisfying (32) and (33) is a 2-dimensional sphere in E^4.

Suppose that there is a point $m \in M$ which is not umbilical. In a convenient neighbourhood U of m we can choose, according to (19) and the relation $k = 0$, a field of orthonormal frames of M in such a way that

$$a_1 - c_1 \neq 0\,, \quad b_1 = 0\,, \quad a_2 - c_2 = 0\,, \quad b_2 = 0\,. \tag{34}$$

The equations (4) thus have the form

$$\begin{aligned} &da_1 - a_2\omega_3^4 = 3\gamma_1\omega^1 + \beta_1\omega^2\,, \quad da_2 + a_1\omega_3^4 = 3\gamma_2\omega^1 + \beta_2\omega^2\,,\\ &(a_1 - c_1)\,\omega_1^2 = \beta_1\omega^1 + \gamma_1\omega^2\,, \quad 0 = \beta_2\omega^1 + \gamma_2\omega^2\,,\\ &dc_1 - a_2\omega_3^4 = \gamma_1\omega^1 + 3\beta_1\omega^2\,, \quad da_2 + c_1\omega_3^4 = \gamma_2\omega^1 + 3\beta_2\omega^2\,, \end{aligned} \tag{35}$$

because of (33). From (35) we have immediately $\beta_2 = 0$, $\gamma_2 = 0$ and, because of (33), $\alpha_2 = 0$, $\delta_2 = 0$. Consequently, it is $\omega_3^4 = 0$. As $H_1^2 + H_2^2 \neq 0$, it follows from (27) that $a_2 + c_2 = 0$ and this, together with (34), implies $a_2 = 0$, $c_2 = 0$. Thus (35) is reduced to

$$\begin{aligned} da_1 &= 3\gamma_1\omega^1 + \beta_1\omega^2\,,\\ (a_1 - c_1)\,\omega_1^2 &= \beta_1\omega^1 + \gamma_1\omega^2\,,\\ dc_1 &= \gamma_1\omega^1 + 3\beta_1\omega^2 \end{aligned} \tag{36}$$

and (32) has the form

$$(H - 4K)\,K = 4(\beta_1^2 + \gamma_1^2)\,. \tag{37}$$

By exterior differentiation of (36) and using Cartan's lemma we get the existence of a function ϱ in U such that

$$\begin{aligned} d\beta_1 + \gamma_1\omega_1^2 &= (\varrho - a_1c_1^2)\,\omega^2\,,\\ d\gamma_1 - \beta_1\omega_1^2 &= (\varrho - a_1^2c_1)\omega^1\,. \end{aligned}$$

Repeated exterior differentiation yields the relation

$$d\varrho = 3c_1^2\gamma_1\omega^1 + 3a_1^2\beta_1\omega^2$$

and hence, differentiating this equation and using again Cartan's lemma, we obtain

$$(a_1 - c_1)\,\beta_1\gamma_1 = 0\,.$$

Now suppose $\beta_1 = 0$, $\gamma_1 \neq 0$, the case $\beta_1 \neq 0$, $\gamma_1 = 0$ being symmetric. The relation (37) then yields

$$(a_1 - c_1)^2\,a_1c_1 = 4\gamma_1^2$$

and thus, by successive differentiations of this equation, we get

$$a_1^3 + 13a_1^2c_1 - 9a_1c_1^2 + 3c_1^3 = 8\varrho\,,$$
$$11a_1^2 + 30a_1c_1 - 21c_1^2 = 0\,,$$
$$2a_1 + c_1 = 0\,,$$
$$7\gamma_1 = 0\,.$$

The last relation contradicts our supposition.

Thus we have $\beta_1 = 0$, $\gamma_1 = 0$, and (34), (36) yield $\omega_1^2 = 0$. However, exterior differentiation of this equation yields $K\omega^1 \wedge \omega^2 = 0$ and hence $K = 0$ in U. This being a contradiction to (i), $m \in M$ must be an umbilical point of M.

To be able to introduce some consequences of Theorem 2, we define normal vector fields

$$\xi_1 = (V_1\xi)^N\,, \quad \xi_2 = (V_2\xi)^N\,,$$

ξ being the mean curvature vector field in $N(M)$, $V_1, V_2 \in T(M)$ orthonormal vector fields on M and $(X)^N$ the normal component of the vector field X.

Corollary 1. *Let M be a surface in E^4 satisfying the conditions* (i), (ii) *and* (iv) *of Theorem* 2. *Let*

(iii) $4(H - 4K)HK \geqq \langle\xi, \xi_1\rangle^2 + \langle\xi, \xi_2\rangle^2 > 0$ *on* M.

Then M is a part of a 2-dimensional sphere in E^4.

Proof. In each $m \in M$ choose orthonormal frames of M in such a way that $V_1 = v_1$, $V_2 = v_2$. It follows from (8) that

$$\mathrm{d}\xi = [(\alpha_1 + \gamma_1)v_3 + (\alpha_2 + \gamma_2)v_4]\,\omega^1 + [(\beta_1 + \delta_1)v_3 + (\beta_2 + \delta_2)v_4]\,\omega^2 \pmod{v_1, v_2}$$

and thus, by the definition,

$$\xi_1 = (\alpha_1 + \gamma_1)v_3 + (\alpha_2 + \gamma_2)v_4\,, \tag{38}$$
$$\xi_2 = (\beta_1 + \delta_1)v_3 + (\beta_2 + \delta_2)v_4\,.$$

Then, because of (8), (27) and (38), we have

$$\langle\xi, \xi_1\rangle = (a_1 + c_1)(\alpha_1 + \gamma_1) + (a_2 + c_2)(\alpha_2 + \gamma_2) = \tfrac{1}{2}H_1\,,$$
$$\langle\xi, \xi_2\rangle = (a_1 + c_1)(\beta_1 + \delta_1) + (a_2 + c_2)(\beta_2 + \delta_2) = \tfrac{1}{2}H_2$$

and hence

$$\langle\xi, \xi_1\rangle^2 + \langle\xi, \xi_2\rangle^2 = \tfrac{1}{4}(H_1^2 + H_2^2)\,.$$

This equation and the condition (iii) together with the assertion of Theorem 2 conclude the proof.

Corollary 2. *Let M be a surface in E^4 with the properties* (i), (ii) *and* (iv) *of Theorem* 2. *Let*

(iii) $4(H - 4K)K \geqq \langle \xi_1, \xi_1 \rangle + \langle \xi_2, \xi_2 \rangle > 0$ *on* M.

Then M is a part of a 2-dimensional sphere in E^4.

Proof. Choose again a field of orthonormal frames of M in such a way that $V_1 = v_1$, $V_2 = v_2$. Then (27) and (38) imply

$$\langle \xi_1, \xi_1 \rangle = (\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2 = \tfrac{1}{4}H^{-1}H_1^2\,,$$
$$\langle \xi_2, \xi_2 \rangle = (\beta_1 + \delta_1)^2 + (\beta_2 + \delta_2)^2 = \tfrac{1}{4}H^{-1}H_2^2$$

and hence

$$\langle \xi_1, \xi_1 \rangle + \langle \xi_2, \xi_2 \rangle = \tfrac{1}{4}H^{-1}(H_1^2 + H_2^2)\,.$$

Thus (iii) and Theorem 2 prove our assertion.

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