Milan Kučera

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A NEW METHOD FOR OBTAINING EIGENVALUES OF VARIATIONAL INEQUALITIES BASED ON BIFURCATION THEORY

MILAN KUČERA, Praha

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0. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and with the corresponding norm $\|\cdot\|$. Let K be a closed convex cone in H with its vertex at the origin. We shall suppose that $A : H \to H$ is a linear symmetric completely continuous operator. We shall consider the following problem:

$$\text{(I)} \qquad u \in K\,,$$

$$\text{(II)} \qquad \langle \lambda u - Au, v - u \rangle \geqq 0 \quad \text{for all} \quad v \in K\,,$$

where λ is a real parameter. A real number λ is said to be an eigenvalue of the variational inequality (I), (II) if there exists a nontrivial u satisfying (I), (II). In this case the element u is said to be the corresponding (to λ) eigenvector of the variational inequality (I), (II). The aim of this paper is to study the existence of eigenvalues and eigenvectors of the variational inequality which are not eigenvalues and eigenvectors of the operator A. The basic idea is the following. We shall introduce a penalty operator β (for the properties of β see Section 2) and consider an eigenvalue $\lambda^{(0)}$ of A corresponding to an eigenvector $u^{(0)} \notin K$ of A. Starting with $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$, we want to prove the existence of branches λ_ε, u_ε ($\varepsilon \in \langle 0, +\infty)$) satisfying the equation with the penalty

$$\lambda_\varepsilon u_\varepsilon - Au_\varepsilon + \varepsilon \beta u_\varepsilon = 0$$

and converging to an eigenvalue λ_∞ and an eigenvector u_∞ of (I), (II). The original idea was to prove the existence of such functions λ_ε, u_ε on the basis of the abstract implicit function theorem. A result of this type was announced in [8] (without proof) and a complete version of this part of the theory is given in [9]. However, this approach requires very strong assumptions (it is supposed that the linear operator

$A - \varepsilon\,\beta'(u)$ for an arbitrary fixed $u \in H$ and $\varepsilon \geqq 0$ has only simple eigenvalues) and only very simple examples covered by the theory are known to the author. In the course of investigation, it turned out that it is possible to use the known global results of the bifurcation theory to prove the existence of branches of eigenvalues and eigenvectors of the equation with the penalty. This approach seems to be substantially more effective. Under certain assumptions it is possible to start with the eigenvalue $\lambda^{(0)}$ of A of an arbitrary multiplicity and with the corresponding eigenvector $u^{(0)} \notin K$ and to prove the existence of a closed connected (in a certain sense) and unbounded in ε set S_0 of triplets $[\lambda, u, \varepsilon] \in R \times H \times R$ satisfying the conditions $[\lambda^{(0)}, u^{(0)}, 0] \in S_0$,

$$\|u\| = 1\,, \quad \lambda_* \leqq \lambda \leqq \lambda^*\,, \quad u \notin K\,,$$

$$\lambda u - Au + \varepsilon\beta u = 0\,,$$

where λ_*, λ^* are some suitable eigenvalues of A. Such a set S_0 contains at least one sequence $[\lambda_n, u_n, \varepsilon_n]$ such that $\lambda_n \to \lambda_\infty$, $u_n \to u_\infty$, where λ_∞ and u_∞ is an eigenvalue and an eigenvector of (I), (II), respectively. Moreover, $u_\infty \in \partial K$ and it is a "new eigenvector of (I), (II)", i.e. u_∞ is not an eigenvector of A. In certain cases, this method yields an infinite sequence of eigenvalues and "new" eigenvectors of (I), (II). In special cases the set S_0 can be described by smooth functions λ_ε, u_ε (see [9]).

In this paper we shall study the case of a simple initial eigenvalue $\lambda^{(0)}$. It is easier than the case of a multiple eigenvalue $\lambda^{(0)}$ which will be treated in the paper [10]. The proof of existence of an unbounded branch S_0 for a simple eigenvalue $\lambda^{(0)}$ is based on a global bifurcation result of E. N. Dancer [3] (see Section 3).

A classification of eigenvalues of (I), (II) and of A is given and their basic properties are explained in Section 1 of this paper. The main result is formulated in Section 2 (Theorems 2.1, 2.2, 2.3). Further, general properties of the branches S_0 of the above mentioned type with the exception of the fact that S_0 is unbounded are proved. This represents the first part of the proof of the main results. Section 3 contains an explanation of the above mentioned result of E. N. Dancer [3] (which is a strengthening of Rabinowitz's result [14]). Further, on the basis of this result it is proved that under certain assumptions the branch S_0 is unbounded. This is the second part of the proof of the main theorems. Applications to the case of variational inequalities describing a beam which is supported by fixed obstacles are given in Section 4.

A very special situation occurs if K is a halfspace. This corresponds to the case of "one point obstacle" (i.e. $n = 1$ in the notation of Example 1.1 and Section 4). In this case, the method from the papers [8], [9] can be used and λ_ε is a decreasing function. Moreover, the eigenvalues of (I), (II) can be calculated in concrete examples on the basis of a method given by S. Fučík, J. Milota [7] and therefore our theory has no practical significance for this special case.

The eigenvalue problem for variational inequalities in a more general setting is studied in [11], where we use a modification of the Ljusternik-Schnirelamann theory for the corresponding penalty problem. We obtain formally infinitely many eigenvalues (or critical levels in a more general situation) but it is not clear if they all are mutually different. A better situation occurs again in the case of a halfspace. Using a certain special trick, we prove the existence of an infinite set of mutually different eigenvectors lying on ∂K with the corresponding critical levels (or eigenvalues) converging to zero.

Let us remark that E. MIERSEMANN investigated a more general variational inequality than (I), (II) (with nonlinear operators) on a cone. He proved the existence of n bifurcation points, where n is determined by the parameters of the problem (see [12]). The proof is based on a Krasnoselskij's sup-min principle.

Speaking about the eigenvalue problem for variational inequalities, we should mention also other papers about this topic (for example [1], [2], [4], [5], [6], [13], [15]). However, the approach to the problem in these papers is completely different from that explained above and the existence results are of the other type than in the present paper.

1. TERMINOLOGY AND GENERAL REMARKS

Denote by ∂K and K^0 the boundary and the interior of K, respectively. The sets of all eigenvalues of the operator A and of the variational inequality (I), (II) will be denoted by Λ_A and Λ_V, respectively. Analogously, we shall denote by E_A and E_V the sets of all eigenvectors of the operator A and of the variational inequality (I), (II), respectively. The strong convergence and the weak convergence will be denoted by $\to$ and $\rightharpoonup$, respectively.

Remark 1.1. It is easy to see that $E_A \cap K \subset E_V$, $E_V \cap K^0 = E_A \cap K^0$. The second assertion follows from the fact that if $u \in K^0$ then there exists $\delta > 0$ such that $v = w + u \in K$ for all $w \in H$, $\|w\| \leqq \delta$ and therefore (II) implies

$$\langle \lambda u - Au, w \rangle \geqq 0 \quad \text{for all} \quad w \in H\,, \quad \|w\| \leqq \delta\,.$$

The last inequality holds also for all $w \in H$ which means $\lambda u - Au = 0$.

Definition 1.1. We shall say that

(1) $\lambda \in \Lambda_V$ is a *boundary eigenvalue of* (I), (II) if there exists a corresponding eigenvector $u \in \partial K \cap E_V$ and there is no $u \in K^0 \cap E_V$ corresponding to λ;

(2) $\lambda \in \Lambda_V$ is an *interior eigenvalue of* (I), (II) if λ is not a boundary eigenvalue of (I), (II) and there exists a corresponding eigenvector $u \in K^0 \cap E_V$;

(3) $\lambda \in \Lambda_A$ is a *boundary* (with respect to K) *eigenvalue of A* if there exists a corresponding eigenvector $u \in \partial K \cap E_A$ and there is no $u \in K^0 \cap E_A$ corresponding to λ;

(4) $\lambda \in \Lambda_A$ is an *interior* (with respect to K) *eigenvalue of A* if there exists a corresponding eigenvector $u \in K^0 \cap E_A$;

(5) $\lambda \in \Lambda_A$ is an *external* (with respect to K) *eigenvalue of A* if $u \notin K$ for all the corresponding eigenvectors $u \in E_A$.

The set of all interior eigenvalues of (I), (II) (or A*)) will be denoted by Λ_i. Further, we shall denote by $\Lambda_{V,b}$, Λ_b and Λ_e the set of all boundary eigenvalues of (I), (II), the set of all boundary eigenvalues of A and the set of all external eigenvalues of A, respectively.

Remark 1.2. It is clear that $\Lambda_V = \Lambda_i \cup \Lambda_{V,b}$, $\Lambda_i \cap \Lambda_{V,b} = \emptyset$. Analogously, $\Lambda_A = \Lambda_i \cup \Lambda_b \cup \Lambda_e$, $\Lambda_i \cap \Lambda_b = \emptyset$, $\Lambda_b \cap \Lambda_e = \emptyset$, $\Lambda_i \cap \Lambda_e = \emptyset$. Further, $\Lambda_b \subset \Lambda_{V,b}$. On the other hand, if $\lambda \in \Lambda_{V,b}$, then there are three possibilities (the concrete illustration will be given in Example 1.1):

(α) $\lambda \in \Lambda_b$, i.e. λ is simultaneously a boundary eigenvalue of A; in this case there is a common eigenvector $u \in \partial K \cap E_A \cap E_V$ of A and of (I), (II) corresponding to λ;

(β) $\lambda \in \Lambda_e$, i.e. λ is simultaneously an eigenvalue of A but the corresponding eigenvectors of A are not in K, i.e. they are different from the corresponding eigenvectors of (I), (II);

(γ) $\lambda \notin \Lambda_A$.

Remark 1.3. In general, the set of eigenvectors of (I), (II) corresponding to a given eigenvalue $\lambda \in \Lambda_{V,b}$ need not to be convex. (See Example 1.1.) A certain information about the structure of the set of eigenvalues of (I), (II) corresponding to a given eigenvalue $\lambda \in \Lambda_V \cap \Lambda_A$ is given by Lemma 1.1 below.

Lemma 1.1. *Suppose that $\lambda \in \Lambda_A$ and there is a corresponding eigenvector $u_0 \in E_A \cap K$. If $u_1 \in E_V$ is an arbitrary eigenvector of* (I), (II) *corresponding to λ, then for arbitrary $t_0 \geqq 0$, $t_1 \geqq 0$ the point $u = t_0 u_0 + t_1 u_1$ is an eigenvector of* (I), (II) *corresponding to λ, too. Moreover, if $u_0 \in E_A \cap K^0$, then $E_A(\lambda) \cap K = E_V(\lambda)$, where $E_A(\lambda)$ and $E_V(\lambda)$ denote the sets of all eigenvectors of A and of* (I), (II), *respectively, corresponding to λ.*

Proof. It is easy to see that the conditions (I), (II) are equivalent to the condition (I) and

$$(1.1) \qquad \langle \lambda u - Au, v \rangle \geqq 0 \quad \text{for all} \quad v \in K\,;$$

$$(1.2) \qquad \langle \lambda u - Au, u \rangle = 0\,.$$

*) A number λ is an interior eigenvalue of A if and only if it is an interior eigenvalue of (I) (II). The corresponding eigenvectors of (I), (II) lying in K^0 are those of A. This follows from Remark 1.1.

Clearly, (I) and (1.1) are true for $u = t_0 u_0 + t_1 u_1$ ($t_0 \geqq 0$, $t_1 \geqq 0$). Using (1.2) for u_0, u_1, we obtain

$$\langle \lambda(t_0 u_0 + t_1 u_1) - A(t_0 u_0 + t_1 u_1), t_0 u_0 + t_1 u_1 \rangle =$$
$$= 2 t_0 t_1 \langle \lambda u_0 - A u_0, u_1 \rangle = 0$$

because u_0 is an eigenvector of A. Thus (1.2) is proved and that means $u \in E_V$. Further, let $u_0 \in K^0$. It is clear that $E_A(\lambda) \cap K \subset E_V(\lambda)$. On the other hand, if $u_1 \in E_V(\lambda)$, then we have proved that $t u_0 + (1 - t) u_1 \in E_V(\lambda)$ for all $t \in \langle 0, 1 \rangle$. Moreover, it is clear that $t u_0 + (1 - t) u_1 \in K^0$ for $t \in (0, 1)$ and therefore $t u_0 + (1 - t) u_1 \in E_A(\lambda)$ for all $t \in (0, 1)$ (cf. Remark 1.1). The set $E_A(\lambda)$ is closed and therefore $u_1 \in E_A(\lambda)$.

Remark 1.4. It is possible that there are eigenvalues in $\Lambda_{V,b}$ which are not simple*) even in the case that the operator A has only simple eigenvalues (see Example 1.1 and [6, Section 1]). Nonetheless, it follows from Lemma 1.1 that $\lambda \in \Lambda_i$ is a simple eigenvalue of (I), (II) if and only if λ is a simple eigenvalue of A.

The definitions and assertions mentioned in this section can be best illustrated by the following Example 1.1, in which the set of eigenvalues and eigenvectors of (I), (II) can be completely described in an elementary way (see [6, Section 1]).

Example 1.1. Denote by $H = \mathring{W}_2^1(0, 1)$ the well-known Sobolev space of all absolutely continuous functions on $\langle 0, 1 \rangle$ vanishing at 0 and 1 whose derivatives are square integrable over $\langle 0, 1 \rangle$. Introduce the inner product on H by

$$\langle u, v \rangle = \int_0^1 u'v' \, dx \quad \text{for all} \quad u, v \in H$$

(instead of the usual equivalent inner product $(u, v) = \int_0^1 (u'v' + uv) \, dx$). Set $K = \{u \in H;\ u(x_i) \geqq 0,\ i = 1, \ldots, n\}$, where $x_i \in (0, 1)$ $(i = 1, \ldots, n)$ are given numbers (n is positive integer). Let us define the operator A by

$$(Au, v) = \int_0^1 uv \, dx \quad \text{for all} \quad u, v \in H .$$

It is easy to see that $\lambda \in \Lambda_A$ and a nontrivial u is a corresponding eigenvector from E_A if and only if u has a continuous second derivative on $\langle 0, 1 \rangle$ and

$$\lambda u'' + u = 0 \quad \text{on} \quad (0, 1)\,, \tag{1.3}$$

$$u(0) = u(1) = 0\,. \tag{1.4}$$

*) By a simple eigenvalue of (I), (II) we mean a number $\lambda \in \Lambda_V$ such that there exists only one corresponding eigenvector $u \in E_V$ with $\|u\| = 1$.

Further, denote $x_0 = 0$, $x_{n+1} = 1$, $u'(x_i \pm) = \lim_{x \to x_i \pm} u'(x_i)$. It is easy to show that $\lambda \in \Lambda_V$ and u is a corresponding eigenvector from E_V if and only if u is a nontrivial continuous function on $\langle 0, 1 \rangle$ with a continuous second derivative on (x_i, x_{i+1}) $(i = 0, \ldots, n)$ satisfying

$$\lambda u'' + u = 0 \quad \text{on} \quad (x_i, x_{i+1}), \quad i = 0, \ldots, n, \tag{1.5}$$

$$u(0) = u(1) = 0, \tag{1.6}$$

$$u(x_i) \geqq 0, \quad i = 1, \ldots, n, \tag{1.7}$$

$$u'(x_i-) - u'(x_i+) \geqq 0, \quad i = 1, \ldots, n, \tag{1.8}$$

$$u(x_i)\left[u'(x_i-) - u'(x_i+)\right] = 0, \quad i = 1, \ldots, n. \tag{1.9}$$

Moreover, $\lambda \in \Lambda_{V,b}$ if and only if each corresponding eigenvector u satisfies

$$u(x_i) = 0 \quad \text{at least for one } i.$$

Analogously, $\lambda \in \Lambda_i$ if and only if the corresponding eigenvector satisfies the condition

$$u(x_i) > 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

Let us show that all the situations described in Remark 1.2 are possible.

If we take $n = 1$, $x_1 = \frac{1}{4}$, then $\lambda = (\frac{3}{4})^2 (1/\pi^2) \in \Lambda_{V,b}$ is the second eigenvalue of (I), (II) corresponding to the eigenvector $u_V \in E_V$,

$$u_V(x) = \begin{cases} 0 & \text{on} \quad \langle 0, x_1 \rangle, \\ -\sin \frac{4}{3}\pi(x - \frac{1}{4}) & \text{on} \quad \langle x_1, 1 \rangle, \end{cases}$$

but it is not an eigenvalue of A (see Fig. 1.1).

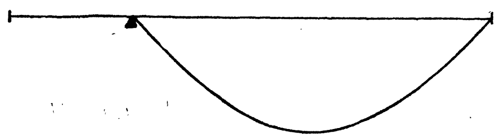

Fig. 1.1

If we choose $n = 2$, $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, then $\lambda = (\frac{1}{2})^2 (1/\pi^2) \in \Lambda_{V,b} \cap \Lambda_e$ is the second eigenvalue of (I), (II) and simultaneously the second eigenvalue of A. However, we have $u_A \notin K$, $-u_A \notin K$, $u_A \neq u_V \neq -u_A$, where

$$u_V(x) = \begin{cases} 0 & \text{on} \quad \langle 0, x_1 \rangle \cup \langle x_2, 1 \rangle, \\ -\sin 2\pi(x - \frac{1}{4}) & \text{on} \quad \langle x_1, x_2 \rangle, \end{cases}$$

$$u_A(x) = \sin 2\pi x \quad \text{on} \quad \langle 0, 1 \rangle,$$

$u_A \in E_A$ is a corresponding eigenvector of A and $u_V \in E_V$ is a corresponding eigenvector of (I), (II) (see Fig. 1.2).

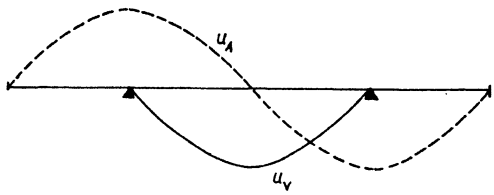

Fig. 1.2

By the same choice of x_1, x_2, the value $\lambda = (\frac{1}{4})^2 (1/\pi^2) \in \Lambda_b$ (i.e. also $\lambda \in \Lambda_{V\,b}$) is the fourth eigenvalue of A and simultaneously the third eigenvalue of (I), (II), corresponding to a common eigenvector $u \in E_A \cap E_V$, $u(x) = \sin 4\pi x$.

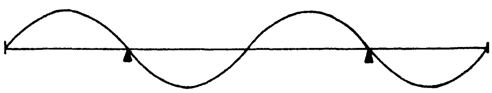

Fig. 1.3

If we set $K = \{u \in \mathring{W}_2^1(0, 1);\ u(\frac{1}{4}) \geqq 0,\ u(\frac{3}{4}) \leqq 0\}$, then the functions u_1, u_2,

$$u_1(x) = \begin{cases} \sin \frac{4}{3}\pi x & \text{on } \langle 0, \frac{3}{4}) , \\ 0 & \text{on } \langle \frac{3}{4}, 1 \rangle , \end{cases}$$

$$u_2(x) = \begin{cases} 0 & \text{on } \langle 0, \frac{1}{4}) , \\ -\sin \frac{4}{3}(x - \frac{1}{4}) & \text{on } \langle \frac{1}{4}, 1 \rangle \end{cases}$$

are eigenvectors of (I), (II) corresponding to the eigenvalue $\lambda = (\frac{3}{4})^2 (1/\pi^2) \in \Lambda_V$ (Fig. 1.4), but for arbitrary $t \in (0, 1)$ the point $tu_1 + (1 - t) u_2$ is not an eigenvector. That means that λ is not simple (although A has only simple eigenvalues) and the set of the corresponding eigenvectors is not convex.

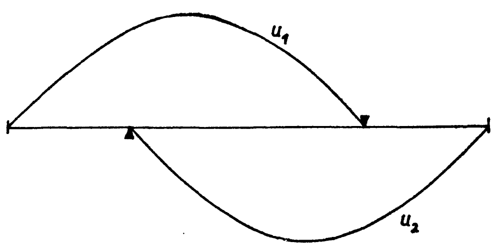

Fig. 1.4

In the sequel, we shall consider a nonlinear continuous operator $\beta : H \to H$ satisfying the following assumptions:

(P) $\beta u = 0$ if and only if $u \in K$, $\langle \beta u, u \rangle > 0$ for all $u \notin K$ (i.e. β is the penalty operator corresponding to K);

(H) $\beta(tu) = t\beta u$ for all $t \geqq 0$, $u \in H$ (i.e. β is positive homogeneous);

(CC) β is completely continuous; moreover, if $\varepsilon_n > 0$, $u_n \in H$ $(n = 1, 2, \ldots)$ are such that the sequence $\{\varepsilon_n \beta u_n\}$ is bounded then $\{\varepsilon_n \beta u_n\}$ contains a strongly convergent subsequence;

(M) $\langle \beta u - \beta v, u - v \rangle \geqq 0$ for all $u, v \in H$ (i.e. β is monotone);

(β, K^0) if $u \in K^0$, $v \notin K$, then $\langle \beta v, u \rangle \neq 0$.

The points $u \in H$ satisfying the following "symmetry condition" will be useful for our further considerations:

(SC) there exists a neighborhood U of u such that

$$\langle \beta u, v \rangle = \langle \beta v, u \rangle \quad \text{for all} \quad v \in U .$$

The eigenvalues $\lambda \in \Lambda_A$ with the following property will play a special role:

(SC′) if u is an arbitrary eigenvector of A corresponding to λ and $u \notin K$, then u satisfies the condition (SC).

Remark 2.1. If H and K are the space and the cone from Example 1.1, then we can define the operator β by the formula

$$\langle \beta u, v \rangle = - \sum_{i=1}^{n} u^-(x_i)\, v(x_i) ,$$

where u^- denotes the negative part of u. It is easy to see that the assumptions (P), (H), (CC), (M), (β, K^0) are fulfilled and that (SC) holds for each $u \in H$. In particular, all eigenvalues of A satisfy the assumption (SC′), where A can be an arbitrary linear completely continuous operator in H.

Now, let us consider the situation from Example 1.1 but with the cone

$$K_1 = \{u \in H;\, u(x) \geqq 0 \text{ on } \langle \tfrac{2}{5}, \tfrac{3}{5} \rangle\}$$

instead of K. We can define the operator β by

$$\langle \beta u, v \rangle = - \int_{2/5}^{3/5} u^-(x)\, v(x)\, dx \quad \text{for all} \quad u, v \in H .$$

It is easy to see that the function $u \in H$ satisfies the condition (SC) if and only if $|u(x)| > 0$ for all $x \in \langle \tfrac{2}{5}, \tfrac{3}{5} \rangle$. All the other assumptions mentioned above are obviously fulfilled. Hence, if $\lambda_1 > \lambda_2 > \ldots$ is the sequence of all eigenvalues of the operator A from Example 1.1, then only the eigenvalues λ_1, λ_3 satisfy the condition (SC′). (The eigenvector u_n corresponding to λ_n is given by $u_n(x) = \sin n\pi x$.)

The main results formulated in Theorems 2.2, 2.3 are somewhat formally complicated and therefore we shall first formulate Existence Theorem 2.1. In fact, Theorem 2.1 is a part of the assertion of Theorems 2.2 and 2.3. Theorems 2.2, 2.3 explain how the eigenvalues and eigenvectors from Theorem 2.1 can be obtained by a limiting process from the branches of eigenvalues and eigenvectors of the equation with penalty.

Theorem 2.1. *Let* $\lambda^{(1)}, \lambda^{(0)} \in \Lambda_i$, $0 < \lambda^{(1)} < \lambda^{(0)}$, $(\lambda^{(1)}, \lambda^{(0)}) \cap (\Lambda_b \cup \Lambda_i) = \emptyset$. *Suppose that* $\lambda^{(0)}, \lambda^{(1)}$ *are simple and* $u^{(0)}$ *is an eigenvector corresponding to* $\lambda^{(0)}$, $u^{(0)} \notin K$, $-u^{(0)} \in K^0$. *Assume that there exists an operator* β *satisfying the conditions* (P), (H), (CC), (M), (β, K^0) *and such that* $\lambda^{(0)}, \lambda^{(1)}$ *satisfy the condition* (SC′). *Then there exists* $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$ *with a corresponding eigenvector* $u_\infty \in \partial K \cap (E_V \setminus E_A)$.

Definition 2.1. We shall denote by S the set of all triplets $[\lambda, u, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying the conditions

(a) $$\|u\| = 1\,, \quad \varepsilon \geqq 0$$

(b) $$\lambda u - Au + \varepsilon\beta u = 0\,.$$

Now we are able to formulate the main results. An additional explanation to Theorems 2.2, 2.3 will be given in Remark 2.2 below.

Theorem 2.2. *Let all the assumptions of Theorem* 2.1 *be fulfilled and let* $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_A = \emptyset$. *Denote by* S_0 *the component of* S *containing the point* $[\lambda^{(0)}, u^{(0)}, 0]$. *Then for each* $\varepsilon > 0$ *there exists at least one couple* $[\lambda, u] \in R \times H$ *such that* $[\lambda, u, \varepsilon] \in S_0$. *For all* $[\lambda, u, \varepsilon] \in S_0$, *the following conditions are satisfied*:

(c) $$u \notin K\,,$$

(d) $$\text{if} \quad [\lambda, u, \varepsilon] \neq [\lambda^{(0)}, u^{(0)}, 0]\,, \quad \text{then} \quad \lambda \in (\lambda^{(1)}, \lambda^{(0)})\,.$$

If $[\lambda_n, u_n, \varepsilon_n] \in S_0$ $(n = 1, 2, \ldots)$ *is an arbitrary sequence such that* $\varepsilon_n \to +\infty$, *then there exists a subsequence of indices* r_n $(n = 1, 2, \ldots)$ *such that* $r_n \to +\infty$, $\lambda_{r_n} \to \lambda_\infty$, $u_{r_n} \to u_\infty$, *where* $\lambda_\infty \in (\Lambda_{V,b} \setminus \Lambda_A) \cap (\lambda^{(1)}, \lambda^{(0)})$ *and* $u_\infty \in (E_V \setminus E_A) \cap \partial K$ *is a corresponding eigenvector of* (I), (II).

Theorem 2.3. *Let all the assumptions of Theorem* 2.1 *be fulfilled. Then there exists a set* $S_0 \subset S$ *having all the properties of* S_0 *from the assertion of Theorem* 2.2 *with* $\lambda_\infty \in \Lambda_{V,b}$ *instead of* $\lambda_\infty \in \Lambda_{V,b} \setminus \Lambda_A$. S_0 *is either closed and connected or* $S_0 = \bigcup_{i=1}^{\varkappa} S_i$ ($\varkappa > 1$ *integer*), *where* S_i *are closed connected sets with the following* *property: there exist* $\lambda_i \in (\lambda^{(1)}, \lambda^{(0)})$, $u_i \notin K$, $\tilde{u}_i \notin K$ $(i = 1, \ldots, \varkappa - 1)$ *such that each* S_i *contains the points* $[\lambda_{i-1}, u_{i-1}, 0]$, $[\lambda_i, \tilde{u}_i, 0]$ *for* $i = 1, \ldots, \varkappa - 1$, *where* $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$, *and* $S_\varkappa$ *contains* $[\lambda_{\varkappa-1}, u_{\varkappa-1}, 0]$ *and is unbounded.*

Remark 2.2. Theorems 2.2, 2.3 guarantee the existence of an unbounded in ε and (in a certain sense) connected branch $S_0 \subset S$ joining the given eigenvalue $\lambda^{(0)}$ and the eigenvector $u^{(0)}$ with an eigenvalue λ_∞ and eigenvector u_∞ of (I), (II). If $(\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_A = \emptyset$, then $[\lambda^{(0)}, u^{(0)}, 0]$ is the only point of the type $[\lambda, u, 0]$ lying on S_0 and the branch S_0 is connected in this case (Theorem 2.2). In the general case, we admit the existence of some external eigenvalues of A in $(\lambda^{(1)}, \lambda^{(0)})$ (Theorem 2.3). In this case the branch S_0 can contain points of the type $[\lambda, u, 0]$, $\lambda \in (\lambda^{(1)}, \lambda^{(0)}) \cap \Lambda_e$, u is the corresponding eigenvector, and the connectedness in the variable u can be violated at these points. In other words, S_0 consists of the (connected) components S_i joining points of the type $[\lambda_{i-1}, u_{i-1}, 0]$, $[\lambda_i, \tilde{u}_i, 0]$, where $\lambda_0 = \lambda^{(0)}$, $u_0 = u^{(0)}$, $\lambda_i \in \Lambda_e \cap (\lambda^{(1)}, \lambda^{(0)})$ and $u_i, \tilde{u}_i$ are the corresponding eigenvectors. The branch S_0 will be obtained in Section 3 by a transformation from a bifurcation branch C_0 for a suitable bifurcation equation (B″) which is an extension of the penalty equation (b). The branch C_0 will be connected in every case and the points $[\lambda, u, \varepsilon] \in S_0$ at which the connectedness of S_0 can be lost will be obtained from the points $[1/\lambda, 0, 0]$.

Remark 2.3. The proof of Theorems 2.2, 2.3 consists of three parts. As we mentioned in Remark 2.2, the existence of S_0 will be proved in Section 3 on the basis of Dancer's global bifurcation result (the last part of the proof). However, for the use of the known bifurcation results, the validity of the basic conditions (c), (d) is essential and therefore we shall prove that the conditions (c), (d) are a priori satisfied on S_0 (if it exists). An investigation of the properties of S_0 is the subject of the next part of this Section. Roughly speaking, the proof of the conditions (c), (d) is based on the following assertions:

(α) S_0 starts at $\lambda^{(0)} > \lambda^{(1)}$, $u^{(0)} \notin K$ (by the assumptions);

(β) the values λ are locally decreasing along S_0 near $\lambda = \lambda^{(0)}$, $\varepsilon = 0$ (Lemma 2.2);

(γ) S_0 cannot intersect the lines $\lambda = \lambda^{(0)}$, $\lambda^{(1)}$ (with the exception of the point $[\lambda^{(0)}, u^{(0)}, 0]$) and it cannot intersect ∂K (Lemmas 2.1, 2.3).

On the whole, the conditions (c), (d) follow from (α − γ) if the branch S_0 under consideration is connected. In the case of Theorem 2.3, (c), (d) will be preserved because the set S_0 will be "connected in λ" and "connected in u except for the points $[\lambda_i, u_i, 0]$, $[\lambda_i, \tilde{u}_i, 0]$" (cf. Remark 2.2) and $u_i, \tilde{u}_i \notin K$ because $\lambda_i \in \Lambda_e$ by the assumptions. The fact that the branch S_0 gives the eigenvalues and the eigenvectors of (I), (II) (for $\varepsilon \to +\infty$) can be proved by a modified penalty method technique (see Lemma 2.4).

Lemma 2.1. *If* $\lambda^{(0)} \in \Lambda_i$ *and the condition* (β, K^0) *is fulfilled, then*

$$\lambda^{(0)} u - Au + \varepsilon \beta u \neq 0 \quad \text{for all} \quad u \notin K\,, \quad \varepsilon > 0\,. \tag{2.1}$$

Proof. There exists an eigenvector $u^{(0)} \in K^0 \cap E_A$ corresponding to $\lambda^{(0)}$. If (2.1) is true, then we have

$$\lambda^{(0)}u - Au + \varepsilon\beta u = 0\,,$$
$$\lambda^{(0)}u^{(0)} - Au^{(0)} = 0$$

for some $u \notin K$, $\varepsilon > 0$. This implies

$$\lambda^{(0)}\langle u, u^{(0)}\rangle - \langle Au, u^{(0)}\rangle + \varepsilon\langle\beta u, u^{(0)}\rangle = 0\,,$$
$$\lambda^{(0)}\langle u^{(0)}, u\rangle - \langle Au^{(0)}, u\rangle = 0\,,$$

and therefore in virtue of the symmetry of A we obtain $\langle\beta u, u^{(0)}\rangle = 0$. But this contradicts the assumption (β, K^0).

Remark 2.4. It is clear from the condition (P) that if $[\lambda, u, \varepsilon] \in S$ and $\varepsilon = 0$ or $u \in K$, then $\lambda \in \Lambda_A$ and u is a corresponding eigenvector. In particular, if $\lambda_0 \in \Lambda_A$, then there exists $\delta > 0$ such that if $[\lambda, u, \varepsilon] \in S$, $0 < |\lambda - \lambda_0| < \delta$, then $\varepsilon > 0$, $u \notin K$. (We use the fact that the eigenvalues of A are isolated.)

Lemma 2.2. *Let* $[\lambda_0, u_0, \varepsilon_0] \in S$, $[\lambda_n, u_n, \varepsilon_n] \in S$, $\varepsilon_n \neq \varepsilon_0$ $(n = 1, 2, \ldots)$, $[\lambda_n, u_n, \varepsilon_n] \to [\lambda_0, u_0, \varepsilon_0]$ *in* $\mathbb{R} \times H \times \mathbb{R}$, *let* u_0 *satisfy the condition* (SC) *and let* (M) *be fulfilled. Then*

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_0}{\varepsilon_n - \varepsilon_0} = -\langle\beta u_0, u_0\rangle \leqq 0\,. \tag{2.2}$$

If $u_0 \notin K$ *and* (P) *is fulfilled, then the last expression is even negative.*

Proof. If $[\lambda, u, \varepsilon] \in S$, then the conditions (a), (b) from Definition 2.1 imply

$$\lambda = \lambda\langle u, u\rangle = \langle Au, u\rangle - \varepsilon\langle\beta u, u\rangle\,.$$

Hence using the symmetry of A, we obtain

$$\begin{aligned}
\lambda_n - \lambda_0 = {} & \langle Au_n, u_n - u_0\rangle - \varepsilon_n\langle\beta u_n, u_n - u_0\rangle + \langle Au_0, u_n - u_0\rangle - \\
& - \varepsilon_0\langle\beta u_0, u_n - u_0\rangle + (\varepsilon_0 - \varepsilon_n)\langle\beta u_0, u_n - u_0\rangle + \\
& + \varepsilon_n(\langle\beta u_0, u_n\rangle - \langle\beta u_n, u_0\rangle) + (\varepsilon_0 - \varepsilon_n)\langle\beta u_0, u_0\rangle = \\
= {} & \lambda_n\langle u_n, u_n - u_0\rangle + \lambda_0\langle u_0, u_n - u_0\rangle + (\varepsilon_0 - \varepsilon_n)\langle\beta u_0, u_n - u_0\rangle + \\
& + \varepsilon_n(\langle\beta u_0, u_n\rangle - \langle\beta u_n, u_0\rangle) + (\varepsilon_0 - \varepsilon_n)\langle\beta u_0, u_0\rangle = \\
= {} & \lambda_n - \lambda_0 + (\lambda_0 - \lambda_n)\langle u_n, u_0\rangle + (\varepsilon_0 - \varepsilon_n)\langle\beta u_0, u_n - u_0\rangle + \\
& + \varepsilon_n(\langle\beta u_0, u_n\rangle - \langle\beta u_n, u_0\rangle) + (\varepsilon_0 - \varepsilon_n)\langle\beta u_0, u_0\rangle\,.
\end{aligned}$$

Dividing this equation by $(\varepsilon_n - \varepsilon_0)$ and using the assumption (SC), we obtain (for $n \geqq n_0$, n_0 sufficiently large)

$$\frac{\lambda_n - \lambda_0}{\varepsilon_n - \varepsilon_0}\langle u_n, u_0\rangle = -\langle\beta u_0, u_n - u_0\rangle - \langle\beta u_0, u_0\rangle\,.$$

This implies (2.2) because of $u_n \to u_0$ and (M). The last assertion of Lemma 2.2 is a consequence of the assumption (P).

Lemma 2.3. *Let the assumptions of Theorem* 2.1 *be fulfilled. Let S_c be a connected subset of S containing a point $[\bar{\lambda}, \bar{u}, 0]$, where $\bar{\lambda} \in (\lambda^{(1)}, \lambda^{(0)}\rangle$, $\bar{u} \notin K$. Then for all $[\lambda, u, \varepsilon] \in S_c$ the conditions* (c), (d) *are fulfilled.*

Proof. Denote by S_1 the component of the set

$$\{[\lambda, u, \varepsilon] \in S_c\,; \quad \lambda \in \langle\lambda^{(1)}, \lambda^{(0)}\rangle\}$$

containing $[\bar{\lambda}, \bar{u}, 0]$. First, we shall prove that (c), (d) are true for all points from S_1. We have $\bar{u} \notin K$, $[\bar{\lambda}, \bar{u}, 0] \in S_1$ and S_1 is connected. Thus, if (c) is not true on S_1, then there exists $[\tilde{\lambda}, \tilde{u}, \tilde{\varepsilon}] \in S_1$ such that $\tilde{u} \in \partial K$. We have $\beta\tilde{u} = 0$ by (P) and (b) implies $\tilde{\lambda} \in \Lambda_b \cap \langle\lambda^{(1)}, \lambda^{(0)}\rangle$, This is a contradiction with the assumptions and hence (c) is proved for the points from S_1. Now, let us suppose that (d) is not true on S_1. Then there exists $[\lambda, u, \varepsilon] \in S_1$ such that either

$$\lambda = \lambda^{(0)}\,, \quad [u, \varepsilon] \neq [u^{(0)}, 0] \tag{2.3}$$

or

$$\lambda = \lambda^{(1)}\,. \tag{2.4}$$

If $\lambda = \lambda^{(0)}$, then $\varepsilon = 0$ with respect to Lemma 2.1 and (c). On the other hand, the only normed eigenvector of A corresponding to $\lambda^{(0)}$ and satisfying (c) is $u^{(0)}$. Thus (2.3) is not possible. If $\lambda = \lambda^{(1)}$, then $\varepsilon = 0$ with respect to Lemma 2.1 again. The set S_1 is connected and therefore there exists a sequence $[\lambda_n, u_n, \varepsilon_n] \in S_1$ such that $\lambda_n > \lambda^{(1)}$, $\varepsilon_n \geqq 0$, $\lambda_n \to \lambda^{(1)}$, $\varepsilon_n \to 0$, $u_n \to u$. In virtue of Remark 2.4 we have $\varepsilon_n > 0$ and $u \in E_A$. But this is not possible due to Lemma 2.2 and (P) because $\lambda^{(1)}$ satisfies the condition (SC′) and $u \notin K$ since (c) holds for the points from S_1. Hence, neither (2.3) nor (2.4) can occur which proves (d) for the points from S_1.

Now we shall show that $S_c = S_1$ and the proof of Lemma 2.3 will be complete. Let us suppose that $S_c \neq S_1$. Then there exists $[\lambda, u, \varepsilon] \in S_c$ such that $\lambda \notin \langle\lambda^{(1)}, \lambda^{(0)}\rangle$. Simultaneously, the set $\{[\lambda, u, \varepsilon] \in S_c; \lambda \in \langle\lambda^{(1)}, \lambda^{(0)}\rangle, [\lambda, u, \varepsilon] \notin S_1\}$ is either empty or separated from the set S_1. This together with the connectedness of S_c implies that there exist $[\lambda_n, u_n, \varepsilon_n] \in S_c$ such that $\lambda_n \notin \langle\lambda^{(1)}, \lambda^{(0)}\rangle$ $(n = 1, 2, \ldots)$, $\lambda_n \to \lambda$, $u_n \to u$, $\varepsilon_n \to \varepsilon$, where $[\lambda, u, \varepsilon] \in S_1$. We have $\lambda \in \langle\lambda^{(1)}, \lambda^{(0)}\rangle$ from the definition of S_1, i.e. we obtain $\lambda = \lambda^{(1)}$ or $\lambda = \lambda^{(0)}$. Moreover, (d) holds for the points from S_1 which implies $\lambda = \lambda^{(0)}$, $u = u^{(0)}$, $\varepsilon = 0$. That means $\lambda_n > \lambda^{(0)}$, $\lambda_n \to \lambda^{(0)}$, $\varepsilon_n > 0$ *), $\varepsilon_n \to 0$ and this contradicts Lemma 2.2 because $\lambda^{(0)}$ satisfies the assumptions (SC′) and $u^{(0)} \notin K$. Hence we have $S_1 = S_c$ and Lemma 2.3 is proved.

Lemma 2.4. *Let $\lambda^{(1)}, \lambda^{(0)} \in \Lambda_I$ be simple, $0 < \lambda^{(1)} < \lambda^{(0)}$ and let the assumptions* (P), (CC) *and* (M) *be fulfilled. Suppose that there exist $\varepsilon_n, u_n, \lambda_n$ $(n = 1, 2, \ldots)$ sat-*

*) We use Remark 2.4 again.

isfying the conditions

(a′) $$\|u_n\| = 1\,, \quad n = 1, 2, \ldots, \quad \varepsilon_n \to +\infty\,,$$

(b′) $$\lambda_n u_n - A u_n + \varepsilon_n \beta u_n = 0\,, \quad n = 1, 2, \ldots,$$

(c′) $$u_n \notin K^0\,, \quad n = 1, 2, \ldots,$$

(d′) $$\lambda_n \in (\lambda^{(1)}, \lambda^{(0)})\,, \quad n = 1, 2, \ldots.$$

If $\{r_n\}$ is an arbitrary sequence of indices such that $r_n \to \infty$, $\lambda_{r_n} \to \lambda_\infty$, $u_{r_n} \rightharpoonup u_\infty$ for some λ_∞, u_∞, then $\lambda_\infty \in \Lambda_{V,b} \cap (\lambda^{(1)}, \lambda^{(0)})$, $u_{r_n} \to u_\infty$ and $u_\infty \in E_V \cap \partial K$ is a corresponding eigenvector of (I), (II).

Remark 2.5. It follows from the boundedness of $\langle \lambda^{(1)}, \lambda^{(0)} \rangle$ and the weak compactness of the unit sphere in H that there exists at least one sequence r_n mentioned in the assumptions of Lemma 2.4. That means that Lemma 2.4 guarantees the existence of at least one couple λ_∞, u_∞.

Proof of Lemma 2.4. The sequences $\{\lambda_n u_n\}$, $\{A u_n\}$ are bounded and therefore $\{\varepsilon_n \beta u_n\}$ is bounded by (b′). The assumption (CC) implies that there exists a strongly convergent subsequence of the sequence $\{\varepsilon_{r_n} \beta u_{r_n}\}$. This together with (b′), (d′) and the fact that A is completely continuous implies that there exists a strongly convergent subsequence of $\{u_{r_n}\}$. But we have $u_{r_n} \rightharpoonup u_\infty$ and therefore $u_{r_n} \to u_\infty$. (If this were not the case, we could obtain another subsequence of $\{u_{r_n}\}$ strongly convergent to the point $\tilde{u}_\infty \neq u_\infty$, which is not possible.) Using the assumptions (b′), (P), (M) we obtain for an arbitrary $v \in K$ that

$$\langle \lambda_\infty u_\infty - A u_\infty, v - u_\infty \rangle = \lim_{n\to\infty} \langle \lambda_{r_n} u_{r_n} - A u_{r_n}, v - u_{r_n} \rangle =$$
$$= \lim_{n\to\infty} \varepsilon_{r_n} \langle \beta v - \beta u_{r_n}, v - u_{r_n} \rangle \geqq 0\,.$$

Further, $\beta u_{r_n} \to 0$ because $\{\varepsilon_n \beta u_n\}$ is a bounded sequence and $\varepsilon_n \to \infty$. Hence we have by (M)

$$\langle \beta v, v - u_\infty \rangle = \lim_{n\to\infty} \langle \beta v - \beta u_{r_n}, v - u_{r_n} \rangle \geqq 0$$

for an arbitrary $v \in H$. Setting $v = u_\infty + tw$ for an arbitrary $t > 0$, $w \in H$, we obtain

$$\langle \beta(u_\infty + tw), w \rangle \geqq 0\,.$$

Passing to the limit for $t \to 0+$, we obtain the last inequality for $t = 0$ and for each $w \in H$. This is equivalent to $\beta(u_\infty) = 0$, i.e. $u_\infty \in K$ by (P). We have proved that λ_∞, u_∞ satisfy (I), (II). Moreover, $u_n \notin K^0$, $\|u_n\| = 1$, $u_{r_n} \to u_\infty$ and therefore $u_\infty \in \partial K$, $\|u_\infty\| = 1$. This together with the assumption $\lambda^{(1)}, \lambda^{(0)} \in \Lambda_i$ implies that neither the case $\lambda_\infty = \lambda^{(1)}$ nor $\lambda_\infty = \lambda^{(0)}$ is possible. (We use also the assumption that $\lambda^{(1)}$, $\lambda^{(0)}$ are simple and Lemma 1.1.) Hence we obtain $\lambda_\infty \in (\lambda^{(1)}, \lambda^{(0)})$ and the proof is complete.

3. USING A GLOBAL BIFURCATION RESULT.

First we shall explain a result of E. N. Dancer [3] which is a strengthening of Rabinowitz's result [14]. Let X be a real Hilbert space*) with an inner product $(\cdot, \cdot)$ and with the corresponding norm $|||\cdot|||$, $L : X \to X$ a linear completely continuous selfadjoint*) operator in X. Further, let G be a nonlinear completely continuous mapping of $\mathbb{R} \times X$ into X such that

$$\text{(3.1)} \qquad \lim_{|||x \to 0|||} \frac{G(\mu, x)}{|||x|||} = 0 \quad \text{uniformly on bounded subsets of } \mathbb{R} .$$

We shall consider the bifurcation problem for the equation

$$\text{(B)} \qquad x - \mu L(x) + G(\mu, x) = 0 ,$$

where μ is a real parameter. A point $[\mu_0, 0]$ is said to be a bifurcation point of (B) (with respect to the line $\{[\mu, 0]; \mu \in \mathbb{R}\}$ of trivial solutions) if for each neighbourhood $U(\mu_0, 0)$ of $[\mu_0, 0]$ in $\mathbb{R} \times X$ there exists $[\mu, x] \in U(\mu_0, 0)$ satisfying (B) and $|||x||| \neq 0$. Denote by $r(L)$ the set of all characteristic values of L, i.e. the set of the reciprocals of the non zero eigenvalues of L:

$$r(L) = \{\mu \in \mathbb{R};\ x - \mu L(x) = 0 \text{ for some } x \in X,\ |||x||| \neq 0\} .$$

Remark 3.1. It is well-known that if $[\mu, 0]$ is a bifurcation point of (B) then $\mu \in r(L)$. Indeed, there exist μ_n, x_n $(n = 1, 2, \ldots)$ such that $|||x_n||| > 0$, $\mu_n \to \mu$, $|||x_n||| \to 0$ and

$$\text{(B')} \qquad x_n - \mu_n L(x_n) + G(\mu_n, x_n) = 0 .$$

We can suppose that $y_n = x_n / |||x_n||| \rightharpoonup y$ for some $y \in X$. (In the opposite case we can pass to suitable subsequences.) Dividing (B') by $|||x_n|||$, passing to the limit with $n \to \infty$, using (3.1) and the complete continuity of L we obtain that $y_n \to y$, $y - \mu L(y) = 0$, $|||y||| = 1$. That means $\mu \in r(L)$.

Now denote by C the closure of the set of all nontrivial solutions of (B), i.e.

$$C = \overline{\{[\mu, x] \in \mathbb{R} \times X;\ |||x||| \neq 0 ,\ \text{(B) is fulfilled}\}} .$$

Remark 3.2. A point $[\mu, 0]$ is a bifurcation point of (B) if and only if $[\mu, 0] \in C$. It follows directly from the previous definitions.

Further, let μ_0 be a given simple characteristic value of L with a corresponding eigenvector x_0, $|||x_0||| = 1$. Then $[\mu_0, 0]$ is a bifurcation point of (B) (see [14]). Denote by C_0 the component of C containing the point $[\mu_0, 0]$. Thus, C_0 is non-empty.

*) In the papers [3], [14], a general Banach space and a non-selfadjoint operator L are considered. We are formulating the results for symmetric operators in a Hilbert space because it is simpler and fully sufficient for our purposes.

Moreover, roughly speaking, C_0 "consists of two branches C_0^+ and C_0^- starting in the direction x_0 and $-x_0$, respectively". This situation will be useful for our purposes and we shall describe it precisely.

Let us choose $\eta \in (0, 1)$ and define

$$\begin{aligned} K_\eta &= \{[\mu, x] \in R \times X;\ |(x, x_0)| > \eta |||x|||\}, \\ K_\eta^+ &= \{[\mu, x] \in K_\eta;\ (x, x_0) > 0\}, \\ K_\eta^- &= K_\eta \setminus K_\eta^+ . \end{aligned}$$

There exists $R > 0$ such that

$$(C \setminus \{[\mu_0, 0]\}) \cap B_R(\mu_0, 0) \subset K_\eta ,$$

where $B_R(\mu_0, 0) = \{[\mu, x] \in R \times X;\ |\mu - \mu_0| + |||x||| \leqq R\}$ (for the proof see [14, Lemma 1.24]). For each $r \in (0, R\rangle$ denote by D_r^+ and D_r^-, respectively, the components of the sets $\{[\mu_0, 0]\} \cup (C \cap B_r(\mu_0, 0) \cap K_\eta^+)$ and $\{[\mu_0, 0]\} \cup (C \cap \cap B_r(\mu_0, 0) \cap K_\eta^-)$ containing $[\mu_0, 0]$. Further, denote by $C_{0,r}^+$ and $C_{0,r}^-$ respectively, the components of $\overline{C_0 \setminus D_r^-}$ and $\overline{C_0 \setminus D_r^+}$ containing $[\mu_0, 0]$. Set

$$C_0^+ = \overline{\bigcup_{0<r\leqq R} C_{0,r}^+}, \quad C_0^- = \overline{\bigcup_{0<r\leqq R} C_{0,r}^-} .$$

This definition of C_0^+, C_0^- is independent of the choice of $\eta \in (0, 1)$ (see [14, Lemma 1.24]), the sets C_0^+, C_0^- are connected and

$$C_0 = C_0^+ \cup C_0^-$$

(for the proof see [14]; cf. [3]). Further, the following implications are true (they follow directly from [14, Lemma 1.24] and from the definition of C_0^+, C_0^-):

(3.2) $\quad$ if $[\mu_n, x_n] \in C_0^+ \setminus K_\eta^- \cap B_\delta(\mu_0, 0)$ for some $\delta > 0$,

$$\mu_n \to \mu_0 , \quad \|x_n\| \to 0 , \quad \text{then} \quad \frac{x_n}{\|x_n\|} \to x_0 ;$$

(3.3) $\quad$ if $[\mu_n, x_n] \in C_0^- \setminus K_\eta^+ \cap B_\delta(\mu_0, 0)$ for some $\delta > 0$,

$$\mu_n \to \mu_0 , \quad \|x_n\| \to 0 , \quad \text{then} \quad \frac{x_n}{\|x_n\|} \to -x_0 .$$

Theorem 3.1. (E. N. Dancer [3, Theorem 2]). *Either C_0^+ and C_0^- are both unbounded or $C_0^+ \cap C_0^- \neq \{[\mu_0, 0]\}$.*

Remark 3.3. Let us consider the situation from Theorems 2.2, 2.3. Let us define $X = H \times R$ and introduce the operators L, G from X into X by

$$\begin{aligned} L(x) &= L([v, \varepsilon]) = [Av, 0) \quad \text{for all} \quad x = [v, \varepsilon] \in X , \\ G(\mu, x) &= G([\mu, v, \varepsilon]) = [\mu\varepsilon\beta v, -\|v\|^2] \quad \text{for all} \quad x = [v, \varepsilon] \in X . \end{aligned}$$

We shall study the situation from the beginning of this Section with these special operators and with $\mu_0 = 1/\lambda^{(0)}$. It is easy to see that L, G satisfy all the assumptions mentioned above. The equation (B) can be written as

$$(B'') \qquad [v, \varepsilon] - \mu[Av, 0] + [\mu\varepsilon\beta v, -\|v\|^2] = 0$$

or in the form

$$(a'') \qquad \|v\|^2 = \varepsilon\,,$$

$$(b'') \qquad v - \mu Av + \mu\varepsilon\beta v = 0\,.$$

In particular, we have

$$C = \overline{\{[\mu, v, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R};\ \varepsilon > 0,\ (a''), (b'') \text{ are fulfilled}\}}\,.$$

Remark 3.4. It is clear that μ is a characteristic value of L with a corresponding eigenvector $[u, \varepsilon]$ if and only if $\varepsilon = 0$ and μ is a characteristic value of A with a corresponding eigenvector u. In this case, the multiplicities of μ as a charactetistic value of L and A are equal. Especially, $\mu_0 = 1/\lambda^{(0)}$ is a simple characteristic value of L with a corresponding eigenvector $[u^{(0)}, 0]$ under the assumptions of Theorem 2.1.

Remark 3.5. If we write $\lambda = 1/\mu$, $u = v/\sqrt{\varepsilon}$, then the conditions (a''), (b'') together with $\mu \neq 0$, $\|v\| > 0$ (or $\varepsilon > 0$) are equivalent to the conditions (a), (b) from Definition 2.1 and $\lambda \neq 0$, $\varepsilon > 0$. This together with Remarks 3.2, 3.4 yiedls

$$\{[\lambda, u, \varepsilon] \in S;\ \lambda \neq 0\} = \left\{\left[\frac{1}{\mu}, \frac{v}{\|v\|}, \varepsilon\right];\ [\mu, v, \varepsilon] \in C,\ \mu \neq 0,\ \varepsilon > 0\right\} \cup$$

$$\cup \{[\lambda, u, 0];\ \lambda \neq 0,\ \lambda \in \Lambda_A,\ u \text{ corresp. eigenvector},\ \|u\| = 1\}\,.$$

Remark 3.6. The implications (3.2), (3.3) are equivalent to the following ones in the situation of Remark 3.3:

$$(3.2') \qquad \text{if}\quad [\mu_n, v_n, \varepsilon_n] \in C_0^+ \setminus K_\eta^- \cap B_\delta(\mu_0, 0) \quad \text{for some}\quad \delta > 0\,,$$

$$\mu_n \to \mu_0\,,\quad \|v_n\| \to 0\,,\quad \text{then}\quad \frac{\varepsilon_n}{\|v_n\|} \to 0\,,\quad \frac{v_n}{\|v_n\|} \to u^{(0)}\,;$$

$$(3.3') \qquad \text{if}\quad [\mu_n, v_n, \varepsilon_n] \in C_0^- \setminus K_\eta^+ \cap B_\delta(\mu_0, 0) \quad \text{for some}\quad \delta > 0\,,$$

$$\mu_n \to \mu_0\,,\quad \|v_n\| \to 0\,,\quad \text{then}\quad \frac{\varepsilon_n}{\|v_n\|} \to 0\,,\quad \frac{v_n}{\|v_n\|} \to -u^{(0)}\,.$$

Remark 3.7. If $[\mu, v, \varepsilon] \in C$ and $\varepsilon = 0$ or $v \in K$, then μ is a characteristic value of A and either $\|v\| = 0$ or v is an eigenvector of A corresponding to μ. This follows from Remarks 3.1, 3.2, 3.4, from the equations (a''), (b'') (see Remark 3.3) and the assumption (P). In particular, if $\mu \in r(A)$, then there exists $\delta > 0$ such that if $[\mu, v, \varepsilon] \in$ $\in C$, $0 < |\mu - \mu_0| < \delta$, then $\varepsilon > 0$, $v \notin K$. (We use the fact that the characteristic values of A are isolated; cf. also Remark 2.4.)

Proof of Theorem 2.3. Let X, L and G be the space and the operators introduced in Remark 3.3, $\mu_0 = 1/\lambda^{(0)}$, $\mu_1 = 1/\lambda^{(1)}$. We shall show on the basis of Theorem 3.1 that the set C_0^+ is unbounded. The set S_0 will be obtained by a transformation from C_0^+ and the conditions (c), (d) for S_0 will be proved. Hence it will follow that S_0 is unbounded in ε and this will be the essential part of the proof.

First, we shall show that

$$(3.4) \qquad C_0^- = \{[\mu, v, \varepsilon] \in R \times H \times R;\ \mu = \mu_0,\ \varepsilon \geqq 0,\ v = -\sqrt{(\varepsilon)}\, u^{(0)}\}\,.$$

It is easy to see that the set on the right-hand side of (3.4) is a subset of C_0^-. (It is sufficient to use the fact that $tu^{(0)} \in K$ for all $t \leqq 0$, i.e. $\beta(tu^{(0)}) = 0$ by (P), and that μ_0 is a characteristic value of A with a corresponding eigenvector $u^{(0)}$.) On the other hand, if C_0^- contains some elements of the other type, then in virtue of the connectedness of C_0^- there exists a sequence $\{[\mu_n, v_n, \varepsilon_n]\} \subset C_0^- \setminus K_\eta^+$ such that

$$(3.5) \qquad \|v_n\| > 0\,, \quad |\mu_n - \mu_0| + \left\|\frac{v_n}{\|v_n\|} + u^{(0)}\right\| > 0\,, \quad \mu_n \to \mu_0\,,$$

$$(3.6) \qquad v_n \to tu^{(0)} \quad \text{for some} \quad t \leqq 0\,.$$

It follows from (3.6) that

$$(3.7) \qquad \frac{v_n}{\|v_n\|} \to -u^{(0)}\,.$$

Indeed, this is clear in the case $t < 0$ while in the case $t = 0$ this follows from (3.3') (see Remark 3.6). But we have $-u^{(0)} \in K^0$ by the assumptions, therefore $v_n/\|v_n\| \in K$ for n sufficiently large by (3.7). The conditions (P), (3.5), (3.7) and (b'') imply that $\mu_n \neq \mu_0$ because μ_0 is simple. This is not possible due to Remark 3.7. Hence (3.4) is proved. Now, it is easy to show by an analogous argument using the definition of C_0^+ that

$$(3.8) \qquad C_0^+ \cap C_0^- = [\mu_0, 0, 0]\,.$$

Theorem 3.1 implies that the set C_0^+ is unbounded.

Now let us consider the set

$$\{[\mu, v, \varepsilon] \in C_0^+;\ \mu \neq 0,\ \|v\| > 0\}\,.$$

This set consists of a system of components C_α ($\alpha \in I$, I is a suitable set of indices). Let us define

$$S_\alpha = \overline{\left\{[\lambda, u, \varepsilon];\ \lambda = \frac{1}{\mu},\ u = \frac{v}{\|v\|},\ [\mu, v, \varepsilon] \in C_\alpha\right\}}\,,$$

$$S_0 = \bigcup_{\alpha \in I} S_\alpha\,.$$

It is $S_\alpha \subset S$ and S_α is closed and connected for each $\alpha \in I$. We have $[\mu_0, 0, 0] \in \bar{C}_{\alpha_0}$ at least for one $\alpha_0 \in I$ and therefore there exist $[\mu_n, v_n, \varepsilon_n] \in C_{\alpha_0}$ such that $[\mu_n, v_n, \varepsilon_n] \to \to [\mu_0, 0, 0]$. It is $C_\alpha \subset C_0^+$ which together with (3.8) implies that $[\mu_n, v_n, \varepsilon_n] \notin \notin K_\eta^- \cap B_\delta(\bar{\mu}_0, 0)$ for some δ sufficiently small. Now we obtain $v_n/\|v_n\| \to u^{(0)}$ by (3.2′) and that means $[\lambda^{(0)}, u^{(0)}, 0] \in S_{\alpha_0}$. Thus Lemma 2.3 implies that (c), (d) are fulfilled for all $[\lambda, u, \varepsilon] \in S_{\alpha_0}$. We shall show that this is true for all S_α, $\alpha \in I$. Let us suppose the contrary. We have $C_0^+ = \bigcup_{\alpha\in I} \bar{C}_\alpha$ and this set is connected. Therefore there exist $\alpha_1, \alpha_2 \in I$ such that $\bar{C}_{\alpha_1} \cap \bar{C}_{\alpha_2} \neq \emptyset$, (c), (d) are fulfilled for all $[\lambda, u, \varepsilon] \in S_{\alpha_1}$ but not for all $[\lambda, u, \varepsilon] \in S_{\alpha_2}$. Let $[\tilde{\mu}, \tilde{v}, \tilde{\varepsilon}] \in \bar{C}_{\alpha_1} \cap \bar{C}_{\alpha_2}$. It follows from the definition of $C_{\alpha_1}, C_{\alpha_2}$ that either $\tilde{\mu} = 0$ or $\|\tilde{v}\| = \tilde{\varepsilon} = 0$. We have $\tilde{\mu} \in \langle \mu_0, \mu_1)$ as (d) holds for the points from S_{α_1} and therefore $\tilde{\mu} \neq 0$, $\|\tilde{v}\| = \tilde{\varepsilon} = 0$. Remarks 3.1, 3.2 imply $\tilde{\mu} \in r(L)$. If $\tilde{\mu} = \mu_0$, then we obtain $[\lambda^{(0)}, u^{(0)}, 0] \in S_{\alpha_1} \cap S_{\alpha_2}$ as above for S_{α_0}. If $\tilde{\mu} \in (\mu_0, \mu_1)$, then there exist $[\mu_n^{(i)}, v_n^{(i)}, \varepsilon_n^{(i)}] \in C_{\alpha_i}$ such that $[\mu_n^{(i)}, v_n^{(i)}, \varepsilon_n^{(i)}] \to [\tilde{\mu}, 0, 0]$ $(i = 1, 2)$ and we obtain $v_n^{(i)}/\|v_n^{(i)}\| \to u_i$, where u_i $(i = 1, 2)$ are eigenvectors of A corresponding to $\tilde{\lambda} = 1/\tilde{\mu} \in \Lambda_A$ (see Remarks 3.1, 3.4). Hence it follows that $[\tilde{\lambda}, u_1, 0] \in S_{\alpha_1}$, $[\tilde{\lambda}, u_2, 0] \in S_{\alpha_2}$. We have $u_1 \notin K$, $u_2 \notin K$ be cause we assume $(\lambda^{(1)}, \lambda^{(0)}) \cap (\Lambda_b \cup \Lambda_i) = \emptyset$. Consequently, in each case S_{α_2} contains an element $[\tilde{\lambda}, \tilde{u}, 0]$ with $\tilde{\lambda} \in (\lambda^{(1)}, \lambda^{(0)}\rangle$, $\tilde{u} \notin K$ and Lemma 2.3 implies that (c), (d) are fulfilled for all $[\lambda, u, \varepsilon] \in S_{\alpha_2}$, which is a contradiction. Hence (c), (d) hold for all $[\lambda, u, \varepsilon] \in S_0 = \bigcup_{\alpha\in I} S_\alpha$.

We have proved that C_0^+ is unbounded. It follows from here and (a), (d) that S_0 is unbounded in ε. Using the connectedness of C_0^+ and the previous considerations, it is easy to see that we can choose a finite subsystem $S_1, \ldots, S_\varkappa$ of the system S_α with the properties mentioned in Theorem 2.3. The last part of the assertion of Theorem 2.3 follows from Lemma 2.4.

Remark 3.8. It is easy to see from the proof of Theorem 2.3 that Theorem 2.2 can be proved in the analogous way, only some steps of the proof will be easier.

4. APPLICATION TO THE SUPPORTED BEAM

Let us denote $H = \{u \in W_2^2(\langle 0, 1\rangle);\ u(0) = u(1) = 0\}$. It is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_0^1 u''(x)\, v''(x)\, dx\,.$$

Let A be an operator in H defined by

$$\langle Au, v\rangle = \int_0^1 u'(x)\, v'(x)\, dx \quad \text{for all} \quad u, v \in H\,.$$

A real λ is an eigenvalue and $u \in H$ is a corresponding eigenvector of A if and only if the function u has a continuous derivative of the fourth order on $\langle 0, 1 \rangle$ and

$$\lambda u^{(4)} + u'' = 0 \quad \text{on} \quad \langle 0, 1 \rangle\,, \tag{4.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0\,. \tag{4.2}$$

The problem describes the behaviour of a beam which is simply fixed on its ends and compressed by a force P (see Fig. 4.1). It is $\lambda = IE/P$, where E is the Young modulus of elasticity and I is the moment of inertia. The beam can bend if and only if the force P is such that λ is an eigenvalue of A (i.e. of (4.1), (4.2)) and the bending is described by a corresponding eigenvector.

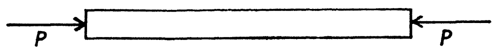

Fig. 4.1

Now let us consider the eigenvalue problem for the variational inequality (I), (II) with the convex closed cone

$$K = \{u \in H;\ u(x_i) \geqq 0,\ i = 1, 2, \ldots, n\}\,,$$

where $x_i \in (0, 1)$ ($i = 1, \ldots, n$, n positive integer) are given numbers.

It is easy to show that λ is an eigenvalue and u is a corresponding eigenvector of (I), (II) if and only if u has a continuous second derivative on $\langle 0, 1 \rangle$, a continuous fourth derivative on (x_i, x_{i+1}) for all $i = 0, 1, \ldots, n$ (where we set $x_0 = 0$, $x_{n+1} = 1$) and

$$\lambda u^{(4)} + u'' = 0 \quad \text{on} \quad (x_i, x_{i+1})\,, \qquad i = 0, 1, \ldots, n\,, \tag{4.3}$$

$$u(x_i) \geqq 0\,, \quad i = 1, \ldots, n\,, \tag{4.4}$$

$$\lim_{x \to x_i -} u'''(x) - \lim_{x \to x_i +} u'''(x) \geqq 0\,, \quad i = 1, \ldots, n\,, \tag{4.5}$$

$$\left[\lim_{x \to x_i -} u'''(x) - \lim_{x \to x_i +} u'''(x)\right] u(x_i) = 0\,, \quad i = 1, \ldots, n\,. \tag{4.6}$$

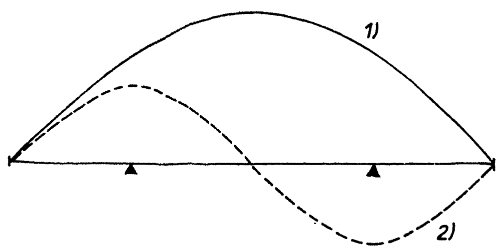

Fig. 4.2. 1) Possible bending 2) Impossible bending

The problem corresponds to a beam which is simply fixed on its ends, compressed by a force P and, moreover, supported by fixed obstacles from below at the points x_i (see Fig. 4.2). The parameter λ has the same meaning as above. The beam can bend if and only if λ is an eigenvalue of the variational inequality (I), (II) (i.e. of (4.3)–(4.6)) and the bending is given by a corresponding eigenvector u of (I), (II).

Let us introduce the penalty operator β by the formula

$$(4.7) \qquad \langle \beta u, v \rangle = - \sum_{i=1}^{n} u^{-}(x_i)\, v(x_i) \quad \text{for all} \quad u, v \in H\,,$$

where u^{-} denotes the negative part of u. It is easy to see that the operators A, β satisfy all the assumptions of Theorems 2.1 and 2.3. The assumption (S) is fulfilled for each $u \in H$ (see also Remark 2.1). The eigenvalues of A (i.e. of (4.1), (4.2)) are the numbers

$$(4.8) \qquad \lambda_k = \frac{1}{k^2 \pi^2}$$

and the corresponding eigenvectors are the functions

$$(4.9) \qquad u_k(x) = \sin k\pi x$$

($k = 1, 2, \ldots$). All eigenvalues of the operator A are simple.

Example 4.1. Let us consider the case $n = 2$, $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$. Then we have

$$\lambda_{4k} \in \Lambda_b\,, \quad \lambda_{4k-3} \in \Lambda_i\,, \quad \lambda_{4k-1} \in \Lambda_i\,, \quad \lambda_{4k-2} \in \Lambda_e\,, \quad k = 1, 2, \ldots,$$

because

$$\sin 4k\pi x_i = 0\,, \quad i = 1, 2\,, \quad k = 1, 2, \ldots,$$
$$\sin (4k-1)\, x_1 = \sin (4k-1)\, x_2 \neq 0\,, \quad \sin (4k-1)\, x_1 = \sin (4k-1)\, x_2 \neq 0\,,$$
$$\sin (4k-2)\, x_1 = - \sin (4k-2)\, x_2 \neq 0\,, \quad k = 1, 2, \ldots.$$

Thus Theorems 2.1 and 2.3 can be used for each couple

$$\lambda^{(1)} = \lambda_{4k-1}\,, \quad \lambda^{(0)} = \lambda_{4k-3}\,.$$

For each $k = 1, 2, \ldots$, we obtain an eigenvalue $\lambda_{k,\infty} \in \Lambda_{V,b} \cap (\lambda_{4k-1}, \lambda_{4k-3})$ with a corresponding eigenvector $u_{k,\infty} \in (E_V \setminus E_A) \cap \partial K$. That means $u_{k,\infty}$ is a "new" eigenvector of the variational inequality (i.e. it is not simultaneously an eigenvector of A) and $u(x_i) \geqq 0$, $i = 1, 2$, $u(x_1)\, u(x_2) = 0$. In particular, there exists an infinite sequence of eigenvalues of (I), (II) such that there exist corresponding eigenvectors which are not eigenvectors of A.

Example 4.2. Let $n \geqq 3$ be arbitrary and let x_i ($i = 1, \ldots, n$) be such that

$$x_i \in (0, \varepsilon\rangle \cup \left\langle \frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2} \right\rangle \cup \langle 1 - \varepsilon, 1)$$

and each of the intervals $(0, \varepsilon\rangle$, $\langle \frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} + \frac{1}{2}\varepsilon\rangle$, $\langle 1 - \varepsilon, 1)$ contains at least one x_i, where $\varepsilon \in (0, \frac{1}{4})$. We shall consider the eigenvalues λ_k with $k < 1/\varepsilon$ only. We have

$$\lambda_{4k} \in \Lambda_e\,, \quad \lambda_{4k-1} \in \Lambda_e\,, \quad \lambda_{4k-2} \in \Lambda_e\,, \quad \lambda_{4k-3} \in \Lambda_i \quad \text{for} \quad k = 1, 2, \ldots, \left[\frac{1}{4\varepsilon}\right]$$

(where $[1/4\varepsilon]$ is the entire part of $1/4\varepsilon$), because

$$\begin{aligned}
&\sin 4k\pi x > 0 \quad \text{on} \quad \langle 0, \varepsilon\rangle\,, \quad \sin 4k\pi x < 0 \quad \text{on} \quad \langle 1 - \varepsilon, 1\rangle\,,\\
&\sin (4k - 1)\,\pi x > 0 \quad \text{on} \quad \langle 0, \varepsilon\rangle \cup \langle 1 - \varepsilon, 1\rangle\,,\\
&\sin (4k - 1)\,\pi x < 0 \quad \langle \tfrac{1}{2} - \tfrac{1}{2}\varepsilon, \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\rangle\,,\\
&\sin (4k - 2)\pi x > 0 \quad \text{on} \quad \langle 0, \varepsilon\rangle\,, \quad \sin (4k - 2)\,\pi x < 0 \quad \text{on} \quad \langle 1 - \varepsilon, 1\rangle\,,\\
&\sin (4k - 3)\,\pi x > 0 \quad \text{on} \quad \langle 0, \varepsilon\rangle \cup \langle \tfrac{1}{2} - \tfrac{1}{2}\varepsilon, \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\rangle \cup (1 - \varepsilon, 1\rangle\,,
\end{aligned}$$

$k = 1, 2, \ldots, [1/4\varepsilon]$. Theorems 2.1, 2.3 can be applied for each couple $\lambda^{(0)} = \lambda_{4k-3}$, $\lambda^{(1)} = \lambda_{4k+1}$, $k = 1, 2, \ldots, [1/4\varepsilon] - 1$. Thus there exists $\lambda_{k,\infty} \in \Lambda_{V,b} \cap (\lambda_{4(k+1)-3}, \lambda_{4k-3})$ with a corresponding eigenvector $u_{k,\infty} \in (E_V \setminus E_A) \cap \partial K$ for $k = 1, 2, \ldots, \ldots, [1/4\varepsilon] - 1$. That means that $u_{k,\infty}$ is a "new" eigenvector of (I), (II) (i.e. it is not simultaneously an eigenvector of A) and $u_{k,\infty}(x_i) \geqq 0$, $i = 1, \ldots, n$, $u(x_1) \cdot \ldots \cdot u(x_n) = 0$.

Analogously, we can consider a beam supported not only in a finite number of points but, for example, on some intervals. This situation corresponds to the variational inequality (I), (II) with the cone

$$K = \{u \in H;\ u(x) \geqq 0 \ \text{for} \ x \in \langle x_i, y_i\rangle,\ i = 1, \ldots, n\}\,,$$

where x_i, y_i are given numbers, $0 < x_1 < y_1 < \ldots < x_n < y_n < 1$. In this case, we can use the penalty operator defined by

$$\langle \beta u, v\rangle = -\sum_{i=1}^{n} \int_{x_i}^{y_i} u^-(x)\, v(x)\, \mathrm{d}x \quad \text{for all} \quad u, v \in H\,.$$

The assumptions of Theorems 2.1, 2.2 are fulfilled again. A point u fulfils the assumption (SC) if and only if $|u(x)| > 0$ on $\langle x_i, y_i\rangle$, $i = 1, \ldots, n$ (cf. Remark 2.1). In particular, an arbitrary interior eigenvalue of A satisfies (SC′).

Example 4.3. Set

$$K = \{u \in H;\ u(x) \geqq 0 \ \text{for} \ x \in \langle \tfrac{1}{2} - \tfrac{1}{2}\delta, \tfrac{1}{2} + \tfrac{1}{2}\delta\rangle\}\,,$$

where $\delta \in (0, \frac{1}{3})$ is given. Then we have

$$\lambda_{2k} \in \Lambda_e\,, \quad \lambda_{2k-1} \in \Lambda_i\,, \quad k = 1, \ldots, [1/2\delta]\,,$$
$$\lambda_k \in \Lambda_e\,, \quad k > [1/\delta]$$

because $\sin 2k\pi x$ for $k = 1, \ldots, [1/2\delta]$ and $\sin k\pi x$ for $k > [1/\delta]$ change their signs on $\langle \frac{1}{2} - \frac{1}{2}\delta, \frac{1}{2} + \frac{1}{2}\delta \rangle$, $\sin (2k - 1)\pi x$ does not change its sign on $\langle \frac{1}{2} - \frac{1}{2}\delta, \frac{1}{2} + \frac{1}{2}\delta \rangle$ for $k = 1, \ldots, [1/2\delta]$. Thus Theorems 2.1 and 2.3 can be used for the couples $\lambda^{(0)} = \lambda_{2k-1}$, $\lambda^{(1)} = \lambda_{2k+1}$, $k = 1, \ldots, [1/2\delta] - 1$. For each $k = 1, \ldots, [1/2\delta] - 1$, we obtain an eigenvalue $\lambda_{k,\infty} \in \Lambda_{V,b} \cap (\lambda_{2k-1}, \lambda_{2k+1})$ with a corresponding eigenvector $u_{k,\infty} \in (E_V \setminus E_A) \cap \partial K$. Hence $u_{k,\infty}$ is a "new" eigenvector of (I), (II) and $u(x) \geqq 0$ on $\langle \frac{1}{2} - \frac{1}{2}\delta, \frac{1}{2} + \frac{1}{2}\delta \rangle$, $u(x) = 0$ at least for one $x \in \langle \frac{1}{2} - \frac{1}{2}\delta, \frac{1}{2} + \frac{1}{2}\delta \rangle$.

Example 4.4. Set

$$K = \{u \in H;\ u(x) \geqq 0 \ \text{ for } \ x \in \langle \tfrac{1}{2}\delta, \delta \rangle \cup \langle \tfrac{1}{2} - \tfrac{1}{2}\delta, \tfrac{1}{2} + \tfrac{1}{2}\delta \rangle \cup \langle 1 - \delta, 1 - \tfrac{1}{2}\delta \rangle\,.$$

Similarly as in Example 4.2, we obtain

$$\lambda_{4k} \in \Lambda_e\,, \quad \lambda_{4k-1} \in \Lambda_e\,, \quad \lambda_{4k-2} \in \Lambda_e\,, \quad \lambda_{4k-3} \in \Lambda_i\,,$$
$$k = 1, \ldots, [1/4\delta]\,, \quad \lambda_k \in \Lambda_e\,, \quad k > [1/4\delta]\,.$$

Hence we obtain $\lambda_{k,\infty} \in \Lambda_{V,b} \cap (\lambda_{4k+1}, \lambda_{4k-3})$, $u_{k,\infty} \in (E_V \setminus E_A) \cap \partial K$, $k = 1, \ldots \ldots, [1/4\delta] - 1$.

Example 4.5. Let

$$K = \{u \in H;\ u(x) \geqq 0 \ \text{ for } \ x \in \langle 0, \delta \rangle\}\,,$$

where $\delta \in (0, \frac{1}{2})$ is given. Then $\lambda_k \in \Lambda_i$ for all $k = 1, 2, \ldots, [1/\delta]$. Theorem 2.2 can be used for each couple $\lambda^{(1)} = \lambda_k$, $\lambda^{(0)} = \lambda_{k-1}$, $k = 1, 2, \ldots, [1/\delta]$ and we obtain $\lambda_{k,\infty} \in (\Lambda_{V,b} \setminus \Lambda_A) \cap (\lambda_k, \lambda_{k-1})$.

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Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).