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NOTE ON OPERATORS PRODUCED BY SESQUILINEAR FORMS

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The purpose of this note is to show some "interior" properties characterizing the operators produced by certain sesquilinear forms which are frequently studied in the theory of elliptic operators.

We shall denote by H an arbitrary complex Hilbert space with a norm $\|\cdot\|$ and scalar product $\langle .,. \rangle$. Further, let $L^+(H)$ be the set of all linear operators from H into itself. The complex number field will be denoted by C.

Let $A \in L^+(H)$. The operator A is called *nondissipative* if $\operatorname{Re} \langle Ax, x \rangle \geqq 0$ for any $x \in D(A)$.

Let $A \in L^+(H)$. The operator A will be called *special* if

(A_1) for every $x, y \in H$ for which there exists a sequence $x_k \in D(A)$, $k \in \{1, 2, \ldots\}$ such that $x_k \to x$, the sequence $\operatorname{Re} \langle Ax_k, x_k \rangle$ is bounded and $\langle Ax_k, z \rangle \to \langle y, z \rangle$ for any $z \in D(A)$, we have $x \in D(A)$ and $Ax = y$,

(B_1) $|\operatorname{Im} \langle Ax, x \rangle| \leqq d[|\operatorname{Re} \langle Ax, x \rangle| + \|x\|^2]$ for any $x \in D(A)$ with a fixed constant $d \geqq 0$.

Let V be a linear space and S a mapping of the set $V \times V$ into C. The mapping S is called a *sesquilinear form on the space* V if

$$S(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 S(x, y) + \alpha_2 S(x_2, y),$$

$$S(x, \alpha_1 y_1 + \alpha_2 y_2) = \bar{\alpha}_1 S(x, y_1) + \bar{\alpha}_2 S(x, y_2)$$

for any $x, y, x_1, x_2, y_1, y_2 \in V$ and $\alpha_1, \alpha_2 \in C$.

Let $A \in L^+(H)$. The operator A will be called sesquilinearizable (or produced by a sesquilinear form) if there exist a Hilbert space V and a sesquilinear form S on V such that

(A_2) V is a dense subset of H,

(B_2) there exists a positive constant $q > 0$ such that $\|x\|_V \geqq q\|x\|$ for any $x \in V$,

(C_2) there exists a nonnegative constant $M \geqq 0$ such that $|S(x, x)| \leqq M\|x\|_V^2$ for every $x \in V$,

(D_2) there exists a positive constant $m > 0$ such that $|\mathrm{Re}\, S(x, x)| \geqq m\|x\|_V^2$ for every $x \in V$,

(E_2) $D(A) \subseteq V$,

(F_2) $\langle Ax, z\rangle + \langle x, z\rangle = S(x, z)$ for every $x \in D(A)$ and $z \in V$,

(G_2) if $x \in V$ and there exists $y \in H$ such that $S(x, z) = (y, z)$ for any $z \in V$, then $x \in D(A)$.

Remark. A very closely related notion is that of "regularly accretive" operators used in [2]. More precisely, an operator $A \in L^+(H)$ is regularly accretive if and only if there is a constant $\omega \in R$ such that $A + \omega I$ is nondissipative and sesquilinearizable.

Lemma 1. *Let V be a pre-Hilbert space and S a sesquilinear form on V. If $|S(x, x)| \leqq K\|x\|^2$ for any $x \in V$, then $|S(x, y)| \leqq 2K\|x\| \cdot \|y\|$ for any $x, y \in V$.*

Proof. See [1], Chap. 12, Cor. 3.2.

Lemma 2. *Let V be a Hilbert space and S a sesquilinear form on V. If there exist constants $0 < m \leqq M$ such that $m\|x\|^2 \leqq S(x, x) \leqq M\|x\|^2$ for any $x \in V$, then for every $\Phi \in V^*$,*

(a) there exists a unique $x \in V$ such that $\Phi(z) = S(z, x)$ for any $z \in V$,

(b) there exists a unique $x \in V$ such that $\overline{\Phi(z)} = S(x, z)$ for any $z \in V$.

Proof. An easy consequence of the Riesz theorem on the representation of continuous linear functionals on Hilbert spaces.

Theorem 1. *Let $A \in L^+(H)$. If the operator A is nondissipative and special, then it is sesquilinearizable.*

Proof. The symbols (A_1), (B_1) and (A_2)–(F_2) refer to the defining properties of special and sesquilinearizable operators, respectively.

Let us now define

(1) $|x| = [|\mathrm{Re}\,\langle Ax, x\rangle| + \|x\|^2]^{1/2}$ for $x \in D(A)$,

(2) $(x, y) = \frac{1}{2}[\langle Ax, y\rangle + \langle x, Ay\rangle]$ for $x, y \in D(A)$.

It is easy to see from (1) and (2) with respect to the nondissipativity of A that for every $x, x_1, x_2, y \in D(A)$ and $\alpha_1, \alpha_2 \in C$ it is

(3) $|x| \geqq \|x\|$,

(4) $|x|^2 = (x, x)$,

(5) $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1(x, y) + \alpha_2(x_2, y)$,

(6) $(x, y) = \overline{(y, x)}$.

The statements (3)–(6) show that

(7) $D(A)$ is a pre-Hilbert space with the norm $|\cdot|$ and the scalar product $(\cdot,\cdot)$.

Let us now choose a fixed constant d for which (B_1) holds, i.e.

(8) $|\text{Im} \langle Ax, x\rangle| \leqq d[\text{Re} \langle Ax, x\rangle + \|x\|^2]$ for any $x \in D(A)$.

We obtain easily from (1) and (8) that

(9) $$|\langle Ax, x\rangle| = [(\text{Re} \langle Ax, x\rangle)^2 + (\text{Im} \langle Ax, x\rangle)^2]^{1/2} \leqq$$
$$\leqq 2[|Re \langle Ax, x\rangle| + |\text{Im} \langle Ax, x\rangle|] \leqq$$
$$\leqq 2[|\text{Re} \langle Ax, x\rangle| + d|\text{Re} \langle Ax, x\rangle| + d\|x\|^2] \leqq$$
$$\leqq 2(1 + d) [|\text{Re} \langle Ax, x\rangle| + \|x\|^2] = 2(1 + d) |x|^2 \quad \text{for every} \quad x \in D(A).$$

Further, let us take

(10) $K = 2(1 + d)$.

It is clear from (7), (9) and (10) that $D(A)$, $\langle A.,.\rangle$ and K fulfil the assumptions of Lemma 1 and consequently

(11) $|\langle Ax, y\rangle| \leqq 2(1 + d) |x| |y|$ for every $x, y \in D(A)$.

Let us now define V as the completion of the pre-Hilbert space $D(A)$ defined by (7). Then

(12) V is a Hilbert space (with the norm $\|\cdot\|_V$ and the scalar product $\langle .,.\rangle_V$),

(13) $D(A)$ is dense in the space V.

Moreover, (3) implies that we can immerse the space V into H in a natural way so that

(14) $\|x\|_V \geqq \|x\|$ for any $x \in V$.

It follows easily from (A_1) that

(15) $D(A)$ is dense in the space H.

Now we conclude from (13) and (15) that

(16) V is a dense subset of H.

We obtain easily from (1), (3), (9), (12) and (13) that there exists a unique sesquilinear form S on V such that

(17) S satisfies the assumptions of Lemma 2 with $M = 2(1 + d)$ and $m = 1$,

(18) $S(x, z) = \langle Ax, z\rangle + \langle x, z\rangle$ for any $x \in D(A)$ and $z \in V$.

Now we shall prove that

(19) for any $y \in H$, there exists $x \in D(A)$ so that $Ax + x = y$.

Let $y \in H$ and let $l(z) = \langle \overline{y, z} \rangle$ for any $z \in H$. Moreover, let Φ be the restriction of l to V. It follows from (14) that $\Phi \in V^*$. By (17), we can apply Lemma 2 and hence there exists $x \in V$ so that

(20) $S(x, z) = \overline{\Phi(z)} = \langle y, z \rangle$ for any $z \in V$.

By (16), there exists a sequence x_k, $k \in \{1, 2, \ldots\}$, such that

(21) $x_k \in D(A)$ for any $k \in \{1, 2, \ldots\}$ and $x_k \to x$ in the space V.

By (18), (20) and (21) we obtain

(22) $\langle x_k, z \rangle + \langle Ax_k, z \rangle \to \langle y, z \rangle$ for every $z \in V$.

Further, we see from (14) and (21) that

(23) $x_k \to x$ in H.

Moreover, (11) and (21) imply that

(24) $\langle Ax_k, x_k \rangle$ is a bounded sequence.

Now we see easily from (13), (16) and (21)–(24) that (A_1) applies to A which proves (19).

Summing up (12), (13), (15)–(19) we see that the operator A is sesquilinearizable, which proves Theorem 1.

Theorem 2. *Let $A \in L^+(H)$. If the operator A is sesquilinearizable, then it is special.*

Proof. The symbols (A_1), (B_1), (C_1) and (A_2)–(F_2) refer to the defining properties of special and sesquilinear operators, respectively.

We shall first prove

(1) $R(I + A) = H$.

Indeed, let $y \in H$ and let us define $\Phi(z) = \langle \overline{y, z} \rangle$. Then (B_2) implies $\Phi \in V^*$. By (C_2) and (D_2) we can apply Lemma 2 and hence there exists $x \in V$ such that

(2) $S(x, z) = \overline{\Phi(z)} = \langle y, z)$ for any $z \in V$.

We see from (E_2)–(G_2) that (2) implies $x \in D(A)$ and $x + Ax = y$ which verifies (1).

Further, we need to prove that

(3) $D(A)$ is dense in V.

Let this be not true. Then there exists $v \in V$ such that

(4) $\quad v \neq 0 \quad$ and $\quad \langle v, x \rangle_V = 0 \quad$ for any $\quad x \in D(A)$.

By Lemma 2, there exists $w \in V$ such that

(5) $\quad \langle v, z \rangle_V = S(w, z) \quad$ for any $\quad z \in V$.

It follows from (4) and (5) that

(6) $\quad S(x, w) = 0 \quad$ for any $\quad x \in D(A)$.

Hence by (F_2) we see from (6) that $\langle x + Ax, w \rangle = 0$ for any $x \in D(A)$ and consequently, by (1), $w = 0$. This implies by (5) that $v = 0$ which contradicts (4). Hence (3) is true.

We shall now prove that

(7) $$|\mathrm{Im} \langle Ax, x \rangle| \leqq \left(\frac{1 + \sqrt{2}}{m}\left(M + \frac{1}{q}\right)\right)(|\mathrm{Re} \langle Ax, x \rangle| + \|x\|^2)$$

for any $\quad x \in D(A)$.

Indeed, we have by (C_2) and (F_2) that for any $x \in D(A)$

$$|\langle Ax, x \rangle + \|x\|^2| \leqq M\|x\|_V^2 .$$

Hence for $x \in D(A)$

$$|\langle Ax, x \rangle| - \|x\|^2 \leqq M\|x\|_V^2$$

which implies according to (B_2)

(8) $$|\langle Ax, x \rangle| \leqq M\|x\|_V^2 + \|x\|^2 \leqq M\|x\|_V^2 + \frac{1}{q}\|x\|_V^2 = \left(M + \frac{1}{q}\right)\|x\|_V^2$$

for any $\quad x \in D(A)$.

On the other hand,

(9) $$|\langle Ax, x \rangle| = [(\mathrm{Re} \langle Ax, x \rangle)^2 + (\mathrm{Im} \langle Ax, x \rangle)^2]^{1/2} \geqq$$
$$\geqq \frac{1}{\sqrt{2}}[|\mathrm{Re} \langle Ax, x \rangle| + |\mathrm{Im} \langle Ax, x \rangle|] \geqq$$
$$\geqq -\frac{1}{\sqrt{2}}|\mathrm{Re} \langle Ax, x \rangle| + \frac{1}{\sqrt{2}}|\mathrm{Im} \langle Ax, x \rangle| \quad \text{for any} \quad x \in D(A).$$

Consequently, (8) and (9) yield

(10) $$|\mathrm{Im} \langle Ax, x \rangle| \leqq \sqrt{2}\,|\langle Ax, x \rangle| + |\mathrm{Re} \langle Ax, x \rangle| \leqq$$
$$\leqq |\mathrm{Re} \langle Ax, x \rangle| + \sqrt{2}\left(M + \frac{1}{q}\right)\|x\|_V^2 \quad \text{for any} \quad x \in D(A).$$

Using now (D_2) and (F_2) we obtain from (10) that

$$|\mathrm{Im}\langle Ax, x\rangle| \leqq |\mathrm{Re}\langle Ax, x\rangle| + \frac{\sqrt{2}}{m}\left(M + \frac{1}{q}\right)(|\mathrm{Re}\langle Ax, x\rangle| + \|x\|^2) \leqq$$
$$\leqq \left[1 + \frac{\sqrt{2}}{m}\left(M + \frac{1}{q}\right)\right](|\mathrm{Re}\langle Ax, x\rangle| + \|x\|^2)$$

which verifies (7).

Suppose that

(11) $x, y \in H$, $x_k \in D(A)$, $x_k \to x$, $\mathrm{Re}\langle Ax_k, x_k\rangle$ is a bounded sequence and $\langle Ax_k, z\rangle \to \langle y, z\rangle$ for any $z \in D(A)$.

Since by the assumption (11) the sequences $\mathrm{Re}\langle Ax_k, x_k\rangle$ and $\|x_k\|$ are bounded we conclude from (D_2) and (F_2) that

(12) the sequence x_k is bounded in V.

The assumption $\langle Ax_k, z\rangle \to \langle y, z\rangle$ for any $z \in D(A)$ and $x_k \to x$ (from (11)) may be rewritten by (F_2) in the form

(13) $S(x_k, z) \to \langle y, z\rangle + \langle x, z\rangle$ for any $z \in D(A)$.

On the other hand, by (12) and (C_2) we have

(14) $|S(x_k, z)| \leqq K(\sup_k \|x_k\|_V) \|z\|_V$ for any $k \in \{1, 2, \ldots\}$ and $y \in V$.

Applying the Banach-Steinhaus theorem, we obtain from (3), (13) and (14) that

(15) $S(x_k, z) \to \langle y, z\rangle$ for every $z \in V$.

Next we shall prove that

(16) the sequence x_k is weakly fundamental in the space V.

Indeed, let $\Phi \in V^*$. By (C_2), (D_2) and Lemma 2 there exists $z \in V$ such that $\Phi(x) = S(x, z)$ for every $x \in V$. Hence (15) implies $\Phi(x_k) \to \langle y, z\rangle$ which proves (16).

On the other hand, since V is assumed to be a Hilbert space, it follows from (16) that

(17) the sequence x_k is weakly convergent in V, i.e. there exists $x_0 \in V$ such that $x_k \to x_0$ weakly in V.

Using (A_2), (B_2), we deduce easily from (18) that

(18) $x_k \to x_0$ weakly in H.

But due to (11) and (18), it is necessarily

(19) $x_0 = x$.

It follows easily from (17) and (19) that

(20) $S(x_k, z) \to S(x, z)$ for any $z \in V$.

The preceding results (13) and (20) imply

(21) $S(x, z) = \langle y, z \rangle + \langle x, z \rangle$ for every $z \in V$.

Using (G_2), we see from (21) that $x \in D(A)$ and using (A_2) and (F_2) we conclude that, moreover, $Ax + x = y + x$. This result enables us to state that

(22) under the assumption (11), $x \in D(A)$ and $Ax = y$.

The proof is complete since the properties (A_1) and (B_1) are verified in (22) and (7).

Proposition. *Every selfadjoint operator is special.*

Proof. First we shall verify (A_1).

Let $x, z \in H$, $x_k \in D(A)$, $x_k \to x$ and $\langle Ax_k, y \rangle \to \langle z, y \rangle$ for every $y \in D(A)$.

Then $\langle Ax_k, y \rangle = \langle x_k, Ay \rangle$ and hence $\langle x, Ay \rangle = \langle z, y \rangle$ for every $y \in D(A)$. This implies that $x \in D(A^*)$ and $A^*x = z$. But this is in fact $x \in D(A)$ and $Ax = z$.

The condition (B_1) is trivial since $\operatorname{Im} \langle Ax, x \rangle = 0$ for any $x \in D(A)$.

References

[1] *Schechter, M.:* Principles of functional analysis, 1971.
[2] *Schechter, M.:* Spectra of partial differential equations, 1971.

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