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EMBEDDING TREES INTO CLIQUE-BRIDGE-CLIQUE GRAPHS

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The paper [2] concerns embedding trees into graphs which have exactly two blocks, each of them being a clique. Now we shall study a similar problem – embedding trees into graphs which consist of two vertex-disjoint cliques and of a bridge between them. Such a graph will be called a clique-bridge-clique graph (shortly *CBC*-graph).

Let n and k be two positive integers, $2 \leqq k \leqq [\frac{1}{2}n]$. By $H_n(k)$ we denote the *CBC*-graph in which one of the mentioned cliques has k and the other $n - k$ vertices. We shall investigate the conditions for a tree with n vertices to be embeddable into $H_n(k)$.

We shall use some concepts from [2]. A median of a tree T with n vertices is a vertex a of T at which the vertex deviation $m_1(a)$ attains its minimum. The vertex deviation is defined by

$$m_1(a) = \frac{1}{n} \sum_{x \in V} d(a, x),$$

where V is the vertex set of T and $d(a, x)$ denotes the distance between a and x in T. A tree has either exactly one median, or exactly two medians which are joined by an edge.

We recall also the definition of a branch. Let a be a vertex of a tree T. We define a binary relation E on the set of vertices of T which are distinct from a such that $(x, y) \in E$ if and only if the vertex a does not separate x from y in T (this means that the path connecting x and y in T does not contain a). The relation E is evidently an equivalence. The subtree of T induced by the union of one class of E with the one-element set $\{a\}$ is called a branch of T with the knag a.

Theorem 1. *Let n be an even positive integer, $n \geqq 4$. A tree T with n vertices can be embedded into $H_n(\frac{1}{2}n)$ if and only if it has two medians.*

Prpof. The weight of a vertex v of a tree T is defined in [1] as the maximal number of edges of a branch with the knag v. In [3] it is proved that a vertex of a tree has the minimal weight if and only if it is a median of this tree. Let T be a tree with n vertices and two medians. By Theorem 3 from [2] it can be embedded into the graph $G_n(\frac{1}{2}n)$ consisting of two blocks which are both cliques, one of them has $\frac{1}{2}n$, the other $\frac{1}{2}n + 1$ vertices. Let the former be B_1, the latter B_2. Let a be the vertex of T which is mapped onto the cut vertex of $G_n(\frac{1}{2}n)$ in this embedding. The weight of a is evidently at most $\frac{1}{2}n$, the weight of any vertex mapped onto a vertex of B_1 which is not a cut vertex is greater than $\frac{1}{2}n$, because there exists a branch with this vertex as a knag which contains all vertices which are embedded into B_2. Thus a is a median of T and the other median a' of T is mapped onto a vertex of B_2. The vertex a cannot be joined in T with other vertex embedded into B_2 than a'. If we delete from $G_n(\frac{1}{2}n)$ all edges joining a with vertices of B_2 except for the edge aa', we obtain the graph $H_n(\frac{1}{2}n)$ and T is embedded into $H_n(\frac{1}{2}n)$. On the other hand, let a tree T be embedded into $H_n(\frac{1}{2}n)$, let a and a' be the vertices of T which are mapped onto the end vertices of the bridge of $H_n(\frac{1}{2}n)$ in this embedding. Then evidently the weights of a and a' are both equal to $\frac{1}{2}n$ and the weights of all other vertices are greater. Therefore a and a' are medians of T.

Theorem 2. *Let T be a tree with $n \geqq 4$ vertices. The tree T can be embedded into $H_n([\frac{1}{2}n])$ if and only if the weight of its median is $[\frac{1}{2}(n + 1)]$.*

Proof. First we shall prove necessity of the condition. If n is even, then $[\frac{1}{2}(n + 1)] = [\frac{1}{2}n] = \frac{1}{2}n$. By Theorem 1 the tree T can be embedded into $H_n(\frac{1}{2}n)$ if and only if it has two medians. Thus let T have two medians a and a'. Let B (or B') be the branch of T with the knag a (or a') which contains a' (or a, respectively). If B has less than $w(a)$ edges (where $w(a)$ denotes the weight of a), then there exists a branch with the knag a other than B which has $w(a)$ edges. It is a proper subtree of B', therefore B' has more than $w(a)$ edges and a' is not a median, which is a contradiction. Therefore B has $w(a)$ edges and analogously B' has $w(a') = w(a)$ edges. The branches B and B' have exactly one common edge aa' and their union is the whole tree T, therefore $n - 1 = 2\,w(a) - 1$ and $w(a) = \frac{1}{2}n$. We have proved necessity of the condition for n even. Now let n be odd. Then $[\frac{1}{2}(n + 1)] = \frac{1}{2}(n + 1)$, $[\frac{1}{2}n] = \frac{1}{2}(n - 1)$. Suppose that the weight $w(a)$ of a median a of T is greater than $\frac{1}{2}(n + 1)$. Let b be the vertex adjacent to a and belonging to the branch with the knag a which has $w(a)$ edges. The branch with the knag b which contains a has $n - w(a)$ edges, the sum of numbers of edges of other branches with the knag b is $w(a) - 1$. Thus $w(b) \leqq \min(w(a) - 1, n - w(a))$. We have $w(a) - 1 > \frac{1}{2}(n + 1) - 1 = \frac{1}{2}(n - 1)$, $n - w(a) < n - \frac{1}{2}(n + 1) = \frac{1}{2}(n - 1)$, therefore $w(b) \leqq \frac{1}{2}(n - 1) < w(a)$ and this is a contradiction with the assumption that a is a median of T. Therefore $w(a) \leqq \frac{1}{2}(n + 1)$. Now let v be a vertex of T which is not a median of T; let again a be a median of T. Let B (or B') be the branch of T with the knag v (or a) which contains a (or v, respectively). If there is a branch with the knag v with $w(v)$ edges other than B, then B' contains all this branch and, moreover, the path connecting a and v, thus it has more than $w(v)$ edges and $w(a) > w(v)$, which is a contradiction. Thus B has $w(v)$ edges. Suppose that $w(v) < \frac{1}{2}(n + 1)$. The sum of numbers of edges of branches with the knag a other than B' is less than $w(v)$, therefore B' has at least $n - w(v)$ edges and $w(a) \geqq n - w(v)$, which implies $w(v) \geqq n - w(a)$. As $w(a) \leqq \frac{1}{2}(n + 1)$, we have $w(v) \geqq n - \frac{1}{2}(n + 1) = \frac{1}{2}(n - 1)$. We have proved that $w(v)$ can be less than $\frac{1}{2}(n - 1)$ only if v is a median of T. Let T be embedded into $H_n(\frac{1}{2}(n - 1))$. Let B_1 be the clique of $H_n(\frac{1}{2}(n - 1))$ with $\frac{1}{2}(n - 1)$ vertices, let u be the vertex of T which is mapped onto the end vertex of the bridge of $H_n(\frac{1}{2}(n - 1))$ belonging to B_1. The vertices of T which are mapped onto vertices of $H_n(\frac{1}{2}(n - 1))$ not belonging to B_1 together with u form a branch of T with the knag u. This branch has $\frac{1}{2}(n + 1)$ edges, thus $w(u) = \frac{1}{2}(n + 1)$ and u is a median of T.

Now we shall prove sufficiency of the condition. Let $w(a) = [\frac{1}{2}(n + 1)]$ for a median a of T. Then evidently T can be embedded into $H_n([\frac{1}{2}n])$ so that a is mapped onto the end vertex of the bridge belonging to the clique with $[\frac{1}{2}n]$ vertices and all vertices of the branch with the knag a having $w(a)$ edges except for a are mapped onto the vertices of the other clique.

Theorem 3. *Let T be a tree with $n \geqq 4$ vertices, let T contain a subtree T' which is a snake and one terminal vertex of which is a median of T. Let T' have $[\frac{1}{2}n]$ vertices. Then T can be embedded into $H_n(k)$ for all $k = 2, \ldots, [\frac{1}{2}n]$.*

Remark. A snake is a tree which consists of vertices and edges of one simple path.

Proof. Let the vertices of the snake T' be $u_0, \ldots, u_m$, where $m = [\frac{1}{2}n]$. Let u_m be the median of T. Then for each $k = 2, \ldots, [\frac{1}{2}n]$ we can embed T into $H_n(k)$ so that the end vertices of the bridge coincide with the vertices u_k and u_{k+1}.

Theorem 4. *Let $n \geqq 4$ be a positive integer, let K be a subset of the number set $\{2, \ldots, [\frac{1}{2}n]\}$. Then there exists a tree with n vertices which can be embedded into $H_n(k)$ for each $k \in K$ and cannot be embedded into $H_n(k)$ for $k \notin K$.*

Proof. We shall use the concept of caterpillar (introduced by F. Harary). A caterpillar is a tree with the property that after deleting all terminal vertices from it a snake is obtained. This snake is called the body of the caterpillar [4]. If the vertices of the body are $u_0, \ldots, u_m$ and the edges u_iu_{i+1} for $i = 0, 1, \ldots, m - 1$, then we denote by α_i the number of terminal vertices of the caterpillar which are adjacent to u_i for $i = 0, 1, \ldots, m$. Thus we assign a vector $[\alpha_0, \ldots, \alpha_m]$ to the caterpillar. For $K \neq \emptyset$ the required tree is a caterpillar with the vector $[\alpha_0, \ldots, \alpha_m]$ which is described as follows. Let $K = \{k_1, \ldots, k_q\}$ and let $k_i < k_j$ for $i < j$. Then $m = 2q - 1$, $\alpha_0 = k_1$, $\alpha_i = k_{i+1} - k_i$ for $i = 1, \ldots, q - 1$. Further $\alpha_{m-i} = \alpha_i$ for $i = 0, 1, \ldots, q - 1$. The caterpillar C can be embedded into $H_n(k_i)$ so that onto the end vertices of the bridge of $H_n(k_i)$ the vertices u_{i-1}, u_i or the vertices u_{m-i+1}, u_{m-i} are mapped. On the other hand, the unique edges of C which can be mapped onto bridges of $H_n(k)$ for some k are the edges of the body of C and to each of them a unique k exists with this property. Thus C cannot be embedded into $H_n(k)$ for $k \notin K$. For $K = \emptyset$ the required tree is a star.

In [4], embedding rooted trees into rooted block graphs was defined. A graph is called rooted, if one of its vertices is fixed and called the root of the graph. By $H_n^*(k)$ for $k = 2, \ldots, n - 2$ we denote the graph consisting of two vertex-disjoint cliques, one with k, the other with $n - k$ vertices, with a bridge between them, which is rooted at a vertex of the clique with k vertices non-incident with the bridge.

Theorem 5. *Let T be a rooted tree with $n \geqq 4$ vertices. The tree T can be embedded into $H_n^*(k)$ for each $k = 2, \ldots, n - 2$ so that its root coincides with the root of $H_n^*(k)$ if and only if T is a snake whose root is a terminal vertex.*

Proof. Let T be a rooted snake with vertices $u_1, \ldots, u_n$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, n - 1$. Let u_1 be its root. Then evidently T can be embedded into $H_n^*(k)$ for $k = 2, \ldots, n - 2$ in the required way so that the edge $u_k u_{k+1}$ is mapped onto the bridge of $H_n^*(k)$. Now suppose that T is a rooted tree whose root is not a terminal vertex. Then this root r has the degree at least 2. The tree T cannot be embedded into $H_n^*(2)$, because the root of $H_n^*(2)$ has the degree 1. Now let T be a rooted tree whose root r is its terminal vertex, but not a snake. Then T contains at least one vertex of a degree greater than 2; let u be such a vertex whose distance from r is minimal. Let $d(r, u) = h$. Then T cannot be embedded into $H_n^*(h + 1)$, because at this embedding all vertices of the path connecting r and u would have to be contained in the clique with $h + 1$ vertices and all vertices adjacent to u and not belonging to this path (they are at least two) would be embedded into the other clique and there would be at least two edges joining vertices of different cliques of $H_n(h + 1)$, which is impossible.

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