Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 104 (1979), No. 3, 267--280

Persistent URL: `http://dml.cz/dmlcz/118022`

Časopis pro pěstování matematiky, roč. 104 (1979), Praha

THE LAPLACE TRANSFORM OF ANALYTIC VECTOR-VALUED FUNCTIONS (COMPLEX CONDITIONS)

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(Received February 28, 1977)

In this paper, we deal with the problem of characteristic properties of the Laplace transform of vector-valued exponentially bounded functions on the positive halfaxis which are analytic in that sense that they are, roughly speaking, developable in power series about the points of the positive halfaxis with linearly increasing radii of convergence. These properties can be described in the following way: the functions in question are infinitely differentiable on the positive halfaxis R^+ and their derivatives satisfy the inequalities B(I) in Theorem 8 with certain constants $M \geqq 0$, $\omega \geqq 0$, $\varrho \geqq 0$.

We give necessary and sufficient representability conditions of complex type in terms of the existence of a certain analytic continuation of the Laplace image to a fan-shaped domain symmetric around a real halfline (with the angle greater than π) as shown in Theorem 8.

On the other hand, the class of analytic functions in the positive halfaxis, which are considered as originals of the Laplace transformation and which are roughly described above, is characterized by the existence of a certain analytic continuation to a wedge-shaped domain around the positive halfaxis (with the angle less than π) as shown in Theorem 10.

The above described problems were studied in the special case of the so called analytic or holomorphic or parabolic semigroups, but in the proof of representability, very special properties of the resolvent were exploited, which is not possible in the general case of the Laplace transform (cf. [1], [2], [3]).

The advantage of complex representability conditions lies in the fact that the behaviour of derivatives of the Laplace image need not be examined.

1. In the sequel, C will denote the complex number field, R the real number field and R^+ the set of all positive numbers. If M_1, M_2 are arbitrary sets, then $M_1 \to M_2$ will denote the set of all mappings of the whole set M_1 into the set M_2.

2. By E we denote a general Banach space over C with the norm $\|\cdot\|$.

3. Proposition. *Let $f \in R^+ \to E$ and let M, ω, ϱ be nonnegative constants. If*

(α) *the function f is infinitely differentiable on R^+,*

(β) $\|f^{(q)}(t)\| \leqq Me^{\omega t}(q!\,\varrho^q/t^q)$ *for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$,*
then there exists a function $\Phi \in \{z; \operatorname{Re} z + 1/2(1+\varrho)\,|\operatorname{Im} z| > \omega\} \to E$ such that

(a) *Φ is analytic in the domain $\{z : \operatorname{Re} z + 1/2(1+\varrho)\,|\operatorname{Im} z| > \omega\}$,*

(b) $\|\Phi(z)\| \leqq 2M/(\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}\,|\operatorname{Im} z| - \omega)$ *for every* $z \in C$,
$\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}\,|\operatorname{Im} z| > \omega$,

(c) $\Phi(z) = \int_0^\infty e^{-z\tau} f(\tau)\,d\tau$ *for every* $z \in C$, $\operatorname{Re} z > \omega$.

Proof. First of all, it follows from (α) and (β) that

$$(1)\quad \left\|\frac{d^q}{dt^q}\, t^q f(t)\right\| = \left\|\sum_{j=0}^{q} \binom{q}{j} \frac{q!}{(q-j)!}\, t^{q-j} f^{(q-j)}(t)\right\| \leqq$$

$$\leqq \sum_{j=0}^{q} \binom{q}{j} \frac{q!}{(q-j)!}\, Me^{\omega t}(q-j)!\,\varrho^{q-j} =$$

$$= Me^{\omega t}\, q! \sum_{j=0}^{q} \binom{q}{j} \varrho^{q-j} = Me^{\omega t}\, q!(1+\varrho)^q \text{ for any } t \in R^+ \text{ and } q \in \{0, 1, \ldots\}.$$

Let us now write

$$(2)\quad \Phi_0(z) = \int_0^\infty e^{-z\tau} f(\tau)\,d\tau \quad \text{for} \quad \operatorname{Re} z > \omega.$$

Then by (α), (β) and (2)

(3) Φ_0 is an analytic function in the domain $\{z : \operatorname{Re} z > \omega\}$.

It follows from (1) and (2) that

$$(4)\quad \|z^q \Phi_0^{(q)}(z)\| = \left\|(-1)^q z^q \int_0^\infty e^{-z\tau}\tau^q f(\tau)\,d\tau\right\| = \left\|(-1)^q \int_0^\infty e^{-z\tau} \frac{d^q}{d\tau^q}(\tau^q f(\tau))\,d\tau\right\| \leqq$$

$$\leqq \int_0^\infty e^{-\operatorname{Re} z\tau} \left\|\frac{d^q}{d\tau^q}(\tau^q f(\tau))\right\| d\tau \leqq \int_0^\infty e^{-\operatorname{Re} z\tau}\, Me^{\omega\tau}\, q!(1+\varrho)^q\,d\tau =$$

$$= \frac{M}{\operatorname{Re} z - \omega}\, q!\,(1+\varrho)^q \quad \text{for any} \quad \operatorname{Re} z > \omega \quad \text{and} \quad q \in \{0, 1, \ldots\}.$$

Now we shall prove that there exists a function
$\Phi \in \{z : \operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}\,|\operatorname{Im} z| > \omega\} \to E$ such that

(5) Φ is analytic in its domain,

(6) $\Phi(z) = \Phi_0(z)$ for any $\operatorname{Re} z > \omega$,

$$(7)\quad \Phi(z) = \sum_{q=0}^{\infty} \frac{\Phi_0^{(q)}(z + \frac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z|)}{q!} (\tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z|)^q \quad \text{for any}$$

$$\mathrm{Re}\, z + \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z| > \omega .$$

To this aim let us first fix $z \in C$ so that

$$(8)\quad \mathrm{Re}\, z + \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z| > \omega .$$

In examining the series (7), we denote for the sake of simplicity

$$(9)\quad z_0 = z + \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z| .$$

Using the notation (9), we can write (7) in the form

$$(10)\quad \Phi(z) = \sum_{q=0}^{\infty} \frac{\Phi_0^{(q)}(z_0)}{q!}(z - z_0)^q .$$

Since $\mathrm{Re}\, z_0 > \omega$ by (8) and (9), the series (10) represents the analytic extension of Φ_0 in its domain of convergence. Hence we need only to prove that z lies in this domain. We have by (4)

$$\left\| \sum_{q=0}^{\infty} \frac{\Phi_0^{(q)}(z_0)}{q!}(z - z_0)^q \right\| \leqq \sum_{q=0}^{\infty} \frac{\|\Phi_0^{(q)}(z_0)\|}{q!} |z - z_0|^q \leqq$$

$$\leqq \frac{M}{\mathrm{Re}\, z_0 - \omega} \sum_{q=0}^{\infty} \left(\frac{(1+\varrho)\,|z - z_0|}{z_0} \right)^q .$$

Consequently, it suffices to prove that

$$(11)\quad |z - z_0| < \frac{|z_0|}{1+\varrho} .$$

We have by (8) and (9)

$$|z - z_0| = \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z| \leqq \tfrac{1}{2}(1+\varrho)^{-1}[\omega^2 + (\mathrm{Im}\, z)^2]^{1/2} \leqq$$

$$\leqq \tfrac{1}{2}(1+\varrho)^{-1}[(\mathrm{Re}\, z + \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z|)^2 + (\mathrm{Im}\, z)^2]^{1/2} =$$

$$= \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Re}\, z + \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z| + \mathrm{i}\,\mathrm{Im}\, z| =$$

$$= \tfrac{1}{2}(1+\varrho)^{-1}\big|z + \tfrac{1}{2}(1+\varrho)^{-1}|\mathrm{Im}\, z|\big| = \tfrac{1}{2}(1+\varrho)^{-1}|z_0| < (1+\varrho)^{-1}|z_0|$$

which verifies (11).

The above considerations prove (5)–(7).

Finally, we estimate by means of (4) and (7)

$$(12)\quad \|\Phi(z)\| \leqq \left\| \sum_{q=0}^{\infty} \frac{\Phi_0^q(z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z|)}{q!} (\tfrac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z|)^q \right\| \leqq$$

$$\leqq \frac{M}{\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z| - \omega} \sum_{q=0}^{\infty} \left(\frac{\frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z|}{|z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z||} \right)^q$$

for any $\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z| > \omega$.

For the series in (12) we obtain further the estimate

$$(13)\quad \sum_{q=0}^{\infty} \left(\frac{\frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z|}{|z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z||} \right)^q =$$

$$= \sum_{q=0}^{\infty} \left(\frac{\frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z|}{[(\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z|)^2 + (\operatorname{Im} z)^2]^{1/2}} \right)^q \leqq \sum_{q=0}^{\infty} (\tfrac{1}{2}(1+\varrho)^{-1})^q =$$

$$= \frac{1}{1 - \frac{1}{2}(1+\varrho)^{-1}} = \frac{2(1+\varrho)}{1+2\varrho} \leqq 2\,.$$

It follows from (12) and (13) that

$$(14)\quad \|\Phi(z)\| \leqq \frac{2M}{\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z| - \omega} \quad \text{for any}$$

$\operatorname{Re} z + \frac{1}{2}(1+\varrho)^{-1}|\operatorname{Im} z| > \omega$.

The statement of Proposition 3 follows from (5), (6) and (14).

4. Remark. The extension constructed in Proposition 3 is not the largest generally accessible. Nonetheless, the construction of a larger extension calls for a more sofisticated technique and hence we restrict ourselves to the above result which is sufficient for our purposes.

5. Proposition. *Let $N, \varkappa$ be two nonnegative constants, μ a positive constant and $\Phi \in \{z : \operatorname{Re} z + \mu|\operatorname{Im} z| > \varkappa\} \to E$. If*

(α) *the function Φ is analytic in the domain $\{z : \operatorname{Re} z + \mu|\operatorname{Im} z| > \varkappa\}$,*

(β) $\|\Phi(z)\| \leqq \dfrac{N}{\operatorname{Re} z + \mu|\operatorname{Im} z| - \varkappa}$ *for any* $z \in C$, $\operatorname{Re} z + \mu|\operatorname{Im} z| > \varkappa$,

then there exists an infinitely differentiable function $f \in R^+ \to E$ such that

$$\text{(a)}\quad \|f^{(q)}\|(t)\| \leqq \frac{Ne^2(4+\mu^2)^{1/2}}{\pi\mu} e^{(\varkappa+\delta)t} \frac{q!}{t^q} \left[\left(1 + \frac{\mu^2}{4}\right) \frac{2}{\mu} \frac{\varkappa + \delta + 1}{\delta} \right]^q$$

for every $t \in R^+$, $q \in \{0, 1, \ldots\}$ *and* $\delta > 0$,

(b) $\Phi(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\,\mathrm{d}\tau$ *for every* $\lambda > \varkappa$.

Proof. By means of Cauchy's formula we can write (a figure helps)

$$(1)\quad \Phi(\lambda) = \frac{r}{2\pi} \int_{-\pi/2-\mathrm{arctg}(\mu/2)}^{\pi/2+\mathrm{arctg}(\mu/2)} \frac{\Phi(\alpha + re^{\mathrm{i}\varphi})}{\alpha + re^{\mathrm{i}\varphi} - \lambda} e^{\mathrm{i}\varphi}\,\mathrm{d}\varphi +$$

$$+ \frac{1}{2\pi\mathrm{i}} \int_{-c(r)}^{c(r)} \frac{(\mathrm{i} - \frac{1}{2}\mu)\,\Phi(\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)}{\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta - \lambda}\,\mathrm{d}\beta \quad (\text{where} \quad c(r) = r(1 + \tfrac{1}{4}\mu^2)^{-1/2})$$

for any $\varkappa < \alpha < \lambda$ and $r > \lambda - \alpha$.

By simple estimates based on (β) we obtain easily that

$$(2)\quad \frac{r}{2\pi} \int_{-\pi/2-\mathrm{arctg}(\pi/2)}^{\pi/2+\mathrm{arctg}(\pi/2)} \frac{\Phi(\alpha + re^{\mathrm{i}\varphi})}{\alpha + re^{\mathrm{i}\varphi} - \lambda} e^{\mathrm{i}\varphi}\,\mathrm{d}\varphi \to 0\ (r \to \infty) \quad \text{for any} \quad \lambda > \alpha > \varkappa.$$

It follows from (1) and (2) that

$$(3)\quad \Phi(\lambda) = \frac{2\mathrm{i} - \mu}{4\pi\mathrm{i}} \lim_{r\to\infty} \int_{-c(r)}^{c(r)} \frac{\Phi(\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)}{\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta - \lambda}\,\mathrm{d}\beta =$$

$$= \frac{\mu - 2\mathrm{i}}{4\pi\mathrm{i}} \lim_{r\to\infty} \int_{-c(r)}^{c(r)} \frac{\Phi(\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)}{\lambda - \alpha + \frac{1}{2}\mu|\beta| - \mathrm{i}\beta}\,\mathrm{d}\beta \quad (\text{where} \quad c(r) = r(1 + \tfrac{1}{4}\mu^2)^{-1/2})$$

for any $\lambda > \alpha > \varkappa$.

By our assumption (β), we can write for any $\lambda > \alpha > \varkappa$

$$\frac{\|\Phi(\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)\|}{|\lambda - \alpha + \frac{1}{2}\mu|\beta| - \mathrm{i}\beta|} \leqq \frac{N}{(\alpha - \varkappa + \frac{1}{2}\mu|\beta|)\,[(\lambda - \alpha + \frac{1}{2}\mu|\beta|)^2 + \beta^2]^{1/2}} \leqq$$

$$\leqq \frac{N}{(\lambda - \varkappa + \frac{1}{2}\mu|\beta|)\,(\lambda - \alpha + \frac{1}{2}\mu|\beta|)}$$

which guarantees

(4) for any $\lambda > \alpha > \varkappa$, the function $\Phi(\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)/(\lambda - \alpha + \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)$ is absolutely integrable on $-\infty < \beta < \infty$.

Now it follows from (3) and (4) that

$$(5)\quad \Phi(\lambda) = \frac{\mu - 2\mathrm{i}}{4\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\Phi(\alpha - \frac{1}{2}\mu|\beta| + \mathrm{i}\beta)}{\lambda - \alpha + \frac{1}{2}\mu|\beta| - \mathrm{i}\beta}\,\mathrm{d}\beta \quad \text{for any} \quad \lambda > \alpha > \varkappa.$$

On the other hand, it follows easily from the assumption (β) that

(6) $$\left|e^{(\alpha-\mu|\beta|/2+i\beta)t}(\alpha - \tfrac{1}{2}\mu|\beta| + i\beta)^q \Phi(\alpha - \tfrac{1}{2}\mu|\beta| + i\beta)\right| \leqq$$
$$\leqq \frac{Ne^{\alpha t}\, e^{-\mu|\beta|t/2}[(\alpha - \frac{1}{2}\mu|\beta|)^2 + \beta]^{q/2}}{\alpha - \varkappa + \frac{1}{2}\mu|\beta|} \quad \text{for any} \quad t \in R^+, \quad \alpha > \varkappa \quad \text{and}$$
$q \in \{0, 1, \ldots\}$.

Further, it is easy to see that

(7) for any $T \in R^+$, $\alpha > \varkappa$ and $q \in \{0, 1, \ldots\}$, the function
$$\frac{e^{-\mu|\beta|T/2}[(\alpha - \frac{1}{2}\mu|\beta|)^2 + \beta^2]^{q/2}}{\alpha - \varkappa + \frac{1}{2}\mu|\beta|} \quad \text{is integrable over} \quad -\infty < \beta < \infty.$$

It follows from (6) and (7) that

(8) $$\left\|e^{(\alpha-\mu|\beta|/2+i\beta)t}(\alpha - \tfrac{1}{2}\mu|\beta| + i\beta)^q \Phi(\alpha - \tfrac{1}{2}\mu|\beta| + i\beta)\right\| \leqq$$
$$\leqq Ne^{3\alpha T/2}\, \frac{e^{-\mu|\beta|T/4}[(\alpha - \frac{1}{2}\mu|\beta|)^2 + \beta^2]^{q/2}}{\alpha - \varkappa + \frac{1}{2}\mu|\beta|} \quad \text{for any } t, \quad T \in R^+ \quad \text{such that}$$
$|t - T| < \frac{1}{2}T$, $\alpha > \varkappa$ and $q \in \{0, 1, \ldots\}$.

Owing to (6) and (7) we define

(9) $$f(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\varkappa+1-\mu|\beta|/2+i\beta)t}\, \Phi(\varkappa + 1 - \tfrac{1}{2}\mu|\beta| + i\beta)\,(i - \tfrac{1}{2}\mu \operatorname{sign} \beta)\, d\beta$$
for $t \in R^+$.

By means of Cauchy's formula, we obtain form (7) and (9) and from the assumption (β) that

(10) $$f(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\alpha-\mu|\beta|/2+i\beta)t}\, \Phi(\alpha - \tfrac{1}{2}\mu|\beta| + i\beta)\,(i - \tfrac{1}{2}\mu \operatorname{sign} \beta)\, d\beta$$
for any $t \in R^+$ and $\alpha > \varkappa$.

It follows easily from (7), (8) and (10) by induction on q that

(11) the function f is infinitely differentiable on R^+,

(12) $f^{(q)}(t) =$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\alpha-\mu|\beta|/2+i\beta)t}(\alpha - \tfrac{1}{2}\mu\,|\beta| + i\beta)^q\,(\alpha - \tfrac{1}{2}\mu\,|\beta| + i\beta)\,(i - \tfrac{1}{2}\mu \operatorname{sign} \beta)\, d\beta$$
for every $t \in R^+$, $\alpha > \varkappa$ and $q \in \{0, 1, \ldots\}$.

Using the assumption (β), we can estimate by (12) as follows:

$$(13)\quad \|f^q(t)\| \leqq \frac{N(4+\mu^2)^{1/2}}{4\pi}\int_{-\infty}^{\infty}\frac{e^{(\alpha-\mu|\beta|/2)t}((\alpha-\frac{1}{2}\mu|\beta|)^2+|\beta|^2)^{q/2}}{\alpha-\varkappa+\frac{1}{2}\mu|\beta|}\,d\beta \leqq$$

$$\leqq \frac{N(4+\mu^2)^{1/2}\,e^{\alpha t}}{2\pi(\alpha-\varkappa)}\int_0^{\infty}e^{-\mu\beta t/2}[(\alpha-\tfrac{1}{2}\mu\beta)^2+\beta^2]^{q/2}\,d\beta =$$

$$= \frac{N(4+\mu^2)^{1/2}\,e^{\alpha t}}{2\pi(\alpha-\varkappa)}\int_0^{\infty}e^{-\mu\beta t/2}[\alpha^2-2\alpha\tfrac{1}{2}\mu\beta+\tfrac{1}{4}\mu^2\beta^2+\beta^2]^{q/2}\,d\beta \leqq$$

$$\leqq \frac{N(4+\mu^2)^{1/2}\,e^{\alpha t}}{2\pi(\alpha-\varkappa)}\int_0^{\infty}e^{-\mu\beta t/2}[\alpha+(1+\tfrac{1}{4}\mu^2)^{1/2}\beta]^q\,d\beta =$$

$$= \frac{N(4+\mu^2)^{1/2}\,e^{\alpha t}}{2\pi(\alpha-\varkappa)}\int_0^{\infty}e^{-\mu\beta t/2}\sum_{j=0}^{q}\binom{q}{j}\alpha^j(1+\tfrac{1}{4}\mu^2)^{(q-j)/2}\beta^{q-j}\,d\beta \leqq$$

$$\leqq \frac{N(4+\mu^2)^{1/2}\,e^{\alpha t}}{2\pi(\alpha-\varkappa)}(1+\tfrac{1}{4}\mu^2)^{q/2}\sum_{j=0}^{q}\binom{q}{j}\alpha^j\int_0^{\infty}e^{\mu\beta t/2}\beta^{q-j}\,d\beta =$$

$$= \frac{N(4+\mu^2)^{1/2}\,e^{\alpha t}}{2\pi(\alpha-\varkappa)}(1+\tfrac{1}{4}\mu^2)^{q/2}\sum_{j=0}^{q}\binom{q}{j}\alpha^j\frac{(q-j)!}{(\frac{1}{2}\mu t)^{q-j+1}}$$

for any $t\in R^+$, $\alpha>\varkappa$ and $q\in\{0,1,\ldots\}$.

Now we take $\alpha=(1+\varkappa t)/t$ in (13) which yields

$$(14)\quad \|f^{(q)}(t)\| \leqq \frac{N(4+\mu^2)^{1/2}\,e^{1+\varkappa t}}{2\pi\left(\dfrac{1+\varkappa t}{t}-\varkappa\right)}(1+\tfrac{1}{4}\mu)^{q/2}\sum_{j=0}^{q}\binom{q}{j}\frac{(1+\varkappa t)^j}{t^j}\frac{(q-j)!}{(\frac{1}{2}\mu t)^{q-j+1}} =$$

$$= \frac{Ne(4+\mu^2)^{1/2}\,e^{\varkappa t}}{2\pi}(1+\tfrac{1}{4}\mu^2)^{q/2}\frac{q!}{t^{q+1}}\sum_{j=0}^{q}\frac{1}{j!}(1+\varkappa t)^j\left(\frac{2}{\mu}\right)^{q-j+1} \leqq$$

$$\leqq \frac{Ne(4+\mu^2)^{1/2}\,e^{\varkappa t}}{\pi\mu}\left[(1+\tfrac{1}{4}\mu^2)^{1/2}\frac{2}{\mu}\right]^q\frac{q!}{t^q}\sum_{j=0}^{q}\frac{(1+\varkappa t)^j}{j!}$$

for any $t\in R^+$ and $q\in\{0,1,\ldots\}$.

By (14),

(15) $\|f^{(q)}(t)\| \leqq$

$$\leqq \frac{Ne(4+\mu^2)^{1/2}\,e^{\varkappa t}}{\pi\mu}\left[(1+\tfrac{1}{4}\mu^2)^{1/2}\frac{2}{\mu}\right]^q \frac{q!}{t^q}\sum_{j=0}^{q}\left(\frac{\varkappa+1}{\delta}\right)^j \frac{\left(\frac{\delta}{\varkappa+1}\right)^j(1+\varkappa t)^j}{j!} \leqq$$

$$\leqq \frac{Ne(4+\mu^2)^{1/2}\,e^{\varkappa t}}{\pi\mu}\left[(1+\tfrac{1}{4}\mu^2)^{1/2}\frac{2}{\mu}\right]^q.$$

$$\cdot\frac{q!}{t^q}\left(\frac{\varkappa+\delta+1}{\delta}\right)^q \sum_{j=0}^{q}\frac{\left[\frac{\delta}{\varkappa+1}(1+\varkappa t)\right]^j}{j!} \leqq$$

$$\leqq \frac{Ne(4+\mu^2)^{1/2}\,e^{\varkappa t}}{\pi\mu}\left[(1+\tfrac{1}{4}\mu^2)\frac{2}{\mu}\frac{\varkappa+\delta+1}{\delta}\right]^q \frac{q!}{t^q}\,e^{\delta(1+\varkappa t)/(\varkappa+\delta+1)} \leqq$$

$$\leqq \frac{Ne^2(4+\mu^2)^{1/2}}{\pi\mu}\,e^{(\varkappa+\delta)t}\,\frac{q!}{t^q}\left[(1+\tfrac{1}{4}\mu^2)\frac{2}{\mu}\frac{\varkappa+\delta+1}{\delta}\right]^q$$

for any $t \in R^+$, $q \in \{0, 1, \ldots\}$ and $\delta > 0$.

Now we need to prove

(16) $$\Phi(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\,d\tau \quad \text{for every} \quad \lambda > \omega.$$

Indeed, by (10), (11) and (12),

(17) $$\int_0^\infty e^{-\lambda\tau} f(\tau)\,d\tau = \frac{2i-\mu}{4\pi i}\int_0^\infty e^{-\lambda\tau}\left(\int_{-\infty}^{\infty} e^{(\alpha-\mu|\beta|/2+i\beta)\tau}\,\Phi(\alpha-\tfrac{1}{2}\mu|\beta|+i\beta)\,d\beta\right)d\tau$$

for any $\lambda > \alpha > \varkappa$.

To justify the change of order of integration we estimate by the assumption (β) as follows:

$$\|e^{-\lambda\tau}e^{(\alpha-\mu|\beta|/2+i\beta)\tau}\,\Phi(\alpha-\tfrac{1}{2}\mu|\beta|+i\beta)\| \leqq e^{-\lambda t}\,e^{(\alpha-\mu|\beta|/2)\tau}\,\frac{N}{\alpha-\varkappa+\frac{1}{2}\mu\beta}.$$

Since the last function is integrable in (τ, β) over $(0, \infty) \times (-\infty, \infty)$, we obtain from (17) by interchanging the order of integration that

(18) $$\int_0^\infty e^{-\lambda\tau} f(\tau)\,d\tau = \frac{2i-\mu}{4\pi i}\int_{-\infty}^{\infty}\left(\int_0^\infty e^{-\lambda\tau}e^{(\alpha-\mu|\beta|/2+i\beta)\tau}\,d\tau\right)\Phi(\alpha-\tfrac{1}{2}\mu|\beta|+i\beta)\,d\beta =$$

$$= \frac{2i-\mu}{4\pi i}\int_{-\infty}^{\infty}\frac{\Phi(\alpha-\frac{1}{2}\mu|\beta|+i\beta)}{\lambda-\alpha+\frac{1}{2}\mu|\beta|-i\beta}\,d\beta$$

for every $\lambda > \alpha > \varkappa$.

The relation (16) follows from (5) and (18).
The statements (a) and (b) are contained in (15) and (16) and the proof is complete.

6. Lemma. *Let $a \geqq 0$ and $b > 0$. Then for every $\xi, \eta \in R$ and $\delta > 0$ such that $\xi + \frac{1}{2}b|\eta| > a + \delta$, the following inequality holds*:

$$\frac{1}{\xi + b|\eta| - a} \leqq \frac{4\left(1 + \dfrac{1}{\delta} + \dfrac{1}{b}\right)}{1 + (\xi^2 + \eta^2)^{1/2}}.$$

Proof. Let us choose $\xi, \eta \in R$ and $\delta > 0$ so that

(1) $\quad \xi + \frac{1}{2}\, b|\eta| > a + \delta.$

First, we have by (1)

(2) $\quad \xi + b|\eta| - a = \xi + \frac{1}{2}\, b|\eta| - a + \frac{1}{2}\, b|\eta| > \delta + \frac{1}{2}\, b|\eta|.$

Further, we obtain from (1) if $\xi \geqq a$, then $\xi + b|\eta| - a > |\xi - a|$ and if $\xi \leqq a$, then $|\xi - a| = a - \xi < \frac{1}{2}\, b|\eta|$ which implies $|\xi - a| = 2|\xi - a| + (\xi - a) < < b|\eta| + \xi - a = \xi + b|\eta| - a$. Summing up we get

(3) $\quad \xi + b|\eta| - a > |\xi - a|.$

On the other hand,

(4) $$|\xi| = |\xi - a + a| \leqq |\xi - a| + a \leqq |\xi - a| + a + \tfrac{1}{2}\delta \leqq$$
$$\leqq \left(1 + \frac{2a}{\delta}\right)\left(|\xi - a| + \tfrac{1}{2}\delta\right).$$

It follows from (3) and (4) that

(5) $$\xi + b|\eta| - a > \frac{\delta}{\delta + 2a}\,|\xi| - \tfrac{1}{2}\delta \geqq |\xi| - \tfrac{1}{2}\delta.$$

By (2) and (5) we conclude

(6) $$\xi + b|\eta| - a > \tfrac{1}{2}[\delta + \tfrac{1}{2}\, b|\eta| + |\xi| - \tfrac{1}{2}\delta] = \tfrac{1}{4}[\delta + |\xi| + b|\eta|].$$

On the other hand,

$$\frac{1 + |\xi| + |\eta|}{\delta + |\xi| + b|\eta|} \leqq \frac{1}{\delta} + 1 + \frac{1}{b}$$

which implies

$$(7) \quad \delta + |\xi| + b|\eta| \geqq \left(1 + \frac{1}{\delta} + \frac{1}{b}\right)^{-1} (1 + |\xi| + |\eta|) \geqq$$

$$\geqq \left(1 + \frac{1}{\delta} + \frac{1}{b}\right)^{-1} (1 + (\xi^2 + \eta^2)^2).$$

Finally, (6) and (7) give the desired inequality.

7. Lemma. *Let $a \geqq 0$, $b > 0$. Then for every $\xi, \eta \in R$ such that $\xi + b\eta > a$ the following inequality holds:*

$$\frac{1}{1 + (\xi^2 + \eta^2)^{1/2}} \leqq \frac{2(1 + b)}{\xi + b|\eta| - a}.$$

Proof. We have

$$\xi + b|\eta| - a \leqq |\xi| + b|\eta| - a \leqq |\xi| + b|\eta| \leqq (1 + b)(|\xi| + |\eta|) \leqq$$

$$\leqq 2(1 + b)(\xi^2 + \eta^2)^{1/2} \leqq 2(1 + b)(1 + (\xi^2 + \eta^2)^{1/2}).$$

8. Fundamental theorem (complex form). *Let $\chi \geqq 0$ and let F be a function defined on a subset of C containing (χ, ∞) with values in E. Then the following two statements (A) and (B) are equivalent:*

(A) *for every $\varkappa > \chi$ there exist $N \geqq 0$ such that*

(I) *$\{z : \operatorname{Re} z + \mu|\operatorname{Im} z| > \varkappa\}$ lies in the domain of F,*

(II) *the function F is analytic in the domain $\{z : \operatorname{Re} z + \mu|\operatorname{Im} z| > \varkappa\}$,*

(III) *$\|F(z)\| \leqq N/(1 + |z|)$ for every $z \in C$ satisfying $\operatorname{Re} z + \mu|\operatorname{Im} z| > \varkappa$;*

(B) *there exists an infinitely differentiable function $f \in R^+ \to E$ such that*

(I) *for every $\omega > \chi$, there exists $M \geqq 0$ and $\varrho > 0$ so that*

$$\|f^{(q)}(t)\| \leqq Me^{\omega t} \frac{q!\, \varrho^q}{t^q}$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$,

(II) *$F(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ for every $\lambda > \chi$.*

Proof. An easy consequence of Propositions 3 and 5 and Lemmas 6 and 7.

9. Remark. In the preceding part of this paper, we studied characteristic properties of the Laplace transform of infinitely differentiable functions whose derivatives satisfy certain growth conditions. In the subsequent Theorem we shall give an equivalent property in terms of the analytic continuation to a wedge-shaped domain symmetric around the real axis and satisfying a certain growth condition.

10. Theorem. *Let $f \in R^+ \to E$. The following two properties* (A), (B) *are equivalent*:

(A) *the function f is infinitely differentiable and there exist three nonnegative constants M, ω, ϱ so that*

$$\|f^{(q)}(t)\| \leqq \frac{M e^{\omega t} q! \, \varrho^q}{t^q}$$

for any $t \in R^+$ and $q \in \{0, 1, \ldots\}$;

(B) *there exist nonnegative constants $N, \varkappa$, a positive constant μ and a function $\varphi \in \{z : |\mathrm{Im}\, z| < \mu \,\mathrm{Re}\, z\} \to E$ so that φ is analytic in the domain $\{z : |\mathrm{Im}\, z| < \mu \,\mathrm{Re}\, z\}$, $\varphi(t) = f(t)$ for any $t \in R^+$ and $\|\varphi(z)\| \leqq N e^{\varkappa |z|}$ for any $z \in C$, $|\mathrm{Im}\, z| < \mu \,\mathrm{Re}\, z$.*

Proof. (A) $\Rightarrow$ (B): We shall suppose $\varrho > 1$ which is always admissible without loss of generality.

Using Taylor's theorem we get from (A) that

$$(1) \quad f(\tau) = \sum_{q=0}^{\infty} \frac{f^{(q)}(t)}{q!} (\tau - t)^q \quad \text{for every} \quad t, \tau \in R^+ \quad \text{for which} \quad |\tau - t| < t/\varrho.$$

Let us now denote $\Omega = \{z : \text{there exists } t \in R^+ \text{ so that } |z - t| < t/\varrho\}$. Clearly

$$(2) \quad R^+ \subseteq \Omega.$$

Using elementary properties of analytic functions we get easily from (1) and (2) that there exists a (unique) function $\varphi \in \Omega \to E$ such that

(3) φ is analytic in Ω,

(4) $\varphi(t) = f(t)$ for every $t \in R^+$.

Let us denote $t(z) = \mathrm{Re}\, z + (1/\sqrt{(\varrho^2 - 1)}) |\mathrm{Im}\, z|$ for arbitrary $z \in C$.

First

(5) $t(z) \in R^+$ for every $z \in C$ with $\mathrm{Re}\, z > 0$,

$$(6) \quad t(z) \leqq |\mathrm{Re}\, z| + \frac{1}{\sqrt{(\varrho^2 - 1)}} |\mathrm{Im}\, z| \leqq \left(1 + \frac{1}{\sqrt{(\varrho^2 - 1)}}\right) |z| \quad \text{for every} \quad z \in C,$$

$$(7) \quad |z - t(z)| = \left|\mathrm{Re}\, z + \mathrm{i}\,\mathrm{Im}\, z - \left(\mathrm{Re}\, z + \frac{1}{\sqrt{(\varrho^2 - 1)}} |\mathrm{Im}\, z|\right)\right| =$$

$$= \left|\mathrm{i}\,\mathrm{Im}\, z - \frac{1}{\sqrt{(\varrho^2 - 1)}} |\mathrm{Im}\, z|\right| = \left[(\mathrm{Im}\, z)^2 + \frac{1}{\varrho^2 - 1} (\mathrm{Im}\, z)^2\right]^{1/2} =$$

$$= \frac{\varrho}{\sqrt{(\varrho^2 - 1)}} |\mathrm{Im}\, z| \quad \text{for every} \quad z \in C.$$

Further, by (7) we have

$$(8)\quad \frac{|z - t(z)|\,\varrho}{t(z)} = \frac{\dfrac{\varrho^2}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|}{\mathrm{Re}\, z + \dfrac{1}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|} <$$

$$< \frac{\dfrac{\varrho^2}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|}{\sqrt{(\varrho^2 - 1)}\,|\mathrm{Im}\, z| + \dfrac{1}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|} = 1$$

for every $z \in C$ satisfying $|\mathrm{Im}\, z| < 1/\sqrt{(\varrho^2 - 1)}\, \mathrm{Re}\, z$.

It follows easily from (1)–(4) and (8) that

$$(9)\quad \left\{z : |\mathrm{Im}\, z| < \frac{1}{\sqrt{(\varrho^2 - 1)}}\, \mathrm{Re}\, z\right\} \subseteq \Omega\,,$$

$$(10)\quad \varphi(z) = \sum_{q=0}^{\infty} \frac{f^{(q)}(t(z))}{q!}\,(z - t(z))^q \quad \text{for every} \quad z \in C \quad \text{such that}$$

$$|\mathrm{Im}\, z| < \frac{1}{\sqrt{(\varrho^2 - 1)}}\, \mathrm{Re}\, z.$$

Further we get

$$(11)\quad \frac{|z - t(z)|\,\varrho}{t(z)} = \frac{\dfrac{\varrho^2}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|}{\mathrm{Re}\, z + \dfrac{1}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|} <$$

$$< \frac{\dfrac{\varrho^2}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|}{2\sqrt{(\varrho^2 - 1)}\,|\mathrm{Im}\, z| + \dfrac{1}{\sqrt{(\varrho^2 - 1)}}\,|\mathrm{Im}\, z|} = \frac{\varrho^2}{2\varrho^2 - 1}$$

for every $z \in C$ such that $|\mathrm{Im}\, z| < \dfrac{1}{2}\dfrac{1}{\sqrt{(\varrho^2 - 1)}}\, \mathrm{Re}\, z.$

Now we get from (5), (6), (10) and (11)

$$(12)\quad \|\varphi(z)\| \leq \sum_{q=0}^{\infty} M e^{\omega t(z)} \left(\frac{|z - t(z)|\,\varrho}{t(z)}\right)^q \leq M e^{\omega t(z)} \sum_{q=0}^{\infty} \left(\frac{\varrho^2}{2\varrho^2 - 1}\right)^q \leq$$

$$\leqq M e^{\omega(1+1/\sqrt{(\varrho^2-1)})|z|} \frac{1}{1-\dfrac{\varrho^2}{2\varrho^2-1}} = M \frac{2\varrho^2-1}{\varrho^2-1} e^{\omega(1+1/\sqrt{(\varrho^2-1)})|z|}$$

for every $z \in C$ such that $|\mathrm{Im}\, z| < \frac{1}{2} \frac{1}{\sqrt{(\varrho^2-1)}} \mathrm{Re}\, z$.

But (2), (3), (4), (9) and (12) prove the assertion (B) if we take

$$N = M \frac{2\varrho^2-1}{\varrho^2-1}, \quad \varkappa = \omega\left(1 + \frac{1}{\sqrt{(\varrho^2-1)}}\right) \quad \text{and} \quad \mu = \frac{1}{2} \frac{1}{\sqrt{(\varrho^2-1)}}.$$

(B) $\Rightarrow$ (A): First we prove the inequality

(13) $|\mathrm{Im}\, z| \leqq \frac{1}{2}\mu\, \mathrm{Re}\, z$ for every $z \in C$ for which there is $t \in R^+$ such that $|z - t| = t\mu/(\mu + 2)$.

Indeed, we have

$$(\mathrm{Re}\, z - t)^2 + (\mathrm{Im}\, z)^2 = t^2 \frac{\mu^2}{(\mu+2)^2}$$

and consequently

$$|\mathrm{Im}\, z| \leqq t \frac{\mu}{\mu+2}.$$

On the other hand,

$$\mathrm{Re}\, z \geqq t - t\frac{\mu}{\mu+2} = t\frac{2}{\mu+2}.$$

Concluding we have

$$|\mathrm{Im}\, z| \leqq \frac{\mu+2}{2} \mathrm{Re}\, z \frac{\mu}{\mu+2} = \frac{\mu}{2} \mathrm{Re}\, z$$

which proves (13).

Now (13) enables us to apply Cauchy's integral theorem and we can write

$$(14) \quad f^{(q)}(t) = \frac{q!}{2\pi} \int_0^{2\pi} \frac{\varphi\left(t \dfrac{\mu}{\mu+2} e^{i\tau}\right)}{\left(t \dfrac{\mu}{\mu+2} e^{i\tau}\right)^{q+1}} t \frac{\mu}{\mu+2} e^{i\tau}\, d\tau$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$.

Consequently, (14) implies

$$(15)\quad \|f^{(q)}(t)\| \leqq \frac{q!}{2\pi}\, t \frac{\mu}{\mu+2} \int_0^{2\pi} \frac{N e^{t(\mu/\mu+2)}}{\left(t \dfrac{\mu}{\mu+2}\right)^{q+1}} \,\mathrm{d}\tau = N e^{(\mu/\mu+2)t} \frac{q!\left(\dfrac{\mu+2}{\mu}\right)^q}{t^q}\,.$$

Taking $M = N$, $\omega = \mu/(\mu + 2)$, $\varrho = (\mu + 2)/\mu$ we get from (15) that (A) holds. The proof is complete.

11. Corollary. *Let F be a function defined on a subset of C with values in E. Then the following two statements are equivalent*:

(A) *there exist constants $N_1 \geqq 0$, $\varkappa_1 \geqq 0$ and $\mu_1 > 0$ so that*

(I) *$\{z : \operatorname{Re} z + \mu_1 |\operatorname{Im} z| > \varkappa_1\}$ lies in the domain of F*,

(II) *the function F is analytic in the domain $\{z : \operatorname{Re} z + \mu_1 |\operatorname{Im} z| > \varkappa_1\}$*,

(III) *$\|F(z)\| \leqq N_1/(1 + |z|)$ for every $z \in C$, $\operatorname{Re} z + \mu_1 |\operatorname{Im} z| > \varkappa_1$*;

(B) *there exist constants $N_2 \geqq 0$, $\varkappa_2 \geqq 0$ and $\mu_2 > 0$ and a function $\varphi \in \{z : : |\operatorname{Im} z| < \mu_2 \operatorname{Re} z\} \to E$ so that*

(I) *φ is analytic in the domain $\{z : |\operatorname{Im} z| < \mu_2 \operatorname{Re} z\}$*,

(II) *$\|\varphi(z)\| \leqq N_2 e^{\varkappa_2 |z|}$ for every $z \in C$, $|\operatorname{Im} z| < \mu_2 \operatorname{Re} z$*,

(III) $\int_0^\infty e^{-\lambda\tau} \varphi(\tau)\, \mathrm{d}\tau = F(\lambda)$ for sufficiently large $\lambda \in R$.

Proof. Immediate consequence of Theorems 8 and 10.

12. Remark. It is useful to compare Theorem 11 with theorems on generation of the so called holomorphic or analytic or parabolic semigroups as presented, e.g., in [1], [2], [3], [4] and [5].

For higher order linear differential equations in Banach spaces see also [6].

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