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STRONG ELEMENTS IN LATTICES

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0. INTRODUCTION

The concept of a strong element has appeared in [2]. There it is used for the lattice-theoretical characterization of certain properties, which were defined for elements of the alphabet of a formal language in the work [1].

In the present note we discuss the relationship between strong elements and those satisfying "the upper covering condition". We study the structure of the system of all strong elements in a given lattice and give two characterization theorems for lattices in which all elements are strong.

In the following, the cardinality of a set S is denoted by card S. For an element $a \in S$ and for an equivalence relation ϱ on S the symbol $[a]\varrho$ denotes the set of all elements $b \in S$ with the property $a \varrho b$.

The lattice operations are denoted by $\vee$, $\wedge$ and the appropriate partial ordering by $\leqq$. The symbols $<$, $\prec$ and $\|$ are used in the standard sense. The reader is expected to be familiar with the notions of an atom and of the greatest (the smallest) element in a lattice. For a subset S of a lattice L we denote by $\inf_L S$ the infimum of S in L. If i is the greatest element in L, we define $\inf_L \emptyset = i$. We take advantage of the lattice-theoretical duality principle.

1. ELEMENTS FULFILLING THE UPPER COVERING CONDITION

1.1. Definition. Let L be a lattice. An element $a \in L$ is called *strong* (in L), if it holds

$$b < c ,\ a \| c \Rightarrow a \vee b < a \vee c$$

for any elements $b, c \in L$. If a is strong in the lattice dual to L, it is called *dually strong* (in L).

We denote by S_L the set of all elements, which are strong in L. If each element of L is strong (dually strong), we call L a *strong* (*dually strong*) lattice.

1.2. Lemma. *Let a be an element of a lattice L. Then $a \in S_L$ if and only if*

$$a \parallel b \Rightarrow b \prec a \vee b$$

for any element $b \in L$.

Proof. Suppose $a \parallel b$, $b \not\prec a \vee b$ for an element $b \in L$. Then there exists an element $c \in L$ with the property $b < c < a \vee b$. It is obvious that $a \parallel c$, $b < c$ and $a \vee c \leqq a \vee b$. Thus $a \notin S_L$.

Now let $a \notin S_L$. Then it holds $b < c$, $a \parallel c$ and $a \vee b = a \vee c$ for some elements b, $c \in L$. Thus $a < a \vee b$, $b < a \vee b$ and then, clearly, $a \parallel b$. Simultaneously, $b < c < < a \vee b$ and the above-mentioned implication does not hold.

1.3. Definition. An element a of a lattice L fulfils the *upper covering condition* (in L), if it holds

$$a \wedge b \prec a \Rightarrow b \prec a \vee b$$

for any element $b \in L$. If all elements of L satisfy this condition, we say that L fulfils the upper covering condition.

1.4. Corollary. *Each strong element of a lattice L fulfils the upper covering condition in L.*

1.5. Lemma. *Let a lattice L have a smallest element. An atom $a \in L$ is strong if and only if it fulfils the upper covering condition.*

Proof. The statement follows from 1.2 and 1.4.

1.6. Corollary. *If the lattice L fulfils the upper covering condition, then each atom in L is strong.*

2. THE SET OF ALL STRONG ELEMENTS IN A LATTICE

2.1. Lemma. *In any lattice the infimum of an arbitrary set of strong elements is strong.*

Proof. Let us put $a = \inf_L M$ for a lattice L and for $M \subseteq S_L$. In case $M = \emptyset$, a is the greatest element in L and, clearly, a is strong.

For $M \neq \emptyset$ suppose $a \notin S_L$. Then it holds $b < c$, $a \parallel c$ and $a \vee b = a \vee c$ for some elements $b, c \in L$. Since $b \nleqq a$, there exists an element $m \in M$ with the property $b \nleqq m$. Then $b \not\prec m \vee b$, because $b < c < a \vee b \leqq m \vee b$ and, obviously, $m \parallel b$. By 1.2, we obtain a contradiction.

2.2. Lemma. *Let a, b be strong elements of a lattice L such that $a \parallel b$. Then it holds*

$$c < a \Leftrightarrow c < b$$

for any $c \in L$.

Proof. Suppose $c < a$ for an element $c \in L$. It follows that $b \vee c < b \vee a$ and, clearly, $b \nleqq c$. Let us admit $b \parallel c$. Then it holds $b < b \vee c < b \vee a$. This implies $a \notin S_L$ according to 1.2 and we have a contradiction. It remains $c < b$.

The converse implication follows by interchanging the symbols a, b in the preceding consideration.

2.3. Theorem. *Let L be a complete lattice. Then S_L is a complete strong lattice satisfying*

$$\inf_L M = \inf_{S_L} M$$

for any set $M \subseteq S_L$.

Proof. Theorem 10 in [3] and 2.1 say that S_L is a complete lattice satisfying the required equalities.

Suppose $a \in S_L$ and $b < c$, $a \parallel c$ for elements $b, c \in S_L$. Then $b < a$ by 2.2. It follows $a \vee b = a < a \vee c$ and a is strong in S_L.

3. CHARACTERIZATIONS OF STRONG LATTICES

3.1. Theorem. *A lattice is strong if and only if it does not contain a sublattice isomorphic to one of the lattices L_5, L_6, the graphic description of which is given in Fig. 1.*

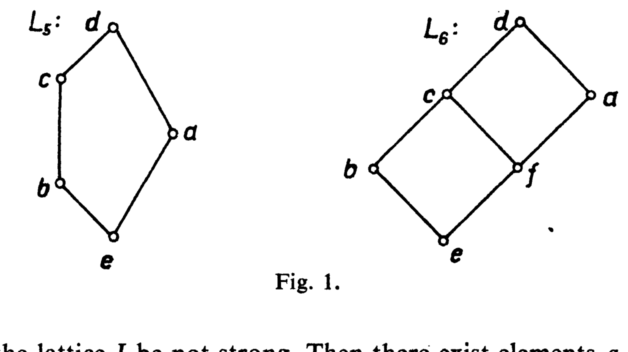

Fig. 1.

Proof. Let the lattice L be not strong. Then there exist elements a, b, c, d in L so that $a \parallel c$, $b < c$ and $a \vee b = a \vee c = d$. Let us denote $e = a \wedge b$, $f = a \wedge c$. Obviously, $e \leqq f$. If $e = f$, then the elements a, b, c, d, e form the sublattice L_5 in L. If $e < f$, then $e \vee x = x$ for $x = a, b, c, d, e, f$ and $f \vee x = x$ for $x = a, c, d, f$. Clearly, $f \vee b \leqq c$.

In case $f \vee b = c$ the elements a, b, c, d, e, f form the sublattice L_6 in L. If $f \vee b < < c$, then the elements $f \vee b, a, c, d, f$ form a sublattice in L isomorphic to L_5.

It is obvious that any sublattice of a strong lattice is isomorphic neither to L_5 nor to L_6.

3.2. Corollary. *A lattice is strong if and only if it is dually strong.*

Proof. The assertion follows from 3.1, the dual statement and from the fact that the lattices L_5 and L_6 are selfdual.

3.3. Definition. Let L be a lattice and let id_L denote the identity relation on L. We put $\|_r = \| \cup \mathrm{id}_L$.

It is clear that the relation $\|_r$ is reflexive and symmetric in any lattice. The following theorem characterizes lattices in which $\|_r$ is an equivalence.

3.4. Theorem. *Let L be a lattice. Then L is strong if and only if $\|_r$ is a transitive relation on L.*

Proof. Let L be strong. Let $a \|_r b$, $b \|_r c$ for elements $a, b, c \in L$. In case $a = b$ and/or $b = c$ we have $a \|_r c$. In the opposite case it holds $a \| b$, $b \| c$. Suppose $a \nparallel_r c$. Then $c < a$ or $a < c$. However, $c < a$, $a \| b$ give $c < b$ by 2.2 and $a < c$, $a \| b$ imply $b < c$ according to the dualization of 2.2. In both cases we have a contradiction.

Now let $\|_r$ be transitive on L. An arbitrary element $a \in L$ is strong: Let $a \| b$ for $b \in L$. If there exists an element $c \in L$ with the property $b < c < a \vee b$, then $a \| c$. The symmetry and transitivity of $\|_r$ give $b \|_r c$, which is a contradiction. Thus, $b \prec a \vee b$ and a is strong by 1.2.

3.5. Lemma. *Let L be a strong lattice and let us assume $a < b$ for $a, b \in L$. Then it holds $a' < b'$ for any elements $a' \in [a]\|_r$, $b' \in [b]\|_r$.*

Proof. Suppose that $a' \|_r a$ and $b' \|_r b$ for $a', b' \in L$. As $\|_r$ is an equivalence and $a \nparallel_r b$, it holds $a' \nparallel_r b'$. That means $a' < b'$ or $b' < a'$. By 2.2, the validity of $b' < a'$ gives $b' < a$. It implies $b' < b$, which is a contradiction. Thus, we have proved $a' < b'$.

3.6. Lemma. *Let a, b be elements of a strong lattice L with the property $a \prec b$. Then*

$$\operatorname{card} [a]\|_r > 1 \Rightarrow \operatorname{card} [b]\|_r = 1 .$$

Proof. Let us suppose $\operatorname{card} [a]\|_r > 1$ and $\operatorname{card} [b]\|_r > 1$. Then there exist elements a', b' in L such that $a' \| a$ and $b' \| b$. By 3.5, it holds $a' < b$, $a' < b'$ and $a < b'$. It follows $a < a \vee a' \leqq b \wedge b' < b$, which contradicts $a \prec b$.

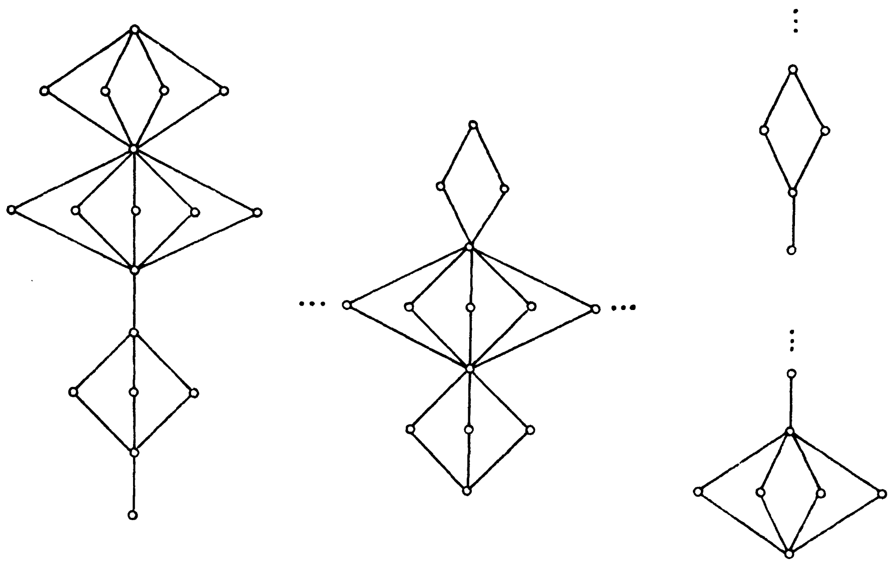

Fig. 2.

The statements 3.4, 3.5 and 3.6 give a clear description of the structure of strong lattices. Three examples of these lattices are given in Fig. 2.

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