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COMPATIBILITY IN ORTHOMODULAR POSETS

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1. NOTATION AND INTRODUCTORY REMARKS

Troughout this paper the letter P will be reserved for an *orthomodular poset* (cf. [3]), that is, a partially ordered set (with an ordering relation $\leqq$) with the greatest element 1 and with a mapping $\perp : P \to P$, $a \mapsto a^{\perp}$ satisfying the conditions:

(i) $a \leqq b$ implies $b^{\perp} \leqq a^{\perp}$;
(ii) $(a^{\perp})^{\perp} = a$ for all $a \in P$;
(iii) for all $a, b \in P$ such that $a \leqq b^{\perp}$ there exists $\sup(a, b)$;
(iv) $\sup(a, a^{\perp}) = 1$ for all $a \in P$;
(v) if $a \leqq b$, then there exists a unique c such that $c \leqq a^{\perp}$ and $\sup(a, c) = b$; in this case we write $c = b - a$.

(The condition (v) is the so-called *orthomodular law*.)

We say that $a, b \in P$ are *orthogonal* and write $a \perp b$ if $a \leqq b^{\perp}$.

The least upper bound or the greatest lower bound of a family $(a_i)_{i \in I}$ will be denoted by $\bigwedge_{i \in I} a_i$ or $\bigwedge_{i \in I} a_i$, respectively. We shall use the notation $\sum_{i \in I} a_i$ for $\bigvee_{i \in I} a_i$ iff $a_i \perp a_j$ for all $i, j \in I$, $i \neq j$.

An orthomodular poset P is said to be *σ-orthoadditive* if the following condition is satisfied:

(vi) if $a_i \perp a_j$, $i, j = 1, 2, \ldots, i \neq j$, then there exists $\sum_{i=1}^{\infty} a_i$.

If, in addition, P is a lattice or a σ-complete lattice, then P is called an *orthomodular lattice* or an *orthomodular σ-complete lattice*, respectively.

Remarks. 1) $0 =^{\text{df}} 1^{\perp}$ is the least element of P and $a \wedge a^{\perp} = 0$ for all $a \in P$.

2) $(\bigvee_{i \in I} a_i)^{\perp} = \bigwedge_{i \in I} a_i^{\perp}$ whenever $\bigwedge_{i \in I} a_i^{\perp}$ or $\bigvee_{i \in I} a_i$ exists.

3) $b - a = b \wedge a^{\perp}$ and $(b - a) \perp a$.

4) The condition (v) implies:

(1) $$\text{if} \quad a + b = 1\,, \quad \text{then} \quad b = a^{\perp}\,.$$

5) It is known (cf. [2]) that every orthomodular lattice with unique complements is a Boolean algebra, where $a \mapsto a^{\perp}$ is the (unique) complementation.

6) The notions "orthomodular subposet" "orthomodular sublattice" etc. of P are used in the same sense as in the general theory of abstract algebraic structures. In particular, a subposet $A \subset P$ is a Boolean subalgebra of P if

   (a) $a^{\perp} \in A$ for all $a \in A$;
   (b) if $a, b \in A$, then there exist $a \vee b$, $a \wedge b$ and $a \vee b \in A$, $a \wedge b \in A$;
   (c) A is a Boolean algebra with respect to the operations $(a, b) \mapsto a \vee b$, $(a, b) \mapsto$ $\mapsto a \wedge b$, $a \mapsto a^{\perp}$.

2. COMPATIBLE SETS OF P

Definition. Elements $a, b \in P$ are said to be *compatible* (and we write $a \leftrightarrow b$) if there are $a_1, b_1, u \in P$ such that

(2) $$a = a_1 + u\,, \quad b = b_1 + u\,, \quad a_1 \perp b_1\,.$$

It is easy to show that the folowing lemmas hold.

Lemma 1. *If $a \leftrightarrow b$, then there exist $a \vee b$, $a \wedge b$. Moreover, we have $a \wedge b = u$, $a \vee b = a_1 + b_1 + u$* (cf. [2]).

Lemma 2. *For all $a, b \in P$ the following conditions are equivalent:*
(a) *$a \leftrightarrow b$;*
(b) *$a^{\perp} \leftrightarrow b$;*
(c) *there exists $u \in P$ such that $u \leqq a$, $u \leqq b$ and $a - u \perp b$.*

Lemma 3. *Let a' be a complement of a in P* (i.e. $a \wedge a' = 0$, $a \vee a' = 1$). *Then $a \leftrightarrow a'$ iff $a' = a^{\perp}$.*

The following theorem (which is due to VARADARAJAN, cf. [4], [5]) holds.

Theorem 1. *Let P be an orthomodular lattice or a σ-complete orthomodular lattice. Let $M \subset P$ be a subset of pairwise compatible elements. Then there exists a maximal subset $B \supset M$ of pairwise compatible elements and B is a Boolean sublagebra or a Bollean σ-complete subalgebra of P, respectively.*

It should be noted that the theorem cited above does not remain valid in the case, when P is an orthomodular poset or a σ-orthoadditive orthomodular poset. This is shown by the following example.

Example. Let X be the set $\{1, 2, \ldots, 2n\}$, where n is a natural number, $n \geqq 5$. Let P be the system of all subsets of X consisting of an even number of elements. Assuming that the ordering in P is given by the set-theoretical inclusion, $M^\perp = P - M$, it is not difficult to see that P is an orthomodular poset (cf. [1]). Although the elements $A = \{1, 2. 3, 4, 5, 6\}$, $B = \{1, 2, 3, 4, 7, 8\}$, $C = \{1, 2, 3, 5, 7, 9\}$ are pairwise compatible, there is no Boolean subalgebra containing A, B, C since $\sup \{A, B, C\}$ does not exist. We are now going to give a generalization of Varadarajan's result cited above. First of all we need a suitable extension of the notion of compatibility. We define what we mean by a compatible set in P.

Definition. Let M be a finite subset of P. A finite family $(e_i)_{1 \leqq i \leqq n}$ is called *an orthogonal covering of the set M* if (i) $e_i \perp e_k$ for all $i \neq k$ and (ii) for each $a \in M$ there is a subfamily (e_{i_j}) such that $a = \sum_j e_{i_j}$.

A finite set M for which there exists its orthogonal covering is called *compatible* in P.

It is clear that each subset of a compatible set in P is compatible in P, thus we may define: A set $Q \subset P$ is called *compatible in P* if each finite subset of Q is compatible in P.

The notion of compatibility just defined is clearly one of those which are of the so-called "finite character". Thus Tukey's lemma implies that for every compatible set $Q \subset P$ there exists a maximal compatible set B in P containing Q. We call every maximal compatible set in P a *block* of P. Our intention is to show that every block $B \subset P$ is a Boolean subalgebra of P.

Remarks. 7) $\{a, b\}$ is compatible iff $a \leftrightarrow b$.

8) Obviously, if $\{a_1, \ldots, a_n\}$ is compatible, then $\bigvee_{i=1}^{n} a_i$ exists. We shall see that $\bigwedge_{i=1}^{n} a_i$ also exists.

9) The set $\{A, B, C\}$ from the previous example is not compatible, although the elements A, B, C are pairwise compatible.

10) If P is an orthomodular lattice, then M is compatible in P iff $a \leftrightarrow b$ for all $a, b \in M$.

The last assertion may be proved easily by induction.

Lemma 4. *Let M be a compatible set in P. Then*

1) *$a \in M$ implies $M \cup \{a^\perp\}$ is compatible in P;*

2) *$a_1, \ldots, a_n \in M$ implies $M \cup \{\bigvee_{i=1}^{n} a_i\}$ is compatible in P;*

3) *$a_1, \ldots, a_n \in M$ implies $M \cup \{\bigwedge_{i=1}^{n} a_i\}$ is compatible in P.*

Proof. We may assume that M is finite and that there exists an orthogonal covering $(e_i)_{1 \leqq i \leqq m}$ of M with the smallest m possible.

1) Let $M = \{b_1, \ldots, b_r, a\}$ and $a = \sum_{i=1}^{s} e_i$ (possibly with a permutation of indices). The remaining elements $e_{s+1}, \ldots, e_m$ are not subelements of a but they are subelements of some elements b_j. Let us denote $e_{m+1} = (b_1 \vee b_2 \vee \ldots b_r \vee a)^\perp$; clearly $e_{m+1} \perp e_i$ for $i = 1, 2, \ldots, m$. Putting $b = e_{s+1} + \ldots + e_{m+1}$, we have $b \perp a$ and $b \vee a = e_{m+1} \vee e_{m+1}^\perp = 1$. Hence $b = a^\perp$ (see Remark 4) and $(e_i)_{1 \leqq i \leqq m+1}$ is an orthogonal covering of $\{b_1, \ldots, b_r, a, a^\perp\}$.

2) It is clear that every orthogonal covering of M is also an orthogonal covering of $M \cup \{\bigvee_{i=1}^{n} a_i\}$.

3) From 1) it follows that $M \cup \{a_1^\perp, \ldots, a_1^\perp\}$ is compatible in P. According to 2) the set $M \cup \{\bigvee_{i=1}^{n} a_i^\perp\} = M \cup \{(\bigwedge_{i=1}^{n} a_i)^\perp\}$ is also compatible in P. From 1) it follows that the set $M \cup \{((\bigwedge_{i=1}^{n} a_i)^\perp)^\perp\} = M \cup \{\bigwedge_{i=1}^{n} a_i\}$ is compatible in P.

Theorem 2. *Let P be an orthomodular poset. Then every block B of P is a Boolean subalgebra of P.*

Proof. According to Lemma 4, B is closed with respect to finite joins and intersections and to the orthocomplementation $\perp$. Therefore B is an orthomodular sublattice of P. From Lemma 3 it follows that every element $a \in B$ has a unique complement $a^\perp$ in B, thus (see Remark 5) B is a Boolean subalgebra of P.

In the remainder of this paper P will be a σ-orthoadditive orthomodular poset.

Lemma 5. *Suppose $c, b_1, b_2, \ldots$ are arbitrary elements of P and the set $\{c, b_1, b_2, \ldots\}$ is compatible in P. If $b = \sum_{i=1}^{\infty} b_i$, then $c \leftrightarrow b$.*

Proof. Clearly $b_i \wedge c \perp b_j \wedge c$ for $i \neq j$. We put $u = \sum_{i=1}^{\infty} (b_i \wedge c)$; obviously $u \leqq b, u \leqq c$. It holds $(c - u)^\perp = c^\perp \vee u \geqq c^\perp \vee (b_i \wedge c) \geqq b_i$ for all $i = 1, 2, \ldots$. Hence $b \leqq (c - u)^\perp$, i.e. $b \perp (c - u)$ and by Lemma 2, $c \leftrightarrow b$.

Lemma 6. *Let $c_i \perp c_j$ for $i \neq j$, $i, j = 1, 2, \ldots, m$. Let $c_i \leftrightarrow b$ $(i = 1, 2, \ldots, m)$. Then $\{c_1, c_2, \ldots, c_m, b\}$ is compatible in P.*

Proof. We can see easily that an orthogonal covering of the set $\{c_1, \ldots, c_m, b\}$ is the family $(u_1, \ldots, u_m, c_1 - u_1, \ldots, c_m - u_m, b - \sum_{i=1}^{m} u_i)$, where $u_i = b \wedge c_i$ $(i = 1, 2, \ldots, m)$.

Lemmas 5, 6 imply immediately

Lemma 7. *Let* $\{c_1, \ldots, c_m, b_1, \ldots, b_n, \ldots\}$ *be compatible in* P, $c_i \perp c_j$, $b_i \perp b_j$ *for* $i \neq j$. *Then the set* $\{c_1, \ldots, c_m, \sum_{i=1}^{\infty} b_i\}$ *is compatible in* P.

Lemma 8. *Let* M *be a compatible set in* P, $b_i \in M$, $i = 1, 2, \ldots$, $b_i \perp b_j$ *for* $i \neq j$. *Then the set* $M \cup \{\sum_{i=1}^{\infty} b_i\}$ *is compatible in* P.

Proof. It follows from the preceding lemmas and from Remark 10.

Theorem 3. *Let* P *be a* σ*-orthoadditive orthomodular poset. Then every block* $B \subset P$ *is a* σ*-complete Boolean subalgebra of* P.

Proof. B is a Boolean subalgebra by Theorem 2. According to Lemma 8, B is closed with respect to countable joins of mutually disjoint elements. Thus B is σ-complete, which completes the proof.

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