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CONCERNING A GEOMETRICAL CHARACTERIZATION OF FRECHET DIFFERENTIABILITY

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INTRODUCTION

In finitely dimensional normed linear spaces, Fréchet differentiability from a geometric point of view has been studied by many authors, see e.g. FLETT [8] and ROETMAN [12]. Unfortunately, the methods used fail if the requirement of finite dimensionality is dropped. In general normed linear spaces, geometric characterizations of Fréchet differentiability were given first by DURDIL [5], [6], and recently by DANEŠ and DURDIL [4]. They do so by means of tangent plane, tangent cone and conic neighbourhood, respectively.

Here, in §1, we present a new characterization, which may be also called a "metric" one. However, in §4 it is observed that our characterization and those in [4]—[6] are virtually the same because each characterization can be obtained by rewriting another one. Nonetheless, the metric characterization is useful for introducing the Fréchet contiguity of sets. This concept is studied in §2. It is proved there (see Theorem 2.1) that the Fréchet contiguity of sets is conserved by a transformation (general enough) of the space. This result is then applied in §3 for deriving some theorems concerning the Fréchet contiguity of mappings. Namely, the Fréchet contiguity of inverses, linear combinations and compositions of mappings is studied. As corollaries, we obtain the corresponding theorems from differential calculus in normed linear spaces.

§ 0. PRELIMINARIES

Throughout the paper $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$, $(U, \|\cdot\|)$, $(W, \|\cdot\|)$ will stand for real (or complex) normed linear spaces. Let M be a nonempty subset of, say, Z. The symbols M', int M will denote the derived set and the interior of M, respectively. For $z \in Z$ we define

$$M \oplus \{z\} = \{m + z \mid m \in M\}, \tag{0.1}$$

$$d(z, M) = \inf\{\|z - m\| \mid m \in M\}. \tag{0.2}$$

Let $|||\cdot|||$ be a norm on Z equivalent with $\|\cdot\|$. Then there are $\alpha, \beta > 0$ such that

$$(0.3) \qquad \forall z \in Z \quad \alpha\|z\| \leqq |||z||| \leqq \beta\|z\| .$$

Hence, denoting $d'(z, M) = \inf\{|||z - m||| \mid m \in M\}$, we can see that

$$(0.4) \qquad \forall z \in Z \quad \alpha\, d(z, M) \leqq d'(z, M) \leqq \beta\, d(z, M) .$$

In what follows, we shall often put $Z = X \times Y$, $W = X \times U$, etc. In such cases, we shall always use the maximum norm, i.e., e.g., for $Z = X \times Y$,

$$(0.5) \qquad \|(x, y)\| = \max(\|x\|, \|y\|), \quad (x, y) \in Z .$$

We shall recall some concepts and notations concerning (multivalued) mappings. Under a (multivalued) mapping from X into Y we understand an arbitrary nonempty subset $F \subset X \times Y$. We then write $F : X \to 2^Y$. (Hence, we do not distinguish between a mapping and its graph!) The domain of F is defined by $D(F) = \{x \in X \mid \{x\} \times Y \cap F \neq \emptyset\}$. For each $x \in X$, we set $Fx = \{y \in Y \mid (x, y) \in F\}$. Let us remark that F can also be defined by fixing the set Fx for each $x \in X$. If Fx consists of one point only, we denote this point by Fx, too. For a set $M \subset X$ we define $F(M) = \bigcup\{Fm \mid m \in M\}$. Especially, we put $R(F) = F(X)$. F is called singlevalued at $x_0 \in X$ if the set Fx_0 is a singleton. F is called singlevalued if, for each $x \in X$, Fx is either a singleton or $Fx = \emptyset$. In this case, we write $F : X \to Y$. The inverse (multivalued) mapping to F is defined by $F^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in F\}$. Hence $F^{-1} : Y \to 2^X$. We say that F is upper semicontinuous (briefly u. s. c.) at $x_0 \in D(F)$ if, to every $\varepsilon > 0$, there exists a $\delta > 0$ such that Fx is contained in the ε-neighbourhood of the set Fx_0 whenever $\|x - x_0\| < \delta$, $x \in D(F)$.

The set of all singlevalued, positively homogeneous, Lipschitzian mappings $L : X \to Y$ with $D(L) = X$ will be denoted by $\mathscr{H}(X, Y)$.

$\boldsymbol{R}$ will stand for the real line with the usual norm.

§ 1. A GEOMETRIC CHARACTERIZATION OF FRECHET DIFFERENTIABILITY

During the last ten years, the concept of Fréchet differentiability has been generalized by many authors in several various directions and ways, see, e.g., [1], [2], [7], [9], [10] and the literature cited there. We shall work with the following slight generalization:

Definition 1.1. Let X, Y be normed linear spaces. Let $F : X \to 2^Y$ be a mapping with $\operatorname{int} D(F) \neq \emptyset$ and take some $x_0 \in \operatorname{int} D(F)$. Suppose that F is singlevalued at x_0 and that there exists $L \in \mathscr{H}(X, Y)$ such that

$$(1.1) \qquad x_0 \neq x \to x_0 \Rightarrow \frac{\sup\{\|y - Fx_0 - L(x - x_0)\| \mid y \in Fx\}}{\|x - x_0\|} \to 0 .$$

Then we say that F is *Fréchet differentiable at* x_0 and write $\mathrm{d}F(x_0) = L$.

It can be easily seen that the Fréchet differentiability of F at x_0 implies the u. s. c. of F at x_0. Furthermore, from the definition, the uniqueness of L follows.

Theorem 1.1. *Let X, Y be normed linear spaces. Let $F : X \to 2^Y$ be Fréchet differentiable at $x_0 \in X$ and denote $Z = X \times Y$, $z_0 = (x_0, Fx_0)$, $L = \mathrm{d}F(x_0)$.*

Then the following two implications hold (in the space Z):

$$(1.2) \qquad (z_0 \neq z \in F \;\&\; z \to z_0) \Rightarrow \frac{\mathrm{d}(z - z_0, L)}{\|z - z_0\|} \to 0\,,$$

$$(1.3) \qquad (0 \neq z \in L \;\&\; z \to 0) \Rightarrow \frac{d(z + z_0, F)}{\|z\|} \to 0\,.$$

Proof. We shall first prove (1.2). Let $z = (x, y) \in F$, $z \neq z_0$. Then $x \neq x_0$ and, by (0.2) and (0.5),

$$d(z - z_0, L) \leqq \|(x, y) - (x_0, Fx_0) - (x - x_0, L(x - x_0))\| =$$

$$= \|y - Fx_0 - L(x - x_0)\|\,,$$

$$\frac{d(z - z_0, L)}{\|z - z_0\|} \leqq \frac{\|y - Fx_0 - L(x - x_0)\|}{\|x - x_0\|}\,.$$

If now $z \to z_0$, then $x \to x_0$ and Definition 1.1 together with the last inequality proves (1.2).

Let us prove (1.3). Suppose

$$0 \neq z = (x, Lx) \in L \;\&\; z \to 0\,.$$

Then $0 \neq x \to 0$ and, for x so small that $x + x_0 \in D(F)$, we have

$$d(z + z_0, F) \leqq \|(x, Lx) + (x_0, Fx_0) - (x + x_0, y)\| = \|Lx + Fx_0 - y\|\,,$$

where $y \in F(x + x_0)$. Hence Definition 1.1 yields

$$\frac{d(z + z_0, F)}{\|z\|} \leqq \frac{\|Lx + Fx_0 - y\|}{\|x\|} \to 0$$

as (1.3) asserts.

It should be noted that, thanks to (0.3) and (0.4), the implications (1.2) and (1.3) are valid for an arbitrary norm on Z which is equivalent to that defined in (0.5).

Theorem 1.2. *Let X, Y be normed linear spaces. Let $F : X \to 2^Y$ be singlevalued and u. s. c. at $x_0 \in \operatorname{int} D(F)$ and denote $Z = X \times Y$, $z_0 = (x_0, Fx_0)$. Moreover, suppose there exists $L \in \mathscr{H}(X, Y)$ such that (1.2) holds.*

Then F is Fréchet differentiable at x_0 and $\mathrm{d}F(x_0) = L$.

Proof. The Lipschitz constant of L will be denoted by C. Let

$$\varepsilon \in \left(0, \frac{1}{2(1 + C)}\right) \tag{1.4}$$

be given. (1.2) implies that there is $\eta > 0$ such that

$$d(z - z_0, L) < \varepsilon \|z - z_0\|$$

whenever $z \in F$, $0 < \|z - z_0\| < \eta$. Fix such a $z = (x, y)$ arbitrarily. In view of (0.2) and (0.5), there is $u_z \in X$ such that

$$\|x - x_0 - u_z\| < \varepsilon \|z - z_0\| , \quad \|y - Fx_0 - Lu_z\| < \varepsilon \|z - z_0\| . \tag{1.5}$$

Now, using (1.4) and (1.5), we can estimate

$$\|u_z\| \leqq \|x - x_0 - u_z\| + \|x - x_0\| < \varepsilon \|z - z_0\| + \|x - x_0\| ,$$

$$\|y - Fx_0\| \leqq \|y - Fx_0 - Lu_z\| + C\|u_z\| <$$

$$< \varepsilon(1 + C) \|z - z_0\| + C\|x - x_0\| < \tfrac{1}{2}\|z - z_0\| + C\|x - x_0\| ,$$

$$\|z - z_0\| = \max (\|x - x_0\|, \|y - Fx_0\|) < \tfrac{1}{2}\|z - z_0\| + (1 + C) \|x - x_0\| ,$$

$$\|z - z_0\| < 2(1 + C) \|x - x_0\| . \tag{1.6}$$

Further, (1.5) and (1.6) yield

$$\|y - Fx_0 - L(x - x_0)\| \leqq \|y - Fx_0 - Lu_z\| + C\|u_z - (x - x_0)\| <$$

$$< \varepsilon(1 + C) \|z - z_0\| < 2\varepsilon(1 + C)^2 \|x - x_0\| .$$

Hence

$$\|y - Fx_0 - L(x - x_0)\| < 2\varepsilon(1 + C)^2 \|x - x_0\| \tag{1.7}$$

for all $z = (x, y) \in F$ fulfilling $0 < \|z - z_0\| < \eta$. Finally, since F is u. s. c. at x_0, there is $\delta \in (0, \eta)$ such that $\|y - Fx_0\| < \eta$ whenever $\|x - x_0\| < \delta$ and $y \in Fx$. Thus for each $x \in X$ fulfilling $0 < \|x - x_0\| < \delta$ and each $y \in Fx$, (1.7) holds, which completes the proof.

Theorem 1.2 says that (1.2) implies (under some assumptions) that F is Fréchet differentiable at x_0. Unfortunately, the same is not generally true for (1.3). This is shown in the following

Example 1.1. Let $X = Y = R$ and define $F : X \to Y$ as follows:

$$F0 = 0 , \quad Fx = x \sin (x^{-2}) \quad \text{for} \quad 0 \neq x \in R .$$

Obviously, F is continuous at 0. But F is not Fréchet differentiable at 0. Indeed, putting

$$x_n = (\pi n + \tfrac{1}{2}\pi)^{-1/2}, \quad n = 1, 2, \dots,$$

we get $x_n \to 0$ while

$$\frac{Fx_n}{x_n} = \sin(x_n^{-2}) = \sin(\pi n + \tfrac{1}{2}\pi) = (-1)^n .$$

On the other hand, we shall show that (1.3) is fulfilled at $z_0 = (0, 0)$ if we take $L = X \times \{0\}$. Let $x \in \boldsymbol{R}$, $|x| \in (0, \frac{1}{2})$. We can find a unique natural number $n_x > 1$ such that

$$n_x - 1 \leqq \frac{1}{\pi x^2} < n_x .$$

Then, denoting

$$u_x = \frac{x}{|x|} (\pi n_x)^{-1/2}$$

and taking into account (0.2), (0.5), we can estimate

$$d((x, 0), F) \leqq \|(x, 0) - (u_x, Fu_x)\| = |x - u_x| ,$$

$$\|(x, 0)\| > (\pi n_x)^{-1/2} = |u_x| .$$

Therefore, if $(0, 0) \neq (x, 0) \to (0, 0)$, then $n_x \to +\infty$ and so

$$\frac{d((x, 0), F)}{\|(x, 0)\|} < \frac{|x - u_x|}{|u_x|} = |x| \sqrt{(\pi n_x)} - 1 \leqq \sqrt{\left(\frac{n_x}{n_x - 1}\right)} - 1 \to 0 .$$

Thus (1.3) is verified.

The first part of Theorem 1.1 together with Theorem 1.2 gives us a geometrical characterization of Fréchet differentiability. We shall formulate it in

Theorem 1.3. *Let X, Y be normed linear spaces and let $F : X \to 2^Y$ be singlevalued and u. s. c. at $x_0 \in \operatorname{int} D(F)$. Finally, let $L \in \mathscr{H}(X, Y)$ and denote $Z = X \times Y$, $z_0 = (x_0, Fx_0)$.*

Then the following two assertions are equivalent:

(i) *F is Fréchet differentiable at x_0 and $\mathrm{d}F(x_0) = L$,*

(ii) $(z_0 \neq z \in F \;\&\; z \to z_0) \Rightarrow \dfrac{d(z - z_0, L)}{\|z - z_0\|} \to 0 .$

Let us observe that the implications (1.2) and (1.3) can be rewritten in a symmetric form. In fact, denoting $L_1 = L \oplus \{z_0\}$, see (0.1), we get respectively

$$(2.1) \qquad (z_0 \neq z \in F \;\&\; z \to z_0) \Rightarrow \frac{d(z, L_1)}{\|z - z_0\|} \to 0\,,$$

$$(2.2) \qquad (z_0 \neq z \in L_1 \;\&\; z \to z_0) \Rightarrow \frac{d(z, F)}{\|z - z_0\|} \to 0\,.$$

It suggests the following

Definition 2.1. Let M, N be two sets in a normed linear space Z such that $M' \cap N' \neq \emptyset$, and let $z_0 \in M' \cap N'$. We say that M is *Fréchet contiguous to* N at z_0 if the following implication holds:

$$(z_0 \neq z \in N \;\&\; z \to z_0) \Rightarrow \frac{d(z, M)}{\|z - z_0\|} \to 0\,.$$

Of course, replacing $\|z - z_0\|$ by $\|z - z_0\|^n$, Fréchet contiguity of n-th order can be defined. Furthermore, let us observe that Definition 2.1 is based only on the concept of distance, without using the linearity of Z. Hence it follows that Fréchet contiguity can be defined in metric spaces, too. It should be also noted that the concept introduced is local and independent of which equivalent norm on Z is taken, see (0.3) and (0.4).

Theorem 2.1. *Let Z, W be normed linear spaces and M, N two subsets of Z. Suppose that M is Fréchet contiguous to N at $z_0 \in M' \cap N'$. Moreover, let there exist a singlevalued mapping $B : Z \to W$ with $D(B) \supset M \cup N \cup \{z_0\}$, Fréchet differentiable at z_0 and such that the following two conditions hold*:

$$(2.3) \qquad (z \in M \cup N \;\&\; Bz \to Bz_0) \Rightarrow z \to z_0\,,$$

$$(2.4) \qquad \exists \delta > 0\ \exists c > 0\ (z \in M \cup N \;\&\; \|Az - Az_0\| < \delta) \Rightarrow$$
$$\Rightarrow \|Az - Az_0\| \geqq 2c\|z - z_0\|\,,$$

where the mapping $A : Z \to W$ is defined by

$$(2.5) \qquad Az = Bz_0 + dB(z_0)(z - z_0)\,, \quad z \in Z\,.$$

Then each of the sets $B(M)$, $A(M)$ is Fréchet contiguous to both $B(N)$, $A(N)$ at the point $w_0 = Bz_0 = Az_0$.

Proof. First we shall show that $w_0 \in A(M)'$, $w_0 \in B(M)'$. Since $z_0 \in M'$, there exists a sequence $\{z_n\} \subset M$ such that $z_0 \neq z_n \to z_0$. The continuity of A and B at z_0 (B is Fréchet differentiable at z_0) implies that $Az_n \to Az_0$, $Bz_n \to Bz_0$. But (2.4) or (2.3) yields respectively $Az_n \neq Az_0$, $Bz_n \neq Bz_0$ for n large enough. Hence $w_0 = Az_0 \in A(M)'$, $w_0 = Bz_0 \in B(M)'$. The relations $w_0 \in A(N)'$, $w_0 \in B(N)'$ can be obtained in the same way.

Next, put for brevity

$$\omega(z - z_0) = Bz - Bz_0 - \mathrm{d}B(z_0)(z - z_0)\,, \quad z \in D(B)\,. \tag{2.6}$$

Hence, the Fréchet differentiability of B at z_0 implies

$$z_0 \neq z \to z_0 \Rightarrow \frac{\|\omega(z - z_0)\|}{\|z - z_0\|} \to 0\,. \tag{2.7}$$

Now let

$$w_0 \neq w \in B(N) \;\&\; w \to w_0 \quad \text{and} \quad w_0 \neq \bar{w} \in A(N) \;\&\; \bar{w} \to w_0\,.$$

To each w and $\bar{w}$, we can find $z \in N$ and $\bar{z} \in N$ such that $w = Bz$ and $\bar{w} = A\bar{z}$, respectively. Then, by (2.3) and (2.4),

$$z_0 \neq z \in N \;\&\; z \to z_0\,, \quad z_0 \neq \bar{z} \in N \;\&\; \bar{z} \to z_0\,. \tag{2.8}$$

Using (2.4)–(2.8) and assuming w and $\bar{w}$ sufficiently close to w_0, we can estimate

$$\begin{aligned} \|w - w_0\| &= \|Bz - Bz_0\| \geqq \|Az - Az_0\| - \|\omega(z - z_0)\| > \\ &> 2c\|z - z_0\| - c\|z - z_0\| = c\|z - z_0\|\,, \\ \|\bar{w} - w_0\| &= \|A\bar{z} - Az_0\| \geqq 2c\|\bar{z} - z_0\|\,. \end{aligned}$$

That is,

$$\|w - w_0\| > c\|z - z_0\|\,, \quad \|\bar{w} - w_0\| > c\|\bar{z} - z_0\|\,. \tag{2.9}$$

Further, since M is Fréchet contiguous to N at z_0, (2.8) implies

$$\frac{d(z, M)}{\|z - z_0\|} \to 0\,, \quad \frac{d(\bar{z}, M)}{\|\bar{z} - z_0\|} \to 0\,.$$

So to each z, $\bar{z}$ there exist $m_z \in M$, $m_{\bar{z}} \in M$ such that

$$\frac{\|z - m_z\|}{\|z - z_0\|} \to 0 \quad \text{and} \quad \frac{\|\bar{z} - m_{\bar{z}}\|}{\|\bar{z} - z_0\|} \to 0\,. \tag{2.10}$$

Hence combining (2.7), (2.8) and (2.10), we get

$$(2.11)\qquad \begin{gathered} \|m_z - z_0\| \to 0\,, \quad \|m_{\bar z} - z_0\| \to 0\,, \\ \frac{\|\omega(m_z - z_0)\|}{\|z - z_0\|} \to 0\,, \quad \frac{\|\omega(m_{\bar z} - z_0)\|}{\|\bar z - z_0\|} \to 0\,. \end{gathered}$$

Now let C be the Lipschitz constant of $\mathrm{d}B(z_0)$. Then using (2.5) and (2.6), we can estimate

$$\begin{aligned} d(w, B(M)) &= d(Bz, B(M)) \leqq \|Bz - Bm_z\| = \|Bz_0 + \mathrm{d}B(z_0)(z - z_0) + \\ &\quad + \omega(z - z_0) - (Bz_0 + \mathrm{d}B(z_0)(m_z - z_0) + \omega(m_z - z_0))\| \leqq \\ &\quad \leqq C\|z - m_z\| + \|\omega(z - z_0)\| + \|\omega(m_z - z_0)\|\,, \\ d(w, A(M)) &= d(Bz, A(M)) \leqq \|Bz - Am_z\| = \|\mathrm{d}B(z_0)(z - z_0) + \\ &\quad + \omega(z - z_0) - \mathrm{d}B(z_0)(m_z - z_0)\| \leqq C\|z - m_z\| + \\ &\quad + \|\omega(z - z_0)\|\,, \\ d(\bar w, B(M)) &= d(A\bar z, B(M)) \leqq \|A\bar z - Bm_{\bar z}\| = \|\mathrm{d}B(z_0)(\bar z - z_0) - \\ &\quad - \mathrm{d}B(z_0)(m_{\bar z} - z_0) - \omega(m_{\bar z} - z_0)\| \leqq C\|\bar z - m_{\bar z}\| + \\ &\quad + \|\omega(m_{\bar z} - z_0)\|\,, \\ d(\bar w, A(M)) &= d(A\bar z, A(M)) \leqq \|A\bar z - Am_{\bar z}\| = \\ &\quad = \|\mathrm{d}B(z_0)(\bar z - z_0) - \mathrm{d}B(z_0)(m_{\bar z} - z_0)\| \leqq C\|\bar z - m_{\bar z}\|\,. \end{aligned}$$

These estimates together with (2.7)–(2.11) yield

$$\begin{aligned} \frac{d(w, B(M))}{\|w - w_0\|} &\leqq \frac{C\|z - m_z\|}{c\|z - z_0\|} + \frac{\|\omega(z - z_0)\|}{c\|z - z_0\|} + \frac{\|\omega(m_z - z_0)\|}{c\|z - z_0\|} \to 0\,, \\ \frac{d(w, A(M))}{\|w - w_0\|} &\leqq \frac{C\|z - m_z\|}{c\|z - z_0\|} + \frac{\|\omega(z - z_0)\|}{c\|z - z_0\|} \to 0\,, \\ \frac{d(\bar w, B(M))}{\|\bar w - w_0\|} &\leqq \frac{C\|\bar z - m_{\bar z}\|}{c\|\bar z - z_0\|} + \frac{\|\omega(m_{\bar z} - z_0)\|}{c\|\bar z - z_0\|} \to 0\,, \\ \frac{d(\bar w, A(M))}{\|\bar w - w_0\|} &\leqq \frac{C\|\bar z - m_{\bar z}\|}{c\|\bar z - z_0\|} \to 0\,. \end{aligned}$$

The proof is thus complete.

Of course, in the above theorem it suffices to assume that B is Fréchet differentiable at z_0 with respect to the set $M \cup N \cup \{z_0\}$ in an appropriate sense.

§ 3. FRECHET CONTIGUITY OF MAPPINGS

Let X, Y be normed linear spaces and $F, G : X \to 2^Y$ two mappings. Putting $Z = X \times Y$, $M = F$, $N = G$ in Definition 2.1, we arrive at

Definition 3.1. Let F, $G : X \to 2^Y$ be two mappings such that int $D(F) \cap$ int $D(G)$ is nonempty and take an x_0 in this set. Moreover, suppose that F, G are singlevalued and u. s. c. at x_0 and that $Fx_0 = Gx_0$. Then we say that the mapping F is *Fréchet contiguous to* G at x_0 if (the set) F is Fréchet contiguous to (the set) G at the point (x_0, Fx_0) in the space $Z = X \times Y$. We say that the mappings F and G are *mutually Fréchet contiguous at* x_0 if F is Fréchet contiguous to G as well as G is Fréchet contiguous to F at x_0.

We use Definition 2.1 here. Hence we must verify that $(x_0, Fx_0) \in F' \cap G'$. But this follows from the fact that F, G are singlevalued and u. s. c. at x_0 and that $Fx_0 = Gx_0$. Definition 3.1 is thus correct.

Obviously, Fréchet contiguity of mappings is a local concept and is independent of which equivalent norm in $X \times Y$ is taken.

Now, bearing in mind Definition 3.1 and the implications (2.1), (2.2), the results of §1 can be formulated as follows:

Theorem 3.1. *Let the assumptions of Theorem* 1.3 *be fulfilled. Then the following three assertions are equivalent*:

(i) *F is Fréchet differentiable at x_0 and* $\mathrm{d}F(x_0) = L$.

(ii) *The mapping $L_1 = L \oplus \{z_0\}$ is Fréchet contiguous to F at x_0, i.e.,* (2.1) *holds.*

(iii) *The mappings L_1 and F are mutually Fréchet contiguous at x_0, i.e., both* (2.1) *and* (2.2) *hold.*

It should be noted that Example 1.1 and Theorem 3.1 show that, if F is Fréchet contiguous to L_1 at x_0, L_1 and F need not be mutually Fréchet contiguous at x_0.

Let F, $G : X \to 2^Y$ be two mappings with $D(F) = D(G)$ and int $D(F) \neq 0$. Suppose there exists $x_0 \in$ int $D(F)$ such that F and G are singlevalued and u. s. c. at x_0 and let $Fx_0 = Gx_0$. Finally, suppose that the following implication holds (see [3, 1.2.1]):

$$(3.1) \qquad x_0 \neq x \to x_0 \Rightarrow \frac{\sup \{\|u - v\| \mid u \in Fx, v \in Gx\}}{\|x - x_0\|} \to 0 .$$

Then, in a similar way as in the proof of Theorem 1.1, we can see that F and G are mutually Fréchet contiguous at x_0. Unfortunately, the converse is not true generally. This can be easily checked taking $X := Y = \boldsymbol{R}$, $F = G = \{(x, y) \in X \times Y \mid |y| \leqq |x|\}$. It need not hold even if we require F and G to be singlevalued. This is shown in the following.

Example 3.1. Let $X = Y = \boldsymbol{R}$ and define the mappings $F, G : X \to Y$ as follows:

$$F0 = G0 = 0\,,$$

$$Fx = x \sin(x^{-2})\,, \quad Gx = x \cos(x^{-2}) \quad \text{for} \quad 0 \neq x \in \boldsymbol{R}\,.$$

Obviously, F, G are continuous at 0. Further, if we put

$$x_n = \left(\pi n + \frac{\pi}{2}\right)^{-1/2}, \quad n = 1, 2, \ldots,$$

then $x_n \to 0$ while

$$\frac{|Fx_n - Gx_n|}{|x_n|} = 1$$

for all $n = 1, 2, \ldots$. It means (3.1) is violated.

Next we shall show that F and G are mutually Fréchet contiguous at 0. Let $x \in \boldsymbol{R}$ be such that $0 < |x| < \frac{1}{2}$ and put

$$u_x = x\left(1 - \frac{\pi}{2} x^2\right)^{-1/2}.$$

Then

$$x^{-2} = u_x^{-2} + \frac{\pi}{2}$$

and so we get

$$|Fx - Gu_x| = |x \sin(x^{-2}) - u_x \cos(u_x^{-2})| =$$

$$= |x \cos(u_x^{-2}) - u_x \cos(u_x^{-2})| \leqq |x - u_x|\,.$$

Hence

$$\frac{d((x, Fx), G)}{\|(x, Fx)\|} \leqq \frac{\|(x, Fx) - (u_x, Gu_x)\|}{\|(x, Fx)\|} \leqq \frac{|x - u_x|}{|x|} = \left(1 - \frac{\pi}{2} x^2\right)^{-1/2} - 1 \to 0$$

as $(x, Fx) \to (0, 0)$. It means G is Fréchet contiguous to F at 0. And, using a similar argument, we get that F is Fréchet contiguous to G at 0, which was to be proved.

Now we are going to derive some results concerning Fréchet contiguity of inverses, linear combinations and compositions of mappings from Theorem 2.1.

Theorem 3.2. *Let a mapping $F : X \to 2^Y$ be Fréchet contiguous to a mapping $G : X \to 2^Y$ at $x_0 \in X$. Suppose, moreover, that the inverse mappings $F^{-1}, G^{-1} : Y \to 2^X$ are singlevalued and u. s. c. at $y_0 = Fx_0 = Gx_0$ and let $y_0 \in \operatorname{int} R(F) \cap \operatorname{int} R(G)$.*

Then F^{-1} is Fréchet contiguous to G^{-1} at y_0.

Proof. Put $Z = X \times Y$, $W = Y \times X$, $M = F$, $N = G$, $z_0 = (x_0, y_0)$ and define the mapping $B : Z \to W$ by

$$B(x, y) = (y, x), \quad (x, y) \in X \times Y.$$

The assumptions of Theorem 2.1 are obviously fulfilled. Therefore, the set $B(M)$ is Fréchet contiguous to the set $B(N)$ at Bz_0. But

$$B(M) = F^{-1}, \quad B(N) = G^{-1}, \quad B(x_0, y_0) = (y_0, x_0),$$
$$D(F^{-1}) = R(F), \quad D(G^{-1}) = R(G).$$

Hence, according to Definition 3.1, the result follows.

Corollary 3.1. *Let $G : X \to 2^Y$ be Fréchet differentiable at $x_0 \in X$ and such that $y_0 = Gx_0 \in \operatorname{int} R(G)$. Also suppose that $G^{-1} : Y \to 2^X$ is singlevalued and u. s. c. at y_0 and let the mapping $(\mathrm{d}G(x_0))^{-1}$ belong to $\mathscr{H}(Y, X)$.*

Then G^{-1} is Fréchet differentiable at y_0 and

$$\mathrm{d}G^{-1}(y_0) = (\mathrm{d}G(x_0))^{-1}.$$

Proof. Put $F = \mathrm{d}G(x_0) \oplus \{(x_0, y_0)\}$, see (0.1). By Theorem 3.1, F is Fréchet contiguous to G at x_0. But

$$F^{-1} = B(F) = B(\mathrm{d}G(x_0)) \oplus \{B(x_0, y_0)\} = (\mathrm{d}G(x_0))^{-1} \oplus \{(y_0, x_0)\}.$$

Hence F^{-1} is singlevalued and continuous at y_0. Theorem 3.2 then says that F^{-1} is Fréchet contiguous to G^{-1} at y_0. Now, making use of Theorem 3.1 again, we get the result.

It should be noted that the above corollary can also be obtained directly from Theorem 1.3 by using (0.2) and (0.5).

Let $F, G : X \to 2^Y$ be two mappings and let λ be a given number. Then we define the mappings $\lambda F : X \to 2^Y$, $F + G : X \to 2^Y$ by

$$\lambda F = \{(x, \lambda y) \mid (x, y) \in F\},$$
$$F + G = \{(x, y_1 + y_2) \mid (x, y_1) \in F, (x, y_2) \in G\}.$$

Remark that the mappings λF and $F + G$ are different from the λ-multiple of the set F and the sum of the sets F and G in the space $X \times Y$, respectively.

Theorem 3.3. *If $F : X \to 2^Y$ is Fréchet contiguous to $G : X \to 2^Y$ at $x_0 \in X$, then λF is Fréchet contiguous to λG at x_0, too.*

Proof. The case $\lambda = 0$ is trivial. If $\lambda \neq 0$, put $Z = W = X \times Y$, $M = F$, $N = G$, $z_0 = (x_0, Fx_0)$ and define the mapping $B : Z \to W$ as follows:

$$B(x, y) = (x, \lambda y), \quad (x, y) \in X \times Y.$$

The verification of the hypotheses of Theorem 2.1 is easy. Further, $B(F) = \lambda F$, $B(G) = \lambda G$. Hence the result follows.

Corollary 3.2. *If $G : X \to 2^Y$ is Fréchet differentiable at $x_0 \in X$, then so is λG and*

$$\mathrm{d}(\lambda G)(x_0) = \lambda\, \mathrm{d}G(x_0)\,.$$

Proof. Denote $F = \mathrm{d}G(x_0) \oplus \{(x_0, Gx_0)\}$, see (0.1). Then Theorems 3.1 and 3.3 yield the result.

Next we shall need a slight generalization of Theorem 2.1. Checking its proof we can easily see that this theorem remains valid also for a multivalued mapping $B : Z \to 2^W$ if (2.3) is replaced by

$$(z \in M \cup N \;\&\; w \in Bz \;\&\; w \to w_0) \Rightarrow z \to z_0\,. \tag{3.2}$$

In the rest of the section we shall use Theorem 2.1 in this more general setting.

Theorem 3.4. *Let $F,\ G,\ H : X \to 2^Y$ be three mappings such that F is Fréchet contiguous to G at $x_0 \in X$ and H is Fréchet differentiable at x_0.*

Then each of the mappings $F + H$, $F + \varphi$ is Fréchet contiguous to both $G + H$, $G + \varphi$ at x_0, where

$$\varphi = \mathrm{d}H(x_0) \oplus \{(x_0, Hx_0)\}\,, \quad \text{i.e.,} \quad \varphi(x) = \mathrm{d}H(x_0)(x - x_0) + Hx_0\,, \quad x \in X\,.$$

Proof. Since $x_0 \in \operatorname{int} D(F) \cap \operatorname{int} D(G) \cap \operatorname{int} D(H)$ and Fréchet contiguity is a local concept, we may assume without loss of generality that $D(F) \cup D(G) \subset D(H)$. Put $Z = W = X \times Y$, $M = F$, $N = G$, $y_0 = Fx_0 = Gx_0$, $z_0 = (x_0, y_0)$ and define the mapping $B : Z \to 2^W$ by

$$B(x, y) = \{(x, y + v) \mid v \in Hx\}\,, \quad (x, y) \in D(H) \times Y\,. \tag{3.3}$$

Let us verify the hypotheses of Theorem 2.1. Obviously, $D(B) = D(H) \times Y \supset M \cup N \cup \{z_0\}$. A simple computation yields that B is Fréchet differentiable at z_0 and that

$$\mathrm{d}B(z_0)(h, k) = (h, k + \mathrm{d}H(x_0)\,h)\,, \quad (h, k) \in X \times Y\,. \tag{3.4}$$

It remains to show the validity of (3.2) and (2.4). So, let $z = (x, y) \in M \cup N$, $w \in Bz$, $w \to w_0$. According to (3.3), $w = (x, y + v) \to (x_0, y_0 + Hx_0)$, where $v \in Hx$. Hence $x \to x_0$ and the fact that F, G are singlevalued and u. s. c. at x_0 yields $y \to y_0$. Thus $z \to z_0$, i.e. (3.2) holds. In order to show (2.4) suppose the contrary. Then there exists a sequence $\{z_n\} = \{(x_n, y_n)\} \subset M \cup N$ such that

$$Az_0 \neq Az_n \to Az_0 \;\&\; \frac{\|Az_n - Az_0\|}{\|z_n - z_0\|} \to 0\,.$$

(It should be noted that $Az = Az_0$ implies $z = z_0$.) Hence, bearing in mind (2.5) and (3.4), $x_n \to x_0$ and

$$\frac{\max(\|x_n - x_0\|, \|y_n - y_0 + \mathrm{d}H(x_0)(x_n - x_0)\|)}{\max(\|x_n - x_0\|, \|y_n - y_0\|)} \to 0 .$$

This implies

$$\frac{\|x_n - x_0\|}{\|y_n - y_0\|} \to 0 , \quad \frac{\|y_n - y_0 + \mathrm{d}H(x_0)(x_n - x_0)\|}{\|y_n - y_0\|} \to 0 .$$

Further, denoting the Lipschitz constant of $\mathrm{d}H(x_0)$ by C,

$$0 = \lim_{n\to\infty} \frac{\|y_n - y_0 + \mathrm{d}H(x_0)(x_n - x_0)\|}{\|y_n - y_0\|} \geqq \lim_{n\to\infty}\left(1 - C\frac{\|x_n - x_0\|}{\|y_n - y_0\|}\right) = 1 ,$$

which is impossible. (2.4) is thus proved.

Now we may apply Theorem 2.1. (2.5), (3.3) and (3.4) yield

$$\begin{aligned} B(M) &= B(F) = \bigcup\{B(x, y) \mid (x, y) \in F\} = \\ &= \{(x, y + v) \mid (x, y) \in F, (x, v) \in H\} = F + H , \\ A(M) &= A(F) = \{A(x, y) \mid (x, y) \in F\} = \\ &= \{B(x_0, y_0) + \mathrm{d}B(x_0, y_0)((x, y) - (x_0, y_0)) \mid (x, y) \in F\} = \\ &= \{(x_0, y_0 + Hx_0) + (x - x_0, y - y_0 + \mathrm{d}H(x_0)(x - x_0)) \mid (x, y) \in F\} = \\ &= \{(x, y + \mathrm{d}H(x_0)(x - x_0) + Hx_0) \mid (x, y) \in F\} = F + \varphi . \end{aligned}$$

Similarly for N,

$$B(N) = B(G) = G + H , \quad A(N) = A(G) = G + \varphi .$$

Thus our theorem follows from Theorem 2.1 and Definition 3.1.

Corollary 3.3. *If mappings $G, H : X \to 2^Y$ are Fréchet differentiable at $x_0 \in X$, then so is $G + H$ and*

$$\mathrm{d}(G + H)(x_0) = \mathrm{d}G(x_0) + \mathrm{d}H(x_0) .$$

Proof. Put

$$F = \mathrm{d}G(x_0) \oplus \{(x_0, Gx_0)\} , \quad \text{i.e.,} \quad Fx = \mathrm{d}G(x_0)(x - x_0) + Gx_0 , \quad x \in X .$$

Then

$$Fx + \varphi(x) = (\mathrm{d}G(x_0) + \mathrm{d}H(x_0))(x - x_0) + (G + H)x_0 , \quad x \in X ,$$

i.e.,

$$F + \varphi = (dG(x_0) + dH(x_0)) \oplus \{(x_0, (G + H)\, x_0)\}.$$

The result now follows from Theorems 3.1 and 3.4.

If $F : X \to 2^Y$, $H : Y \to 2^U$ (U being a normed linear space) are two mappings, the composition $H \circ F$ of F and H is defined by

$$H \circ F = \{(x, u) \mid \exists y \in Y\ (x, y) \in F \ \&\ (y, u) \in H\}.$$

Hence $H \circ F : X \to 2^U$.

Theorem 3.5. *Let $F, G : X \to 2^Y$ be two mappings such that F is Fréchet contiguous to G at $x_0 \in X$. Let $H : Y \to 2^U$ be Fréchet differentiable at $y_0 = Fx_0 = Gx_0$. Moreover, suppose that for every sequence $\{(x_n, y_n)\} \subset F \cup G$ with $x_0 \neq x_n \to x_0$, either*

$$(3.5)\qquad \liminf_{n\to\infty} \frac{\|y_n - y_n\|}{\|x_n - x_0\|} < +\infty \quad or \quad \liminf_{n\to\infty} \frac{\|y_n - y_0\|}{\|dH(y_0)\,(y_n - y_0)\|} < +\infty .$$

Then each of the mappings $H \circ F$, $\psi \circ F$ is Fréchet contiguous to both $H \circ G$, $\psi \circ G$ at the point x_0, where

$$\psi = dH(y_0) \oplus \{(y_0, Hy_0)\} \quad i.e., \quad \psi(y) = dH(y_0)\,(y - y_0) + Hy_0\,, \quad y \in Y.$$

Proof. Without loss of generality we may restrict ourselves to the case $R(F) \cup R(G) \subset D(H)$. Put $Z = X \times Y$, $W = X \times U$, $z_0 = (x_0, y_0)$, $M = F$, $N = G$ and define the mapping $B : Z \to 2^W$ as follows:

$$(3.6)\qquad B(x, y) = \{(x, u) \mid (y, u) \in H\}, \quad (x, y) \in X \times D(H).$$

We have to verify the assumptions of Theorem 2.1. Obviously, $D(B) = X \times D(H) \supset M \cup N \cup \{z_0\}$. Also we can easily show that B is Fréchet differentiable at z_0 and that

$$(3.7)\qquad dB(z_0)\,(h, k) = (h, dH(y_0)\, k), \quad (h, k) \in X \times Y.$$

(3.2) follows at once from the fact that F, G are singlevalued and u. s. c. at x_0. (2.4) can be proved from (3.5) in a similar way as in the proof of Theorem 3.4.

Further, (2.5), (3.6) and (3.7) yield

$$\begin{aligned}
B(M) &= B(F) = \bigcup\{B(x, y) \mid (x, y) \in F\} = \\
&= \{(x, u) \mid \exists y \in Y\ (x, y) \in F \ \&\ (y, u) \in H\} = H \circ F, \\
A(M) &= A(F) = \{A(x, y) \mid (x, y) \in F\} = \\
&= \{B(x_0, y_0) + dB(x_0, y_0)((x, y) - (x_0, y_0)) \mid (x, y) \in F\} = \\
&= \{(x_0, Hy_0) + (x - x_0, dH(y_0)\,(y - y_0)) \mid (x, y) \in F\} = \\
&= \{(x, \psi(y)) \mid (x, y) \in F\} = \psi \circ F,
\end{aligned}$$

and similarly,

$$B(N) = B(G) = H \circ G\,, \quad A(N) = A(G) = \psi \circ G\,.$$

Hence the result follows from Theorem 2.1 and Definition 3.1.

Corollary 3.4. *If $G : X \to 2^Y$ is Fréchet differentiable at $x_0 \in X$ and $H : Y \to 2^U$ is Fréchet differentiable at $y_0 = Gx_0$, then the composition $H \circ G : X \to 2^U$ is Fréchet differentiable at x_0 and*

$$\mathrm{d}(H \circ G)\,(x_0) = \mathrm{d}H(y_0) \circ \mathrm{d}G(x_0)\,.$$

Proof. Put $F = \mathrm{d}G(x_0) \oplus \{(x_0, Gx_0)\}$. Then

$$\begin{aligned}\psi \circ F &= \{(x, u) \mid \exists y \in Y\ (x, y) \in F \ \&\ (y, u) \in \psi\} = \\ &= \{(x, u) \mid \exists y \in Y\ (x - x_0, y - y_0) \in \mathrm{d}G(x_0) \ \&\ (y - y_0, u - Hy_0) \in \mathrm{d}H(y_0)\} = \\ &= \{(x, u) \mid (x - x_0, u - Hy_0) \in \mathrm{d}H(y_0) \circ \mathrm{d}G(x_0)\} = \\ &= \mathrm{d}H(y_0) \circ \mathrm{d}G(x_0) \oplus \{(x_0, (H \circ G)\, x_0\}\,.\end{aligned}$$

Hence, observing that the Fréchet differentiability of F, G implies the first condition in (3.5), Theorems 3.1 and 3.5 yield the result.

§ 4. COMPARISON

In this section we are going to compare the geometric characterizations of Fréchet differentiability given in [4]–[6] with our one formulated in Theorem 1.3. In order to do it we shall introduce some concepts and notations.

Let Z be a normed linear space. Under a cone (in Z) we understand every subset C of Z such that $C \neq \emptyset$, $C \neq \{0\}$, and $\lambda z \in C$ whenever $z \in C$ and $\lambda \geqq 0$. Observe that every $L \in \mathscr{H}(X, Y)$ is a cone in $Z = X \times Y$.

Let C be a cone in Z and $\varepsilon \in (0, 1)$. We define the following two kinds of conic ε-neighbourhoods of C, see [4], [6]:

$$V_\varepsilon(C) = \{z \in Z \mid d(z, C) < \varepsilon\|z\|\} \cup \{0\}\,,$$
$$U_\varepsilon(C) = \{\lambda(c + z) \mid c \in C\,, \quad \|c\| = 1\,, \quad z \in Z\,, \quad \|z\| < \varepsilon\,, \quad \lambda \geqq 0\}\,.$$

Both these sets are cones. The relations between them are established in the following

Proposition 4.1. *Let $\varepsilon \in (0, 1)$ and let C be a cone in Z. Then*

$$U_\varepsilon(C) \subset V_\varepsilon(C) \subset U_\Delta(C)\,,$$

where $\Delta = 2\varepsilon/(1 - \varepsilon)$.

Proof. Let $0 \neq w \in U_\varepsilon(C)$. We can then write $w = \lambda(c + z)$, where $c \in C$, $\|c\| = 1$, $z \in Z$, $\|z\| < \varepsilon$, $\lambda > 0$. We have

$$d\left(\frac{w}{\|w\|}, C\right) \leqq d\left(\frac{w}{\|w\|}, \{\alpha c \mid \alpha \geqq 0\}\right) =$$

$$= d(c, \{\alpha w \mid \alpha \geqq 0\}) \leqq \left\|c - \frac{1}{\lambda} w\right\| = \|z\| < \varepsilon .$$

Hence $d(w, C) < \varepsilon\|w\|$, i.e., $w \in V_\varepsilon(C)$.

Further, let $0 \neq w \in V_\varepsilon(C)$ and denote $z = w/\|w\|$. Then $d(z, C) < \varepsilon$ and so there exists $c \in C$, $c \neq 0$ such that $\|z - c\| < \varepsilon$. Hence

$$1 - \varepsilon < \|z\| - \|z - c\| \leqq \|c\| \leqq \|z\| + \|z - c\| < 1 + \varepsilon ,$$

$$\frac{-\varepsilon}{1 - \varepsilon} = 1 - \frac{1}{1 - \varepsilon} < 1 - \frac{1}{\|c\|} < 1 - \frac{1}{1 + \varepsilon} = \frac{\varepsilon}{1 + \varepsilon} ,$$

$$\left|1 - \frac{1}{\|c\|}\right| < \frac{\varepsilon}{1 - \varepsilon} ,$$

$$\left\|z - \frac{c}{\|c\|}\right\| \leqq \|z - c\| + \left\|c - \frac{c}{\|c\|}\right\| < \varepsilon + \|c\| \left|1 - \frac{1}{\|c\|}\right| < \varepsilon + (1 + \varepsilon)\frac{\varepsilon}{1 - \varepsilon} =$$

$$= \frac{2\varepsilon}{1 - \varepsilon} = \Delta .$$

Thus, according to the definition of $U_\Delta(C)$,

$$z = \frac{c}{\|c\|} + \left(z - \frac{c}{\|c\|}\right) \in U_\Delta(C) , \quad w = \|w\|\, z \in U_\Delta(C)$$

and the proof is complete.

Further, let $M \subset Z$, $M' \neq \emptyset$ and take $z_0 \in M'$. For $r > 0$ we define the following cone:

$$C_r(M, z_0) = \{\lambda(z - z_0) \mid z \in M, \|z - z_0\| < r, \lambda \geqq 0\} .$$

Finally, for $z_0 \in Z$ and $\delta > 0$, set

$$B_Z(z_0, \delta) = \{z \in Z \mid \|z - z_0\| < \delta\} .$$

Now, using the above notation and Proposition 4.1, we can easily rewrite the assertion (ii) in Theorem 1.3. We arrive at

Theorem 4.1. *Let X, Y be normed linear spaces and $F : X \to 2^Y$ a mapping with* $\operatorname{int} D(F) \neq \emptyset$, *singlevalued and u. s. c. at $x_0 \in \operatorname{int} D(F)$. Let $L \in \mathscr{H}(X, Y)$ and denote* $Z = X \times Y$, $z_0 = (x_0, Fx_0)$.

Then the following four assertions are equivalent:

(i) *F is Fréchet differentiable at x_0 and $\mathrm{d}F(x_0) = L$,*

(ii) $(z_0 \neq z \in F \;\&\; z \to z_0) \Rightarrow \dfrac{\mathrm{d}(z - z_0, L)}{\|z - z_0\|} \to 0$,

(iii) $\forall \varepsilon > 0 \;\exists \delta > 0 \;\; F \cap B_Z(z_0, \delta) \subset V_\varepsilon(L) \oplus \{z_0\}$,

(iv) $\forall \varepsilon > 0 \;\exists r > 0 \;\; C_r(F, z_0) \subset U_\varepsilon(L)$.

In the papers [4]–[6], a slightly restricted situation is considered, namely, F is assumed to be singlevalued and L linear and continuous. Bearing in mind such F and L, we can compare our results with those of [4]–[6]. (i) $\Leftrightarrow$ (iii) says the same as [4, Theorems 1, 2]. It is also shown there (see [4, Lemma 1]) that this equivalence is a reformulation of [5, Theorem 1]. (i) $\Leftrightarrow$ (iv) is included in [6, Theorem 1]. It follows that the geometric characterizations of Fréchet differentiability given in [4]–[6] and here, in §1, are virtually the same.

Let us remind that [6, Theorem 1] asserts in addition that (i) implies

$$\text{(v)} \qquad \forall \varepsilon > 0 \;\forall r > 0 \;\; L \subset U_\varepsilon(C_r(F, z_0)).$$

Hence the following equivalence holds:

$$\text{(i)} \Leftrightarrow (\text{(iv)} \;\&\; \text{(v)}).$$

Regarding the fact that $C_r(F, z_0) \subset C_s(F, z_0)$ whenever $0 < r < s$, we can even rewrite ((iv) & (v)) as follows:

$$\text{(vi)} \qquad \forall \varepsilon > 0 \;\exists \delta > 0 \;\forall r \in (0, \delta) \;\; C_r(F, z_0) \subset U_\varepsilon(L) \;\&\; L \subset U_\varepsilon(C_r(F, z_0)).$$

It should be noted that the equivalence (i) $\Leftrightarrow$ (vi) suggests to introduce the concept of the Fréchet cone of a set as well as of a mapping, see [7].

At the end of the section we shall show that (under the hypotheses of Theorem 4.1) the implication (i) $\Rightarrow$ (v) can be derived from the results of §1. In fact, by Theorem 1.1, (i) implies (1.3), i.e.,

$$(0 \neq z \in L \;\&\; z \to 0) \Rightarrow \frac{d(z + z_0, F)}{\|z\|} \to 0.$$

Hence, we shall obtain the result if we prove the following

Proposition 4.2. *Under the assumptions of Theorem* 4.1,

$$(1.3) \Rightarrow (\mathrm{v}).$$

Proof. Let $\varepsilon \in (0, 1)$ and $r > 0$ be fixed. Put $\Delta = \varepsilon/(1 + \varepsilon)$ and choose $\delta > 0$ such that $(\Delta + 1)\,\delta < r$ and that

$$(0 \neq z \in L \;\&\; \|z\| < \delta) \Rightarrow d(z + z_0, F) < \Delta\|z\| .$$

To each z from this implication, take $w_z \in F$ so that

$$\|z + z_0 - w_z\| < \Delta\|z\| . \tag{4.1}$$

We can then estimate

$$\|w_z - z_0\| \leqq \|z + z_0 - w_z\| + \|z\| < \Delta\|z\| + \|z\| < (\Delta + 1)\,\delta < r . \tag{4.2}$$

Further, (4.1) implies that $w_z \neq z_0$ because $\Delta < 1$. Then, denoting

$$b = \frac{z + z_0 - w_z}{\|w_z - z_0\|} ,$$

we can write

$$z = \|w_z - z_0\| \left(\frac{w_z - z_0}{\|w_z - z_0\|} + b\right) , \tag{4.3}$$

where, by (4.1),

$$\|b\| \leqq \frac{\|z + z_0 - w_z\|}{\|z\| - \|z + z_0 - w_z\|} < \frac{\Delta}{1 - \Delta} = \varepsilon . \tag{4.4}$$

Now, (4.2)–(4.4) together yield that $z \in U_\varepsilon(C_r(F, z_0))$ for all $z \in L$ fulfilling $\|z\| < \delta$. But both L and $U_\varepsilon(C_r(F, z_0))$ are cones. Hence $L \subset U_\varepsilon(C_r(F, z_0))$ as required.

However, the reverse implication $(\mathrm{v}) \Rightarrow (1.3)$ may be false.

Example 4.1. Let $\{e_n\}_1^\infty$ be a system of orthonormal elements of a real Hilbert space. Let X be the linear span of $\{e_n\}_1^\infty$, i.e., the set of all elements $x = \sum_{i=1}^{\infty} \alpha_i e_i$ such that $\alpha_i = 0$ for each $i = 1, 2, \ldots$ except for a finite number of i. Put $Y = \boldsymbol{R}$ and define the mapping $F : X \to Y$ as follows:

$$Fx = F\Big(\sum_{i=1}^{\infty} \alpha_i e_i\Big) = \begin{cases} 0 & \text{if } \alpha_i < i^{-2} \text{ for all } i = 1, 2, \ldots , \\ \max_i |\alpha_i| & \text{otherwise} . \end{cases}$$

F is continuous at 0. We shall show that (v) holds for $L = X \times \{0\}$ and $z_0 = (0, 0)$. Indeed, let $r > 0$. For each $x = \sum_{i=1}^{\infty} \alpha_i e_i \in X$ there exists $t > 0$ such that $t^2 \sum_{i=1}^{\infty} \alpha_i^2 < r^2$

and $t|\alpha_i| < i^{-2}$ for all $i = 1, 2, \ldots$. Hence $F(tx) = 0$ and $(tx, 0) \in C_r(F, z_0)$. It means $L \subset C_r(F, z_0)$ and so (v) is verified.

On the other hand, (1.3) is violated. In fact, let

$$z_n = \frac{1}{n}(e_n, 0), \quad n = 2, 3, \ldots.$$

Then $z_n \to (0, 0)$ and for each $x = \sum_{i=1}^{\infty} \alpha_i e_i \in X$ we have

$$\|z_n - (x, Fx)\| = \max\left(\left\|\frac{1}{n} e_n - x\right\|, |Fx|\right) \geqq \frac{1}{2}\left|\frac{1}{n} - \alpha_n\right| + \frac{1}{2}|Fx|.$$

If $\alpha_n < n^{-2}$, then

$$\|z_n - (x, Fx)\| \geqq \frac{1}{2}\left|\frac{1}{n} - \alpha_n\right| > \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n^2}\right) \geqq \frac{1}{4} \cdot \frac{1}{n} = \frac{1}{4}\|z_n\|.$$

If $\alpha_n \geqq n^{-2}$, then

$$\|z_n - (x, Fx)\| \geqq \frac{1}{2}\left|\frac{1}{n} - \alpha_n\right| + \frac{1}{2}|\alpha_n| \geqq \frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2}\|z_n\|.$$

Hence

$$d(z_n, F) > \tfrac{1}{4}\|z_n\|, \quad n = 2, 3, \ldots.$$

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