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COMPARISON OF THE MOST TYPICAL PROBABILITY DISTRIBUTIONS WITH RESPECT TO SPREAD

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1. Introduction. A partial ordering on a set of probability distributions with respect to a property of global dispersion – shortly spread – is introduced and studied. The definition and some properties of this partial ordering were established by BICKEL and LEHMANN [1c] and they are shortly described in Section 2 of the present paper. In Section 3, the conclusions of Section 2 are applied to a finite set of selected distribution shapes: normal, double exponential, exponential, logistic, uniform, Gumbel, Cauchy and triangular. For any one of those distributions, necessary and sufficient conditions for its being more spread out than any other continuous probability distribution are derived. The comparisons are then made for the comparable pairs of distributions of the above set. Finally, it is shown that the ordering with respect to dispersion, introduced by Bickel and Lehmann for symmetrical distributions, is weaker than the ordering with respect to spread.

2. Some conditions for ordering with respect to spread. Let us start with the definition of Bickel and Lehmann.

Definition 2.1. Let X, Y be random variables with distribution functions F, G. We shall call the distribution G more spread out than F (denoting $G >^r F$) if

$$G^{-1}(v) - G^{-1}(u) \geqq F^{-1}(v) - F^{-1}(u) \quad \forall 0 < u < v < 1\,, \tag{2.1}$$

where $F^{-1}(t) = \sup\{x : F(x) \leqq t\}$.

We could see easily from (2.1) that $G >^r F$ iff any two percentage points of G are at last so apart as the corresponding percentage points of F. The relation

$$F \sim^r G \Leftrightarrow F >^r G \,\&\, G >^r F \tag{2.2}$$

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is an equivalence on any set $\mathscr{F}$ of probability distributions. The corresponding classes of equivalence are described in the following easy proposition.

Proposition 2.2. *Let F, G be distribution functions. Then $F \sim^r G$ if and only if $F(x - a) \equiv G(x)$ for some $a \in E_1$.*

The following propositions give equivalent versions of Definition 2.1 under different conditions on F and G.

Proposition 2.3. *If F and G are such that F^{-1} and G^{-1} are differentiable, then*

$$G >^r F \quad \textit{iff} \quad g[G^{-1}(v)] \leqq f[F^{-1}(v)]\,, \quad 0 < v < 1\,, \tag{2.3}$$

where f, g are the respective densities corresponding to F and G.

Proof. (2.1) implies the inequality

$$\lim_{u \to v} \frac{F^{-1}(v) - F^{-1}(u)}{v - u} \leqq \lim_{u \to v} \frac{G^{-1}(v) - G^{-1}(u)}{v - u}\,,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}v}[G^{-1}(v)] \geqq \frac{\mathrm{d}}{\mathrm{d}v}[F^{-1}(v)]\,, \quad 0 < v < 1\,,$$

which gives (2.3).

The following equivalence of Definition 2.1 is due to Bickel and Lehmann.

Proposition 2.4. *If X, Y are random variables with strictly increasing distribution functions F and G, respectively, then G is more spread out than F if and only if there exists a strictly increasing function $h(x)$ such that*

(i) *$x - x' \leqq h(x) - h(x')$ for any x, x' such that $x > x'$,*
(ii) *$h(X)$ has the distribution function G.*

Proof. see [1c].

Corollary. *Let X and Y be random variables with strictly increasing distribution functions F and G, respectively. Than $F <^r G$ if and only if*

$$G^{-1}[F(x)] - x \tag{2.4}$$

is nondecreasing on $I = \{x : 0 < F(x) < 1\}$.

Remark. This form of partial ordering of distributions was mentioned for the first time by HÁJEK in [2].

Proof. Regarding that F and G are strictly increasing, we see that (2.1) holds if and only if the function

$$h(x) = G^{-1}[F(x)]$$

is strictly increasing and

$$x' - x \leqq h(x') - h(x) \quad \text{for all} \quad x < x'.$$

The rest of the proof follows from Proposition 2.4.

Table 1.

Distribution	$f(x)$	$f[F^{-1}(y)]$
doubly exponential	$\frac{1}{2a} \cdot e^{-\lvert x/a \rvert}$ $\quad x \in E_1$	$a^{-1} \cdot y \quad y \in (0, \frac{1}{2}\rangle$ $a^{-1} \cdot (1 - y) \quad y \in \langle \frac{1}{2}, 1)$
exponential	$0 \quad x \leqq 0$ $a^{-1} \cdot e^{-x/a} \quad x > 0$	$a^{-1} \cdot (1 - y) \quad y \in (0, 1)$
logistic	$\frac{a^{-1} \cdot e^{-x/a}}{[1 + e^{-x/a}]^2} \quad x \in E_1$	$a^{-1} \cdot y \cdot (1 - y) \quad y \in (0, 1)$
uniform on $(0, a)$	$\frac{1}{a} \quad x \in (0, a)$ $0 \quad x \notin (0, a)$	$a^{-1} \quad y \in (0, 1)$
Gumbel's	$a^{-1} . \exp\left(\frac{x}{a} - e^{xa^{-1}}\right) \quad x \in E_1$	$a^{-1} \cdot (y - 1) \cdot \ln(1 - y)$ $y \in (0, 1)$
$N(0, a^2)$	$\frac{1}{a \cdot \sqrt{(2\pi)}} \exp\left(-\frac{x^2}{2a^2}\right)$ $x \in E_1$	$f[a\, \Phi^{-1}(y)] \quad y \in (0, 1)$
Cauchy	$\frac{1}{a\pi} \frac{1}{1 + \left(\frac{x}{a}\right)^2} \quad x \in E_1$	$\{a\pi \cdot [1 + \mathrm{tg}^2\, \pi(y - \frac{1}{2})]\}^{-1}$ $y \in (0, 1)$
triangular	$0 \quad x \notin (-1, 1)$ $1 - \lvert x \rvert \quad x \in (-1, 1)$	$\sqrt{(2y)} \quad y \in (0, \frac{1}{2}\rangle$ $\sqrt{(2 - 2y)} \quad y \in \langle \frac{1}{2}, 1)$

3. Comparison of the selected distributions. Considering a specific distribution F, we may be interested in what conditions must satisfy any other distribution G to be more spread out than F.

Let us consider this problem for a finite set of selected shapes: double exponential, exponential, logistic, uniform, Gumbel, triangular, normal and Cauchy; each one in the standard ($a = 1$) as well as in the parametric form (see Table 1).

(1) Let a random variable X has a double exponential distribution F. Then it follows from (2.3) that a random variable Y with a distribution function G and a density g is more spread out than X if and only if

$$\frac{y}{a} \geqq g[G^{-1}(y)], \quad y \in (0, \tfrac{1}{2}\rangle \quad \text{and} \quad \frac{1-y}{a} \geqq g[G^{-1}(y)], \quad y \in \langle \tfrac{1}{2}, 1).$$

Regarding the relation $G(x) = y \Leftrightarrow x = G^{-1}(y)$ we get:

$$\frac{G(x)}{a} \geqq g\{G^{-1}[G(x)]\} = g(x), \quad x \leqq 0 \quad \text{and}$$

$$\frac{1 - G(x)}{a} \geqq g\{G^{-1}[G(x)]\} = g(x), \quad x \geqq 0.$$

Thus a random variable Y with a distribution G given by a density g is more spread out than X if and only if

$$g(x) \leqq \begin{cases} a^{-1} \,.\, G(x), & x \leqq 0, \\ a^{-1} \,.\, [1 - G(x)], & x \geqq 0. \end{cases} \tag{3.1}$$

(2) Let F be the cdf of the exponential distribution with a density f. Then a distribution G with a density g is more spread out than F if and only if

$$g(x) \leqq a^{-1} \,.\, [1 - G(x)], \quad x \geqq 0. \tag{3.2}$$

(3) Let F be the cdf of the logistic distribution with a density f. Then a distribution G with a density g is more spread out than F if and only if

$$g(x) \leqq a^{-1} \,.\, [G(x)] \,.\, [1 - G(x)], \quad x \in E_1. \tag{3.3}$$

(4) Let F be the cdf of the uniform distribution on $(0, a)$. Then a distribution G with a density g is more spread out than F if and only if

$$g(x) \leqq a^{-1}, \quad x \in E_1. \tag{3.4}$$

(5) Let F be the cdf of Gumbel's distribution with a density f. Then a distribution G with a density g is more spread out than F if and only if

$$g(x) \leqq a^{-1} \,.\, [G(x) - 1] \,.\, \ln [1 - G(x)], \quad x \in E_1. \tag{3.5}$$

(6) Let F be the cdf of the triangular distribution with a density f. Then a distribution G with a density g is more spread out than F if and only if

$$(3.6) \qquad g(x) \leqq \begin{cases} \sqrt{[2 \cdot G(x)]}, & x \leqq 0, \\ \sqrt{\{2 \cdot [1 - G(x)]\}}, & x \geqq 0. \end{cases}$$

(7) Let Φ be the normal distribution $N(0, a^2)$. Then a distribution G with a density g is more spread out than Φ if and only if

$$(3.7) \qquad g(x) \leqq a^{-1} \cdot f[\Phi^{-1}(G(x))], \quad x \in E_1,$$

where f is the density of the normal distribution and Φ^{-1} is the inverse distribution function.

(8) Let F be the cdf of Cauchy's distribution with a density f. Then a distribution G with a density g is more spread out than F if and only if

$$(3.8) \qquad g(x) \leqq \{a\pi[1 + \mathrm{tg}^2\, \pi(G(x) - \tfrac{1}{2})]\}^{-1}, \quad x \in E_1.$$

The opposite inequalities in 3.1–3.8 mean that the distribution G is less spread out than the distribution F.

We could now try to compare the pairs of distributions of Table 1 by specifying the conditions (3.1)–(3.8) to the distributions of Table 1 in the role of G. Some pairs are comparable in their standard form, the others only under some conditions on a (e.g. Cauchy and logistic distributions); some pairs are not comparable (e.g. Gumbel and exponential distributions). Generally speaking, the results confirm our intuitive feeling that the more spread out distributions are those with heavy tails.

The results of the comparisons of the distributions of Table 1 follow.

(3.9) The double exponential distribution is more spread out than $N(0, a^2)$ for $a \leqq \sqrt{(2/\pi)}$.

(3.10) The parametrized Cauchy's distribution with $a \geqq 4/\pi$ is more spread out than the logistic distribution.

(3.11) The logistic distribution is more spread out than the double exponential distribution.

(3.12) The double exponential distribution is more spread out than the exponential distribution.

(3.13) All distributions from Table 1 are more spread out than the uniform distribution on (0, 1).

(3.14) Gumbel's distribution is more spread out than the triangular distribution.

(3.15) Logistic distribution is more spread out than Gumbel's distribution.

(3.16) Normal distribution $N(0, 1)$ is more spread out than the triangular distribution.

(3.17) On the other hand, it is impossible to compare with respect to spread the following distributions:

a) exponential and triangular distributions

b) Gumbel's distribution with exponential and double exponential distributions.

4. Relations between spread and dispersion. As was mentioned above, the concept of spread is a generalization of the dispersion. We are now interested in the relation between both the concepts. Everywhere in this part $\mathscr{F}_1$ will denote the class of random variables with symmetric distributions.

Definition 4.1. Let $X, Y \in \mathscr{F}_1$, the distribution of X being symmetric around μ and that of Y around ν. We shall say that Y is more dispersed around ν than X around μ, if $|Y - \nu|$ is stochastically larger than $|X - \mu|$ (notation $|X - \mu| \preccurlyeq |Y - \nu|$), i.e.

$$P\{|Y - \nu| \geqq x\} \geqq P\{|X - \mu| \geqq x\}, \tag{4.1}$$

and exist at least one x_0 so that strict inequality occurs in (4.1). The relation of both types of ordering is described in the next proposition.

Proposition 4.2. *Let random variables $X, Y \in \mathscr{F}_1$ have distribution functions F, G. Then (4.1) is equivalent to*

$$G^{-1}(v) - G^{-1}(\tfrac{1}{2}) \gtreqqless F^{-1}(v) - F^{-1}(\tfrac{1}{2}) \quad v \gtreqqless \tfrac{1}{2}, . \tag{4.2}$$

For proving this proposition we need the following lemma.

Lemma 4.3. *If X, Y are random variables with distributions symmetric about zero, F, G are their distribution functions and F^{-1}, G^{-1} inverse distributions functions, then*

$$|X| \preccurlyeq |Y| \Leftrightarrow \begin{cases} G^{-1}(v) \leqq F^{-1}(v), & 0 < v \leqq \tfrac{1}{2}, \\ G^{-1}(v) \geqq F^{-1}(v), & \tfrac{1}{2} \leqq v < 1, \end{cases} \tag{4.3}$$

where the inequalities are strict on a set U of positive Lebesgue measure.

Proof of lemma. Let $|X| \preccurlyeq |Y|$; then it follows from the symmetry of F and G that

$$F(x) \leqq G(x), \quad x \leqq 0 \quad \text{and} \quad F(x) \geqq G(x), \quad x \geqq 0, \tag{4.4}$$

with the strict inequality for some x_0.

Thus

$$G^{-1}(v) \leqq F^{-1}(v) \quad \text{for} \quad 0 < v \leqq \tfrac{1}{2} \quad \text{and} \quad G^{-1}(v) \geqq F^{-1}(v) \quad \text{for} \quad \tfrac{1}{2} \leqq v < 1$$

with strict inequalities on a set U of positive Lebesgue measure.

Conversely, the right-hand side of (4.3) implies (4.4) and this is equivalent to $|X| \preccurlyeq |Y|$.

Proof of Proposition 4.2. We may put $\mu = \nu = 0$ without loss of generality. (4.1) is then equivalent to $|X| \preccurlyeq |Y|$ and (4.2) follows directly from Lemma 4.3.

On the other hand $F^{-1}(\frac{1}{2}) = G^{-1}(\frac{1}{2}) = 0$ [because $\mu = \nu = 0$], so that (4.2) implies

$$G^{-1}(v) - F^{-1}(v) \gtreqless 0, \quad v \gtreqless \tfrac{1}{2},$$

and now (4.1) follows from Lemma 4.3.

Comparing the conditions 2.1 and 4.2, we see that the ordering with respect to spread is stronger than that with respect to dispersion and 2.1 obviously implies 4.2 for symmetric distributions. The next example shows that the converse is not true.

Example. Let random variables X, Y have the respective densities f, g:

$$\begin{aligned} f(x) &= 0, \quad |x| > 2, & g(x) &= 0, \quad |x| > 2, \\ &= \tfrac{1}{8}, \quad 1 < |x| \leqq 2, & &= \tfrac{3}{8}, \quad 1 < |x| \leqq 2, \\ &= \tfrac{3}{8}, \quad |x| \leqq 1, & &= \tfrac{1}{8}, \quad |x| \leqq 1. \end{aligned}$$

Then $|X| \preccurlyeq |Y|$, but $F >^r G$ on $(0, \frac{3}{8}\rangle$ and $\langle \frac{5}{8}, 1)$ and at the same time $G >^r F$ on $\langle \frac{3}{8}, \frac{5}{8} \rangle$.

References

[1] *Bickel P. J. - Lehmann E. L.* (1975): Descriptive statistics for nonparameterics models. [a] Introduction. AMS 1975, No. *5*, 1044—1048. [b] Dispersion. AMS 1975, No. *5*, 1049 to 1068. [c] Spread. Will be published in Hájek's Memorial volume.

[2] *Hájek J.* (1969): A course in nonparametrics statistics. Holden day. San Francisco.

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