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A TOPOLOGICAL PROOF OF DENJOY-STEPANOFF THEOREM

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In 1915, A. DENJOY proved that any Lebesgue measurable function is approximately continuous almost everywhere. (A simple proof using the Lusin theorem on characterization of measurable functions is due to W. SIERPIŃSKI [4], 1922.) The converse of this theorem is also true (see W. STEPANOFF [5] 1924, and for a simplified proof E. KAMKE [2] 1927), so that measurable functions are completely characterized as those approximately continuous at almost all points.

The purpose of this short note is to give a slightly different proof of the mentioned theorems. In what follows, $\mathbb{R}$ will denote the set of reals, λ stands for the Lebesgue measure in $\mathbb{R}$.

The family of all measurable sets on $\mathbb{R}$ having density one at each of its points forms a certain topology which will be called the *density topology*. Thus, a set M is open in the density topology iff M is measurable and $\lim_{h \to 0} (2h)^{-1} \lambda([x - h, x + h] \cap \cap M) = 1$ for any $x \in M$. (For a collection of its properties see e.g. [6].)

The following two propositions are crucial in our investigation.

Proposition A. *The following families of sets coincide*:

(i) *Lebesgue null sets,*
(ii) *nowhere dense sets in the density topology,*
(iii) *sets of the first category in the density topology,*
(iv) *(closed) discrete sets in the density topology.*

Proof. The proof is easy and is left to the reader (cf. [3], Theorem 22.6, [6], Theorem 2.7). ■

Proposition B ([3], Theorem 22.7, [6], Theorem 2.6). *The following families of sets coincide*:

(i) *sets of the type G_δ in the density topology,*
(ii) *Borel sets in the density topology,*
(iii) *sets with the Baire property in the density topology,*
(iv) *Lebesgue measurable sets.*

Proof. Recall that the family of Borel sets in a topology is a smallest σ-algebra containing all open sets, and the sets with the Baire property form the smallest σ-algebra containing all open sets as well the sets of the first category. Since any nowhere dense set is closed (in the density topology), the Borel sets coincide with sets with the Baire property. Thus, it is sufficient to prove (iv) $\Rightarrow$ (i) only. But any Lebesgue measurable set is of the form $G \setminus N$, where G is of type G_δ (in the Euclidean topology) and N has Lebesgue measure zero. It follows (use Proposition A, (i) $\Rightarrow$ (iv)) that any such set is G_δ in the density topology. ■

A real function f on a topological space X has the *Baire property* if $f^{-1}(U)$ has the Baire property for any open set $U \subset \mathbb{R}$. The following assertion substitutes the Lusin theorem in the proof of the Denjoy theorem and forms, in fact, its categorial counterpart.

Proposition C. *A real function f on X has the Baire property if and only if there exists a set M of the first category such that the restriction of f to $X \setminus M$ is continuous.*

Proof. See e.g. [3], Theorem 8.1. ■

Continuous functions in the density topology are exactly *approximately continuous* functions. Now, we are able to pass to the proof of the main theorem.

Theorem (Denjoy-Stepanoff). *A real function is Lebesgue measurable if and only if it is approximately continuous almost everywhere.*

Proof. A real function f is Lebesgue measurable iff it has the Baire property in the density topology (Proposition B). In view of Propositions C and A, f is Lebesgue measurable iff there is a null set $N \subset \mathbb{R}$ such that the restriction of f to $\mathbb{R} \setminus N$ is approximately continuous on $\mathbb{R} \setminus N$. But this is exactly the case when f is approximately continuous at all points of density-open set $\mathbb{R} \setminus N$. ■

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