Časopis pro pěstování matematiky

Ivan Netuka; Jiří Veselý

An inequality for finite sums in R^m

Časopis pro pěstování matematiky, Vol. 103 (1978), No. 1, 73--77

Persistent URL: http://dml.cz/dmlcz/117961

Časopis pro pěstování matematiky, roč. 103 (1978), Praha

AN INEQUALITY FOR FINITE SUMS IN R^m

IVAN NETUKA, JIŘÍ VESELÝ, Praha

(Received June 15, 1976)

In Rudin's book [9] the following inequality is proved: If $z_1, \ldots, z_n$ are complex, then there is a subset I of $\{1, \ldots, n\}$ such that

$$\left|\sum_{j\in I} z_j\right| \geqq (\tfrac{1}{\pi}) \sum_{j=1}^{n} |z_j| .$$

Various modifications and generalizations can be found in literature (for references, see below). In this note we establish an inequality of this type for finite sets of points in R^m.

For $x, y \in R^m (m > 1)$ we denote by $x \,.\, y$ the scalar product of x and y; we write $x = (x^1, \ldots, x^m)$, $|x| = (x \,.\, x)^{1/2}$ and for a finite set $P = \{p_1, \ldots, p_n\} \subset R^m$,

$$\sum P = \sum_{i=1}^{n} p_i , \quad \sum |P| = \sum_{i=1}^{n} |p_i| .$$

Given $\delta \in \langle 0, 1 \rangle$ and a unit vector $u \in R^m$, we shall denote by $T(u, \delta)$ the cone $\{x \in R^m; x \,.\, u \geqq \delta |x|\}$. Finally, put

$$C(m, \delta) = \frac{\Gamma(\frac{1}{2}m)\,(1 - \delta^2)^{(m-1)/2}}{2\sqrt{(\pi)}\,\Gamma((m + 1)/2)} .$$

In this note we shall prove the following

Theorem. *For any finite set $P \subset R^m$ with $\sum |P| > 0$ there is a unit vector u such that*

(1) $$\left|\sum [P \cap T(u, \delta)]\right| > C(m, \delta) \sum |P| .$$

The number $C(m, \delta)$ cannot be replaced by any larger one.

Remark. The Gauss-Green theorem is used to evaluate $C(m, \delta)$. An application of the Krein-Milman theorem shows that $C(m, \delta)$ is the best such constant.

The theorem represents a generalization of analogous inequalities established in [4], p. 85 and 113, [3], [8], p. 330–332 and [6]. The constant $C(m, 0)$ appears in [7] where some deeper results concerning measures are obtained.

In what follows, τ stands for the surface measure in R^m (i.e. τ is the $(m-1)$-dimensional Hausdorff measure), $S = \{x \in R^m; |x| = 1\}$ and $S(u, \delta) = S \cap T(u, \delta)$. We put $a = \tau(S)$ and $\sigma = a^{-1}\tau$. Note that $a = 2\pi^{m/2}/\Gamma(m/2)$.

Lemma. *If $\delta \in \langle 0, 1)$ and $u \in S$, then*

$$\int_{S(u,\delta)} x^j \, d\sigma(x) = C(m, \delta)\, u^j, \quad j = 1, \ldots, m.$$

Proof. Fix $u \in S$, $\delta \in \langle 0, 1)$, $1 \leqq j \leqq m$ and put $V = \{x \in R^m; x \cdot u > \delta, |x| < 1\}$. Then V has a piecewise smooth boundary ∂V and the set $B = \partial V - S(u, \delta)$ is isometric with an $(m-1)$-ball with radius $(1-\delta^2)^{1/2}$. Consequently, $\tau(B) = \pi^{(m-1)/2} \cdot (1-\delta^2)^{(m-1)/2}/\Gamma((m+1)/2)$. Let $n(x)$ denote the exterior normal to V at $x \in \partial V$, if it exists; otherwise let $n(x) = 0$. Note that $n(x) = x$ for $x \in S(u, \delta)$ with $x \cdot u > \delta$, while $n(x) = -u$ for $x \in B$. Consider now a constant vector function w with $w^j = 1$, $w^k = 0$ for $k \neq j$. Applying the Gauss-Green theorem, we obtain

$$0 = \int_V \operatorname{div} w(x)\, dx = \int_{\partial V} n^j(x)\, d\tau(x)$$

so that

$$\int_{S(u,\delta)} x^j \, d\sigma(x) = a^{-1} u^j \tau(B) = C(m, \delta)\, u^j.$$

Proof of the theorem. Let $P = \{p_1, \ldots, p_n\} \subset R^m$. We may suppose $p_j \neq 0$ for all j. Define $q_j = p_j/|p_j|$ and

$$g(u) = |\sum[P \cap T(u, \delta)]|$$

for all $u \in S$. There is only a finite number of $u \in S$ which are positive multiples of some vectors $\sum Q$ with a nonemtpy $Q \subset P$. For all other $u \in S$ we have $|\sum[P \cap T(u, \delta)]| > (\sum[P \cap T(u, \delta)]) \cdot u$. Consequently, for such points u,

$$g(u) > \sum_{i=1}^{n} |p_i| \, (q_i \cdot u)\, \chi_i(u)$$

where χ_i stands for the characteristic function of the set $S(q_i, \delta)$. Integrating g over S with respect to σ, we obtain by the lemma

$$\int_S g(u)\, d\sigma(u) > \sum_{i=1}^{n} |p_i| \left(\sum_{j=1}^{m} q_i^j \int_{S(q_i,\delta)} u^j \, d\sigma(u) \right) = C(m, \delta) \sum |P|.$$

Since $\sigma(S) = 1$, it follows that there is a $u \in S$ with $g(u) > C(m, \delta) \sum|P|$. (This part of the proof is a slight modification of the reasoning used in [3].)

Suppose now that (1) holds with a number c instead of $C(m, \delta)$ whenever $P \subset R^m$ is a finite set with $\sum|P| > 0$. We are going to prove that $c \leqq C(m, \delta)$.

Denote by $\mathscr{M}$ the convex set of all probability measures on S and by D the set of (finite) convex combinations of Dirac measures concentrated on S. Observe that $\sigma \in \mathscr{M}$ and note that μ is an extreme point of $\mathscr{M}$ if and only if μ coincides with the Dirac measure ε_x for an $x \in S$ (see [2], p. 21). Suppose now that $\mu \in D$, $\mu = \sum_{i=1}^{n} c_i \varepsilon_{x_i}$ where $c_i \geqq 0$, $\sum_{i=1}^{n} c_i = 1$ and $x_i \in S$. Putting $p_i = c_i x_i$ and $P = \{p_1, \ldots, p_n\}$, we have $\sum|P| = 1$ and

$$\int_{S(u,\delta)} x \, \mathrm{d}\mu(x) = \sum[P \cap T(u, \delta)]\,, \quad u \in S\,.$$

Thus we can find to any $\mu \in D$ a vector $u \in S$ such that

$$\left|\int_{S(u,\delta)} x \, \mathrm{d}\mu(x)\right| > c\,.$$

By the Krein-Milman theorem (cf. [2], p. 22; see also [5], Vol. II, p. 112) there are measures $\mu_k \in D$ converging vaguely to σ. (This means that

$$\int_S f \mathrm{d}\mu_k \to \int f \, \mathrm{d}\sigma$$

for every function f continuous on S.) We know that there exist vectors $u_k \in S$ such that

$$\left|\int_{S(u_k,\delta)} x \, \mathrm{d}\mu_k(x)\right| > c\,.$$

We may suppose $u_k \to u_0$ by passing, if necessary, to a suitably chosen subsequence. If we find that

$$\int_{S(u_k,\delta)} x \, \mathrm{d}\mu_k(x) \to \int_{S(u_0,\delta)} x \, \mathrm{d}\sigma(x)\,, \quad k \to \infty\,, \tag{2}$$

the proof will be completed, because we have then by the lemma

$$C(m, \delta) = \left|\int_{S(u_0,\delta)} x \, \mathrm{d}\sigma(x)\right| \geqq c\,.$$

But (2) follows from the following lemma for $f(x) = x^j$, $j = 1, \ldots, m$.

Lemma. *Let $\delta \in \langle 0, 1)$, $u_k \in S$ and* $\lim u_k = u_0$. *Let μ_k be positive Borel measures on S converging vaguely to σ. Then*

$$\int_{S(u_k,\delta)} f \, d\mu_k \to \int_{S(u_0,\delta)} f \, d\sigma$$

whenewer f is a continuous function on S.

Proof. Recall that

$$\int_S h \, d\mu_k \to \int_S h \, d\sigma$$

provided h is a function continuous σ-a.e. on S (see e.g. [1], p. 196). Clearly

$$\left| \int_{S(u_k,\delta)} f \, d\mu_k - \int_{S(u_0,\delta)} f \, d\sigma \right| \leqq$$

$$\leqq \left| \int_{S(u_k,\delta)} f \, d\mu_k - \int_{S(u_0,\delta)} f \, d\mu_k \right| + \left| \int_{S(u_0,\delta)} f \, d\mu_k - \int_{S(u_0,\delta)} f \, d\sigma \right|$$

and the product of the function f and the characteristic function of $S(u_0, \delta)$ is a function continuous σ-a.e. on S.

Consequently, the second term tends to 0 for $k \to \infty$ and it is sufficient to prove

$$\lim_{k \to \infty} \mu_k(T_k) = 0$$

where T_k denotes the symmetric difference of the sets $S(u_k, \delta)$ and $S(u_0, \delta)$. For $\eta > 0$ put

$$Q_\eta = \{x \in S;\ \delta - \eta < x \, . \, u_0 < \delta + \eta\}$$

and observe that $T_k \subset Q_\eta$ for all k large enough. Moreover, $\sigma(Q_\eta) \to 0$ for $\eta \to 0+$. Fix $\varepsilon > 0$ and choose $\eta > 0$ such that $\sigma(Q_\eta) < \varepsilon$. The characteristic function of Q_η is continuous σ-a.e. on S and so we have $\mu_k(Q_\eta) < \varepsilon$ for all k sufficiently large. We see that there is a positive integer k_0 such that both conditions $T_k \subset Q_\eta$ and $\mu_k(Q_\eta) < \varepsilon$ are satisfied provided $k \geqq k_0$. For those k we have

$$\mu_k(T_k) \leqq \mu_k(Q_\eta) < \varepsilon$$

and the proof of the theorem is complete.

References

[1] *R. B. Ash:* Measure, Integration and Functional Analysis, Academic Press, New York, 1972.
[2] *H. Bauer:* Konvexität in topologischen Vektorräumen, Universität Hamburg, 1964.
[3] *W. W. Bledsoe:* An Inequality about Complex Numbers, Amer. Math. Monthly *77* (1970), 180—182.
[4] *N. Bourbaki:* Topologie générale, 2nd ed., Paris, 1955.
[5] *G. Choquet:* Lectures on Analysis, W. A. Benjamin, Inc., New York, 1969.
[6] *D. E. Daykin* and *A. Wilansky:* Sets of Complex Numbers, Math. Mag. *47* (1974), 228—229.
[7] *R. P. Kaufman* and *N. W. Rickert:* An Inequality concerning Measures, Bull. Amer. Math. Soc. *72* (1966), 672—676.
[8] *D. S. Mitrinović:* Analytic Inequalitis, Springer Verlag, New York, 1970.
[9] *W. Rudin:* Real and Complex Analysis, McGraw-Hill, Inc., New York, 1974.

Authors' address: 186 00 Praha 8 - Karlín, Sckolovská 83 (Matematickc-fyzikální fakulta UK).