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ON AN INTEGRAL OPERATOR IN THE SPACE OF FUNCTIONS WITH BOUNDED VARIATION, II

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In this note the considerations from [3] concerning the Fredholm-Stieltjes integral equations in the space $BV_n[0, 1]$ of all n-vector functions of bounded variation on the interval $[0, 1]$ are continued.

Let us denote by R_n the n-dimensional real space of all column n-vectors. By a star the transpose of a vector or a matrix will be denoted. For $\boldsymbol{x} = (x_1, \ldots, x_n)^* \in R_n$ we define the norm $\|\boldsymbol{x}\| = \max_{i=1,\ldots,n} |x_i|$. The set of all $n \times n$-matrices let be denoted by $L(R_n)$. For an $n \times n$-matrix $\boldsymbol{A} = (a_{ij})$, $i, j = 1, \ldots, n$ we set $\|\boldsymbol{A}\| = \max_{i=1,\ldots,n} \sum_{j=1}^{n} |a_{ij}|$. The relation for $\|\boldsymbol{A}\|$ defines the usual operator norm which corresponds to the norm in R_n given above.

We denote by $BV_n[0, 1] = BV_n$ the set of all column n-vector functions $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))^*$, $t \in [0, 1]$ for which

$$\|\boldsymbol{x}\|_{BV_n} = \|\boldsymbol{x}(0)\| + \operatorname{var}_0^1 \boldsymbol{x} < \infty$$

where $\operatorname{var}_0^1 \boldsymbol{x}$ means the usual variation of the function $\boldsymbol{x}$ on the interval $[0, 1]$. By $\|\cdot\|_{BV_n}$ a norm in BV_n is given and the linear space BV_n equipped with this norm is a Banach space. If $\boldsymbol{\varphi} \in BV_n$ then the one-sided limits $\lim_{\tau \to t+} \boldsymbol{\varphi}(\tau) = \boldsymbol{\varphi}(t+)$, $t \in [0, 1)$ and $\lim_{\tau \to t-} \boldsymbol{\varphi}(\tau) = \boldsymbol{\varphi}(t-)$, $t \in (0, 1]$ exist. Further, let NBV_n be the subspace of all elements $\boldsymbol{\varphi} \in BV_n$ for which $\boldsymbol{\varphi}(t+) = \boldsymbol{\varphi}(t)$ if $t \in (0, 1)$ and $\boldsymbol{\varphi}(0) = \boldsymbol{0}$. NBV_n is a closed subspace in BV_n and, consequently, NBV_n is also a Banach space if it is equipped with the norm of BV_n, i.e. $\|\boldsymbol{\varphi}\|_{NBV_n} = \operatorname{var}_0^1 \boldsymbol{\varphi}$.

Let us set

$$(1) \qquad \langle \boldsymbol{x}, \boldsymbol{\varphi} \rangle = \int_0^1 \boldsymbol{x}^*(t) \, d\boldsymbol{\varphi}(t) = \sum_{i=1}^{n} \int_0^1 x_i(t) \, d\varphi_i(t)$$

for $\boldsymbol{x} \in BV_n$, $\boldsymbol{\varphi} \in NBV_n$ where the integration is taken in the Perron-Stieltjes sense. The integrals occurring in this definition exist (see [4]).

The relation $\langle \cdot, \cdot \rangle$ evidently defines a bilinear form on $BV_n \times NBV_n$.

1. Lemma. *If $\varphi \in NBV_n$ and $\langle \mathbf{x}, \varphi \rangle = 0$ for every $\mathbf{x} \in BV_n$ then $\varphi = \mathbf{0}$. If $\mathbf{x} \in BV_n$ and $\langle \mathbf{x}, \varphi \rangle = 0$ for every $\varphi \in NBV_n$ then $\mathbf{x} = \mathbf{0}$.*

Proof. Assume that $\varphi \neq \mathbf{0}$. Then there exists an index $i = 1, \ldots, n$ such that either a) there is an $\alpha \in (0, 1)$ such that $\varphi_i(\alpha-) \neq \varphi_i(\alpha)$ or b) $\varphi_i(t-) = \varphi_i(t)$ for all $t \in (0, 1)$ and

1) $\varphi_i(0+) \neq 0 = \varphi_i(0)$

or

2) $\varphi_i(0+) = 0, \ \varphi_i(1) \neq \varphi_i(1-)$

or

3) φ_i is continuous on $[0, 1]$ and there exist $0 \leqq \beta < \gamma \leqq 1$ such that $\varphi_i(\beta) = \varphi_i(\gamma)$.

For the cases a), b1), b2) let us define $x_j(t) = 0, j \neq i, t \in [0, 1], x_i(t) = 0, t \in [0, 1]$, $t \neq \alpha, t \neq 0, t \neq 1$ and $x_i(\alpha) = 1, x_i(0) = 1, x_i(1) = 1$ respectively. Then we have

$$\langle \mathbf{x}, \varphi \rangle = \int_0^1 x_i(t) \, d\varphi_i(t) = x_i(\alpha) [\varphi_i(\alpha+) - \varphi_i(\alpha-)] = \varphi_i(\alpha) - \varphi_i(\alpha-) \neq 0$$

by Proposition 2,1 from [3] in the case a) and similarly $\langle \mathbf{x}, \varphi \rangle \neq 0$ in the cases b1) and b2). In the case b3) let us set $x_i(t) = 1$ for $t \in [\beta, \gamma]$, $x_i(t) = 0$ for $t \in [0, 1] \setminus [\beta, \gamma]$. By the same Proposition 2,1 from [3] it can be easily shown that in this case we have also $\langle \mathbf{x}, \varphi \rangle \neq 0$. Hence the first assertion of our lemma is proved.

For proving the second part let us assume that $\mathbf{x} \in BV_n$, $\mathbf{x} \neq \mathbf{0}$. Then for some $i = 1, \ldots, n$ either there exists an $\alpha \in (0, 1]$ such that $x_i(\alpha) \neq 0$ or $x_i(t) = 0$ for every $t \in (0, 1]$ and $x_i(0) \neq 0$. In the first case we set $\varphi_i(t) = 0$ for $t \in [0, \alpha)$, $\varphi_i(t) = 1$ for $t \in [\alpha, 1]$ and $\varphi_j(t) = 0$ for all $t \in [0, 1]$ and $j = 1, \ldots, n$, $j \neq i$. Evidently $\varphi \in NBV_n$ and by Proposition 2,1 from [3] we get $\langle \mathbf{x}, \varphi \rangle = \int_0^1 x_i(t) \, d\varphi_i(t) = x_i(\alpha) \neq 0$. In the second case we set $\varphi_i(t) = 1$, $t \in (0, 1]$, $\varphi_i(0) = 0$ and Proposition 2,1 [3] implies also in this case $\langle \mathbf{x}, \varphi \rangle = x_i(0) \neq 0$.

2. Proposition. *The pair of the spaces BV_n, NBV_n forms a dual system (BV_n, NBV_n) with respect to the bilinear form $\langle \mathbf{x}, \varphi \rangle$ given by the relation (1).*

This proposition is an immediate consequence of Lemma 1 and the definition of a dual system, see [1], § 15.

Let us denote $J = [0, 1] \times [0, 1]$ and assume that $\mathbf{K}(t, s) : J \to L(R_n)$ is an $n \times n$-matrix valued function defined on the square J such that

$$v_J(\mathbf{K}) < \infty \tag{2}$$

where $v_J(\mathbf{K})$ denotes the two-dimensional (Vitali) variation of $\mathbf{K}$ on J (see [3]). Further, we assume that

$$\operatorname{var}_0^1 \mathbf{K}(0, \cdot) < \infty . \tag{3}$$

These assumptions assure that for every fixed $t \in [0, 1]$ the variation $\operatorname{var}_0^1 \mathbf{K}(t, \cdot)$ is finite and, consequently, for any $\mathbf{x} \in BV_n$ the integral

$$(4) \qquad \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{K}\mathbf{x}$$

exists for every $t \in [0, 1]$. In this way the relation (4) defines a linear operator on the space BV_n which maps BV_n into itself (see [3], Proposition 2,3).

The function $\mathbf{K}(t, s) : J \to L(R_n)$ which determines the operator $\mathbf{K}$ by the relation (4) is called the kernel of the operator $\mathbf{K}$. In some situations the operator $\mathbf{K}$ remains unchanged if the kernel $\mathbf{K}(t, s) : J \to L(R_n)$ is altered.

3. Proposition. *Let us assume that* $\mathbf{K}(t, s)\ J\colon \to L(R_n)$ *satisfies* (2) *and* (3) *and define a new kernel* $\mathbf{K}^*(t, s)$ *by the relations* [1])

$$(5) \qquad \mathbf{K}^*(t, s) = \mathbf{K}(t, s+) - \mathbf{K}(t, 0) = \lim_{\sigma \to s+} \mathbf{K}(t, \sigma) - \mathbf{K}(t, 0) \quad \textit{if} \quad s \in (0, 1),$$

$$\mathbf{K}^*(t, 0) = 0, \quad \mathbf{K}^*(t, 1) = \mathbf{K}(t, 1) - \mathbf{K}(t, 0).$$

Then

(i) $v_J(\mathbf{K}^*) < \infty$, $\operatorname{var}_0^1 \mathbf{K}^*(0, \cdot) < \infty$, $\operatorname{var}_0^1 \mathbf{K}^*(\cdot, 0) < \infty$,

(ii) $\mathbf{K}\mathbf{x} = \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \int_0^1 \mathrm{d}_s[\mathbf{K}^*(t, s)]\, \mathbf{x}(s)$ *for every* $\mathbf{x} \in BV_n$,

(iii) *the integral* $\int_0^1 \mathbf{K}^*(t, s)\, \mathrm{d}\psi(t)$ *exists for every* $\psi \in BV_n$, $s \in [0, 1]$ *and*

$$(6) \qquad \int_0^1 \mathbf{K}^*(t, 0)\, \mathrm{d}\psi(t) = \mathbf{0},$$

$$(7) \qquad \lim_{\delta \to 0+} \int_0^1 \mathbf{K}^*(t, s + \delta)\, \mathrm{d}\psi(t) = \int_0^1 \mathbf{K}^*(t, s)\, \mathrm{d}\psi(t) \quad \textit{for any} \quad s \in (0, 1).$$

Proof. Let us assume that $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ is an arbitrary subdivision of the interval $[0, 1]$ and let us create the corresponding net-type subdivision

$$J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j], \quad i.j = 1, \ldots, k$$

of the interval J. Let us set $\mathbf{K}(t, s) = \mathbf{K}(t, 1)$ for every $t \in [0, 1]$, $s > 1$. For any given $\delta > 0$ we have

$$\sum_{i=1}^{k} \|\mathbf{K}(\alpha_i, \alpha_1 + \delta) - \mathbf{K}(\alpha_i, \alpha_0) - \mathbf{K}(\alpha_{i-1}, \alpha_1 + \delta) + \mathbf{K}(\alpha_{i-1}, \alpha_0)\| +$$
$$+ \sum_{j=2}^{k} \sum_{i=1}^{k} \|\mathbf{K}(\alpha_i, \alpha_j + \delta) - \mathbf{K}(\alpha_i, \alpha_{j-1} + \delta) - \mathbf{K}(\alpha_{i-1}, \alpha_j + \delta) +$$
$$+ \mathbf{K}(\alpha_{i-1}, \alpha_{j-1} + \delta)\| \leqq v_J(\mathbf{K}).$$

[1]) Let us mention that the limit $\mathbf{K}(t, s+)$ exists for every $t \in [0, 1]$, $s \in [0, 1)$ if $\mathbf{K}(t, s)$ satisfies (2) and (3) since for every $t \in [0, 1]$ $\mathbf{K}(t, s)$ is of bounded variation in the second variable.

Passing to the limit $\delta \to 0+$ we get by the definition (5) of $\boldsymbol{K}^*$ the inequality

$$\sum_{j=1}^{k} \sum_{i=1}^{k} \|\boldsymbol{K}^*(\alpha_i, \alpha_j) - \boldsymbol{K}^*(\alpha_i, \alpha_{j-1}) - \boldsymbol{K}^*(\alpha_{i-1}, \alpha_j) + \\ + \boldsymbol{K}^*(\alpha_{i-1}, \alpha_{j-1})\| \leqq v_J(\boldsymbol{K}).$$

This holds for every net-type subdivision J_{ij} of J and, consequently, by the definition of the Vitali variation we obtain

$$v_J(\boldsymbol{K}^*) \leqq v_J(\boldsymbol{K}) < \infty .$$

Further, we have

$$\operatorname{var}_0^1 \boldsymbol{K}^*(0, \cdot) = \operatorname{var}_0^1 (\boldsymbol{K}(0, t+) - \boldsymbol{K}(0, 0)) =$$
$$= \operatorname{var}_0^1 (\boldsymbol{K}(0, t+) - \boldsymbol{K}(0, t) + \boldsymbol{K}(0, t) - \boldsymbol{K}(0, 0)) \leqq$$
$$\leqq \operatorname{var}_0^1 (\boldsymbol{K}(0, t+) - \boldsymbol{K}(0, t)) + \operatorname{var}_0^1 \boldsymbol{K}(0, t) \leqq 2 \operatorname{var}_0^1 \boldsymbol{K}(0, \cdot) < \infty .$$

Clearly also $\operatorname{var}_0^1 \boldsymbol{K}^*(\cdot, 0) = 0$. In this way (i) is proved.

Since $\operatorname{var}_0^1 \boldsymbol{K}(t, \cdot) < \infty$ for every $t \in [0, 1]$ (see (2,14 a) in [3]) we obtain from the well known properties of functions with bounded variation that $\boldsymbol{K}(t, s+) - \boldsymbol{K}(t, s) = \boldsymbol{0}$ holds for every $s \in (0, 1)$ except an at most countable set of points in the interval $(0, 1)$. Hence for the difference $\boldsymbol{W}(t, s) = \boldsymbol{K}^*(t, s) - \boldsymbol{K}(t, s)$ we have $\boldsymbol{W}(t, s+) - \boldsymbol{W}(t, s-) = \boldsymbol{0}$ for any $s \in (0, 1)$ and it can be shown also that $\boldsymbol{W}(t, 0+) = \boldsymbol{W}(t, 0)$, $\boldsymbol{W}(t, 1) = \boldsymbol{W}(t, 1-)$. By Corollary 2,2 in [3] we obtain

$$\int_0^1 \mathrm{d}_s[\boldsymbol{W}(t, s)]\, \boldsymbol{x}(s) = \int_0^1 \mathrm{d}_s[\boldsymbol{K}^*(t, s)]\, \boldsymbol{x}(s) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{0}$$

for all $t \in [0, 1]$ and for any $\boldsymbol{x} \in BV_n$. Hence (ii) is proved.

Since by (i) we have $v_J(\boldsymbol{K}^*) < \infty$ and $\operatorname{var}_0^1 \boldsymbol{K}^*(\cdot, 0) < \infty$, it is also $\operatorname{var}_0^1 \boldsymbol{K}^*(\cdot, s) < \infty$ for every $s \in [0, 1]$ and the integral $\int_0^1 \boldsymbol{K}^*(t, s)\, \mathrm{d}\psi(t)$ exists for every $\psi \in BV_n$ (see e.g. [4]). The relation (6) is clear from $\boldsymbol{K}^*(t, 0) = 0$, $t \in [0, 1]$. For every $t \in [0, 1]$, $s \in (0, 1)$ we have $\|\boldsymbol{K}^*(t, s + \delta) - \boldsymbol{K}^*(t, s)\| \leqq \|\boldsymbol{K}^*(0, s + \delta) - \boldsymbol{K}^*(0, s)\| + \operatorname{var}_0^1 (\boldsymbol{K}^*(\cdot, s + \delta) - \boldsymbol{K}^*(\cdot, s))$. Hence $\lim_{\delta \to 0+} \sup_{t \in [0,1]} \|\boldsymbol{K}^*(t, s + \delta) - \boldsymbol{K}^*(t, s)\| = 0$ (see Remark 2,3 in [3]) and consequently

$$\lim_{\delta \to 0+} \left\| \int_0^1 (\boldsymbol{K}^*(t, s + \delta) - \boldsymbol{K}^*(t, s))\, \mathrm{d}\psi(t) \right\| \leqq$$
$$\leqq \lim_{\delta \to 0+} \sup_{t \in [0,1]} \|\boldsymbol{K}^*(t, s + \delta) - \boldsymbol{K}^*(t, s)\| \operatorname{var}_0^1 \psi = 0 .$$

This proves (iii) and also the proposition.

4. Corollary. *Let us assume that* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2) *and* (3). *Let us define*

$$\mathbf{K}'\varphi = \int_0^1 (\mathbf{K}^\#)^* (t, s) \, \mathrm{d}\varphi(t) , \quad \varphi \in BV_n$$

where $(\mathbf{K}^\#)^*$ *is the transposed matrix to* $\mathbf{K}^\#$ *defined by* (5). *Then* $\mathbf{K}'$ *is a linear operator which maps* BV_n *into* NBV_n.

Proof. For an arbitrary subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ of the interval $[0, 1]$ we have

$$\sum_{i=1}^k \left\| \int_0^1 ((\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1})) \, \mathrm{d}\varphi(t) \right\| \leqq$$

$$\leqq \sum_{i=1}^k \sup_{t \in [0,1]} \|(\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1})\| \operatorname{var}_0^1 \varphi \leqq$$

$$\leqq \operatorname{var}_0^1 \varphi \, . \, (v_J((\mathbf{K}^\#)^*) + \operatorname{var}_0^1 (\mathbf{K}^\#)^* (0, \cdot))$$

since (see (2,12) in [3]) we have

$$\sum_{i=1}^k \|(\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1})\| \leqq$$

$$\leqq \sum_{i=1}^k \|(\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1}) - (\mathbf{K}^\#)^* (0, \alpha_i) + (\mathbf{K}^\#)^* (0, \alpha_{i-1})\| +$$

$$+ \sum_{i=1}^k \|(\mathbf{K}^\#)^* (0, \alpha_i) - (\mathbf{K}^\#)^* (0, \alpha_{i-1})\| \leqq$$

$$\leqq \sum_{i=1}^k v_{[0,1] \times [\alpha_{i-1}, \alpha_i]}((\mathbf{K})^\#)^* + \operatorname{var}_0^1 ((\mathbf{K})^\#)^* \leqq v_J((\mathbf{K}^\#)^*) + \operatorname{var}_0^1 (\mathbf{K}^\#)^* (0, \cdot) \, .$$

This implies $\operatorname{var}_0^1 \int_0^1 (\mathbf{K}^\#)^* (t, s) \, \mathrm{d}\varphi(t) < \infty$ because $(\mathbf{K}^\#)^*$ evidently satisfies (i) from Proposition 3. From (iii) of the same proposition and from the definition of NBV_n we obtain that for every $\varphi \in BV_n$ the integral $\int_0^1 (\mathbf{K}^\#)^* (t, s) \, \mathrm{d}\varphi(t)$ as a function of the variable s belongs to NBV_n.

From the results of [3], the following result can be easily deduced:

5. Theorem. *If* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2) *and* (3) *then the relation*

$$\mathbf{K}\mathbf{x} = \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \mathbf{x}(s) , \quad t \in [0, 1] , \quad \mathbf{x} \in BV_n \tag{8}$$

defines a completely continuous operator on BV_n.

The relation

$$\mathbf{K}'\varphi = \int_0^1 (\mathbf{K}^\#)^* (t, s) \, \mathrm{d}\varphi(t) , \quad s \in [0, 1] , \quad \varphi \in NBV_n \tag{9}$$

where $\mathbf{K}^\#$ *is given by* (5) *defines a completely continuous operator on* NBV_n.

Moreover, if $\langle\cdot,\cdot\rangle$ *is the bilinear form on* $BV_n \times NBV_n$ *given by* (1) *then*

$$\langle \mathbf{K}x, \varphi\rangle = \langle x, \mathbf{K}'\varphi\rangle \tag{10}$$

for every $x \in BV_n$ *and* $\varphi \in NBV_n$.

Proof. The complete continuity of $\mathbf{K}$ given by (8) is proved in Theorem 3,1 from [3]. Theorem 3,2 from [3] states that the operator

$$\mathbf{K}'\psi = \int_0^1 (\mathbf{K}^\#)^* (t, s)\, \mathrm{d}\psi(t)\,, \quad \psi \in BV_n$$

is completely continuous on BV_n. Since NBV_n is a closed subspace of BV_n the restriction of this operator onto NBV_n (i.e. the operator $\mathbf{K}'$ given by (9)) is also completely continuous and maps NBV_n into itself (cf. Corollary 4). Hence the second statement is also valid.

By (ii) from Proposition 3 we have $\mathbf{K}x = \mathbf{K}^\# x$, where $\mathbf{K}^\# x = \int_0^1 \mathrm{d}_s[\mathbf{K}^\#(t, s)]\, x(s)$, $x \in BV_n$ and $\mathbf{K}^\#$ is given by (5). Hence $\langle \mathbf{K}x, \varphi\rangle = \langle \mathbf{K}^\# x, \varphi\rangle$ for every $x \in BV_n$, $\varphi \in NBV_n$. Using Lemma 2,2 from [3] we interchange the order of integrations and by an easy computation we obtain the equality

$$\langle \mathbf{K}^\# x, \varphi\rangle = \langle x, \mathbf{K}'\varphi\rangle$$

where $\mathbf{K}'$ is given by (9) and $x \in BV_n$, $\varphi \in NBV_n$ are arbitrary, i.e. (10) holds for all $x \in BV_n$, $\varphi \in NBV_n$.

In the subsequent considerations we use the usual notation: for a given linear operator $\mathbf{A}$ acting on a Banach space X we set

$$N(\mathbf{A}) = \{x \in X;\ \mathbf{A}x = \mathbf{0}\}$$

(the null space of $\mathbf{A}$) and

$$R(\mathbf{A}) = \{y \in X;\ y = \mathbf{A}x,\ x \in X\}$$

(the range of $\mathbf{A}$). We define the index $\operatorname{ind} \mathbf{A}$ of the operator $\mathbf{A}$ by the relation

$$\operatorname{ind} \mathbf{A} = \dim N(\mathbf{A}) - \operatorname{codim} R(\mathbf{A})$$

if the difference on the right hand side of this equality is defined.

Using this notation we state the following

6. Theorem. *If* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2) *and* (3) *then*

$$\operatorname{ind} (\mathbf{I} - \mathbf{K}) = \operatorname{ind} (\mathbf{I} - \mathbf{K}') = 0 \tag{11}$$

where $\mathbf{I}$ *stands for the identity operator in the corresponding Banach space and the operators* $\mathbf{K}, \mathbf{K}'$ *are given by* (8), (9) *respectively.*

Moreover, we have

$$\dim N(\boldsymbol{I} - \boldsymbol{K}) = \dim N(\boldsymbol{I} - \boldsymbol{K}') \tag{12}$$

and the Fredholm-Stieltjes integral equation

$$\boldsymbol{x}(t) = \int_0^1 \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) + \boldsymbol{f}(t)\,, \quad t \in [0, 1]\,, \quad \boldsymbol{f} \in BV_n \tag{13}$$

has a solution in BV_n if and only if

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = 0$$

for all solutions $\boldsymbol{\varphi} \in NBV_n$ of the equation

$$\boldsymbol{\varphi}(s) = \int_0^1 (\boldsymbol{K}^{\#})^* (t, s)\, \mathrm{d}\boldsymbol{\varphi}(t)\,, \quad s \in [0, 1]\,. \tag{14}$$

Similarly, the equation

$$\boldsymbol{\varphi}(s) = \int_0^1 (\boldsymbol{K}^{\#})^* (t, s)\, \mathrm{d}\boldsymbol{\varphi}(t) + \boldsymbol{\psi}(s)\,, \quad s \in [0, 1]\,, \quad \boldsymbol{\psi} \in NBV_n \tag{15}$$

has a solution in NBV_n if and only if

$$\langle \boldsymbol{x}, \boldsymbol{\psi} \rangle = 0$$

for every solution $\boldsymbol{x} \in BV_n$ of the homogeneous Fredholm-Stieltjes integral equation

$$\boldsymbol{x}(t) = \int_0^1 \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s)\,, \quad t \in [0, 1]\,. \tag{16}$$

Proof. The equality (11) follows immediately from the complete continuity of the operators $\boldsymbol{K}$, $\boldsymbol{K}'$ stated in Theorem 5 (see e.g. [1], Theorem 40,1).

Since (BV_n, NBV_n) is a dual system with respect to the bilinear form (1) and (10) is satisfied we have

$$\langle \boldsymbol{x} - \boldsymbol{K}\boldsymbol{x}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{x}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{K}\boldsymbol{x}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{x}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{x}, \boldsymbol{K}'\boldsymbol{\varphi} \rangle = \langle \boldsymbol{x}, \boldsymbol{\varphi} - \boldsymbol{K}'\boldsymbol{\varphi} \rangle\,.$$

All the assumptions of Satz 40.2 from [1] are satisfied and, consequently, the result follows immediately from this Satz.

Remark. Theorem 6 is essentially a comprehensive version of the results from [3]. In [3], the quotient space BV_n/S_n was used instead of NBV_n. The version of the Fredholm theory for the equation (13) and the corresponding conjugate equation (15) given in Theorem 6 seems to be more natural than the version given in [3].

For the linear operator $\boldsymbol{K} : BV_n \to BV_n$ defined by (8) we have $\operatorname{ind}(\boldsymbol{I} - \boldsymbol{K}) = 0$ and consequently, if $\dim N(\boldsymbol{I} - \boldsymbol{K}) = 0$, i.e. if $N(\boldsymbol{I} - \boldsymbol{K}) = \boldsymbol{0}$ then $BV_n/R(\boldsymbol{I} - \boldsymbol{K}) = \boldsymbol{0}$

and also $R(\mathbf{I} - \mathbf{K}) = BV_n$. In this situation the Bounded Inverse Theorem applies, i.e. the inverse operator $(\mathbf{I} - \mathbf{K})^{-1}$ exists and is bounded (see [2]). This yields the following

7. Lemma. *Let us assume that $\mathbf{K} : J \to L(R_n)$ satisfies (2), (3) and that $N(\mathbf{I} - \mathbf{K}) = \mathbf{0}$, i.e. the homogeneous integral equation (16) has only the trivial solution $\mathbf{x} = \mathbf{0}$ in BV_n. Then there exists a constant $C \geqq 0$ such that for every $\mathbf{f} \in BV_n$ the inequality*

$$\|\mathbf{x}\|_{BV_n} \leqq C\|\mathbf{f}\|_{BV_n}$$

holds for the unique solution $\mathbf{x} \in BV_n$ of the nonhomogeneous equation (13). (Let us mention that $C = \|(\mathbf{I} - \mathbf{K})^{-1}\|$.)

Remark. As was mentioned above, when the assumptions of Lemma 7 are satisfied the inverse operator $(\mathbf{I} - \mathbf{K})^{-1}$ exists. In the sequel we prove that this inverse operator has the form $\mathbf{I} + \boldsymbol{\Gamma}$ where $\boldsymbol{\Gamma} : BV_n \to BV_n$ is a linear integral operator of the same type as the operator $\mathbf{K}$ given by (8).

8. Theorem. *Let us assume that $\mathbf{K} : J \to L(R_n)$ satisfies (2), (3). If the homogeneous equation (16) has only the trivial solution $\mathbf{x} = \mathbf{0} \in BV_n$ then there exists a uniquely determined $n \times n$-matrix valued function $\boldsymbol{\Gamma} : J \to L(R_n)$ such that*

$$\boldsymbol{\Gamma}(t, s) = \mathbf{K}(t, s) - \mathbf{K}(t, 0) + \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)]\, \boldsymbol{\Gamma}(r, s) \tag{17}$$

for all $t, s \in [0, 1]$,

$$\operatorname{var}_0^1 \boldsymbol{\Gamma}(0, \cdot) < \infty\,, \tag{18}$$

$$\boldsymbol{\Gamma}(t, 0) = \mathbf{0} \quad \text{for every} \quad t \in [0, 1]\,, \tag{19}$$

$$v_J(\boldsymbol{\Gamma}) < \infty \tag{20}$$

and for any $\mathbf{f} \in BV_n$ the unique solution $\mathbf{x} \in BV_n$ of (13) is given by the resolvent formula

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \mathbf{f}(s)\,. \tag{21}$$

Proof. Let us denote by $\mathbf{y}_l$ the l-th column of the $n \times n$-matrix $\mathbf{y} \in L(R_n)$. Then the relation (17) can be written in the form

$$\boldsymbol{\Gamma}_l(t, s) = \mathbf{K}_l(t, s) = \mathbf{K}_l(t, 0) + \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)]\, \boldsymbol{\Gamma}_l(r, s)\,, \quad l = 1, 2, \ldots, n\,. \tag{21}$$

We have evidently

$$\operatorname{var}_0^1 (\mathbf{K}(\cdot, s) - \mathbf{K}(\cdot, 0)) \leqq v_J(\mathbf{K}) < \infty$$

for every $s \in [0, 1]$. Hence for any fixed $s \in [0, 1]$ and $l = 1, \ldots, n$ we have $\operatorname{var}_0^1 (\boldsymbol{K}_l(\cdot, s) - \boldsymbol{K}_l(\cdot, 0)) < \infty$. This implies by the assumptions and by Theorem 6 that for any $l = 1, \ldots, n$, $s \in [0, 1]$ the relation (21) determines uniquely the n-vector $\boldsymbol{\Gamma}_l(t, s)$ and, consequently, also the $n \times n$-matrix valued function $\boldsymbol{\Gamma}(t, s)$ is uniquely determined by (17) for every fixed $s \in [0, 1]$. Moreover, by Lemma 7 we have

$$\|\boldsymbol{\Gamma}_l(\cdot, 0)\|_{BV_n} \leqq C\|\boldsymbol{K}_l(\cdot, 0) - \boldsymbol{K}_l(\cdot, 0)\| = 0 .$$

Hence $\boldsymbol{\Gamma}(t, 0) = \boldsymbol{0}$ for every $t \in [0, 1]$. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ be an arbitrary subdivision of the interval $[0, 1]$. For $\boldsymbol{\Gamma}(t, s) : J \to L(R_n)$ satisfying (17) we have

$$\boldsymbol{\Gamma}(t, \alpha_j) - \boldsymbol{\Gamma}(t, \alpha_{j-1}) =$$
$$= \boldsymbol{K}(t, \alpha_j) - \boldsymbol{K}(t, \alpha_{j-1}) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}(t, r)] (\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1}))$$

for $t \in [0, 1]$, $j = 1, 2, \ldots, k$. Using Lemma 7 and the obvious fact that $\operatorname{var}_0^1 (\boldsymbol{K}(\cdot, \alpha_j) - \boldsymbol{K}(\cdot, \alpha_{j-1})) < \infty$ we get

$$\text{(22)} \qquad \|\boldsymbol{\Gamma}(0, \alpha_j) - \boldsymbol{\Gamma}(0, \alpha_{j-1})\| + \operatorname{var}_0^1 (\boldsymbol{\Gamma}(\cdot, \alpha_j) - \boldsymbol{\Gamma}(\cdot, \alpha_{j-1})) \leqq$$
$$\leqq C[\|\boldsymbol{K}(0, \alpha_j) - \boldsymbol{K}(0, \alpha_{j-1})\| + \operatorname{var}_0^1 (\boldsymbol{K}(\cdot, \alpha_j) - \boldsymbol{K}(\cdot, \alpha_{j-1}))]$$

where $C \geqq 0$ is a constat. Hence

$$\sum_{j=1}^{k} \|\boldsymbol{\Gamma}(0, \alpha_j) - \boldsymbol{\Gamma}(0, \alpha_{j-1})\| \leqq C(\operatorname{var}_0^1 \boldsymbol{K}(0, \cdot) + v_J(\boldsymbol{K})) .$$

Since this inequality holds for any subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ we obtain (18). The inequality (20) can be shown as follows. For the subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ we define the net-type subdivision

$$J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j]$$

$i, j = 1, \ldots, k$ of the interval J. For $\boldsymbol{\Gamma} : J \to L(R_n)$ defined by (17) we have ($i, j = 1, \ldots, k$)

$$m_{\boldsymbol{\Gamma}}(J_{ij}) = m_{\boldsymbol{K}}(J_{ij}) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}(\alpha_i, r) - \boldsymbol{K}(\alpha_{i-1}, r)] (\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1}))$$

where $m_{\boldsymbol{\Gamma}}(J_{ij}) = \boldsymbol{\Gamma}(\alpha_i, \alpha_j) - \boldsymbol{\Gamma}(\alpha_i, \alpha_{j-1}) - \boldsymbol{\Gamma}(\alpha_{i-1}, \alpha_j) + \boldsymbol{\Gamma}(\alpha_{i-1}, \alpha_{j-1})$ and similarly for $m_{\boldsymbol{K}}(J_{ij})$. Usual estimates for the Perron-Stieltjes integral lead to the inequality (see [3], [4])

$$\|m_{\boldsymbol{\Gamma}}(J_{ij})\| \leqq \|m_{\boldsymbol{K}}(J_{ij})\| +$$
$$+ \sup_{r \in [0,1]} \|\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1})\| \operatorname{var}_0^1 (\boldsymbol{K}(\alpha_i, \cdot) - \boldsymbol{K}(\alpha_{i-1}, \cdot))$$

for every $i, j = 1, 2, \ldots, k$ and also to the inequality

$$\sum_{i,j=1}^{k} \|m_\Gamma(J_{ij})\| \leqq v_J(\mathbf{K}) +$$
$$+ \sum_{i=1}^{k} \operatorname{var}_0^1 (\mathbf{K}(\alpha_i, \cdot) - \mathbf{K}(\alpha_{i-1}, \cdot)) \, . \sum_{j=1}^{k} \sup_{r \in [0,1]} \|\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1})\| \, .$$

Since

$$\|\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1})\| \leqq$$
$$\leqq \|\boldsymbol{\Gamma}(0, \alpha_j) - \boldsymbol{\Gamma}(0, \alpha_{j-1})\| + \operatorname{var}_0^1 (\boldsymbol{\Gamma}(\cdot, \alpha_{j-1})) - \boldsymbol{\Gamma}(\cdot, \alpha_{j-1}))$$

for every $r \in [0, 1]$, we have by (22)

$$\sum_{i,j=1}^{k} \|m_\Gamma(J_{ij})\| \leqq v_J(\mathbf{K}) + v_J(\mathbf{K}) \, C[\sum_{j=1}^{k} \|\mathbf{K}(0, \alpha_j) - \mathbf{K}(0, \alpha_{j-1})\| +$$
$$+ \operatorname{var}_0^1 (\mathbf{K}(\cdot, \alpha_j) - \mathbf{K}(\cdot, \alpha_{j-1}))] \leqq v_J(\mathbf{K}) \, [1 + C(\operatorname{var}_0^1 \mathbf{K}(0, \cdot) + v_J(\mathbf{K}))] < \infty \, .$$

Since the subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ of $[0, 1]$ is arbitrary we obtain by the definition of the Vitali variation v_J the inequality (20).[2])

It remains to show that by the formula (21) the unique solution of the equation (13) is given. The integral $\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)] \, \mathbf{f}(s)$ exists for every $\mathbf{f} \in BV_n$ and $t \in [0, 1]$ since (18) and (20) are satisfied (see Proposition 2,3 in [3]). Let us put $\mathbf{x}(t)$ from (21) into the expression $\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \mathbf{x}(s)$. We obtain

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \mathbf{x}(s) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)] \, \mathbf{f}(s) -$$
$$- \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \left(\mathbf{f}(r) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(r, s)] \, \mathbf{f}(s)\right) =$$
$$= \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s) - \mathbf{K}(t, s)] \, \mathbf{f}(s) - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \left(\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(r, s)] \, \mathbf{f}(s)\right).$$

Interchanging the order of integrations in the last integral by Lemma 2,2 in [3] and using (17) we obtain

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \mathbf{x}(s) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s \left\{\boldsymbol{\Gamma}(t, s) - \mathbf{K}(t, s) -\right.$$
$$\left. - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \, \boldsymbol{\Gamma}(r, s)\right\} \mathbf{f}(s) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s \left\{\boldsymbol{\Gamma}(t, s) - \mathbf{K}(t, s) +\right.$$
$$\left. + \mathbf{K}(t, 0) - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \, \boldsymbol{\Gamma}(r, s)\right\} \mathbf{f}(s) = \mathbf{f}(t) \, ,$$

[2]) The fact that only net-type subdivisions of J are taken into account is not essential since evidently every subdivision of J can be refined to a net-type one.

i.e. $\boldsymbol{x}(t)$ given by (21) is really the unique solution of the equation (13) and the theorem is completely proved.

Let us now consider the case when $\boldsymbol{K} : J \to L(R_n)$ satisfies (2) and (3) but the assumption $N(\boldsymbol{I} - \boldsymbol{K}) = \{\boldsymbol{0}\}$ is not satisfied. By Theorem 6 we know that $\dim N(\boldsymbol{I} - \boldsymbol{K}) = \operatorname{codim} R(\boldsymbol{I} - \boldsymbol{K}) = \dim N(\boldsymbol{I} - \boldsymbol{K}') = \operatorname{codim} R(\boldsymbol{I} - \boldsymbol{K}') = r$ where $r > 0$ is an integer. In this case $R(\boldsymbol{I} - \boldsymbol{K}) \neq BV_n$ and the inverse operator $(\boldsymbol{I} - \boldsymbol{K})^{-1}$ cannot be defined on the whole space BV_n. The equation (13) has solutions only for $\boldsymbol{f} \in R(\boldsymbol{I} - \boldsymbol{K})$. Our aim is to show that in this situation there exists also an operator $\boldsymbol{\Gamma}^0$ acting on BV_n such that if $\boldsymbol{f} \in R(\boldsymbol{I} - \boldsymbol{K})$ then $\boldsymbol{f} + \boldsymbol{\Gamma}^0\boldsymbol{f}$ is a solution of the equation (13) and, moreover, that the operator $\boldsymbol{\Gamma}^0$ is an integral operator of the same type as $\boldsymbol{K}$. We prove this fact following a general scheme known from functional analysis.

In the sequel we assume that $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^r \in BV_n$ is a given basis of the r – dimensional null space $N(\boldsymbol{I} - \boldsymbol{K})$ (linearly independent solutions of the homogeneous integral equation (16)) and $\varphi^1, \ldots, \varphi^r \in NBV_n$ is a given basis of $N(\boldsymbol{I} - \boldsymbol{K}')$ (linearly independent solutions of the equation (14)). It is known (see e.g. [1], Satz 15.1) that there exist linearly independent elements $\boldsymbol{\eta}^i$ in NBV_n and $\boldsymbol{y}^i$ in BV_n, $i = 1, \ldots, r$ such that

$$\langle \boldsymbol{x}^j, \boldsymbol{\eta}^i \rangle = \delta_{ij}\,, \quad i, j = 1, \ldots, r\,,$$

$$\langle \boldsymbol{y}^j, \varphi^i \rangle = \delta_{ij}\,, \quad i, j = 1, \ldots, r$$

($\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = 1$).

Let us define the projections

$$\boldsymbol{P}\boldsymbol{x} = \sum_{i=1}^{r} \langle \boldsymbol{x}, \boldsymbol{\eta}^i \rangle\, \boldsymbol{x}^i\,, \quad \boldsymbol{x} \in BV_n\,,$$

$$\boldsymbol{Q}\boldsymbol{x} = \sum_{i=1}^{r} \langle \boldsymbol{x}, \varphi^i \rangle\, \boldsymbol{y}^i\,, \quad \boldsymbol{x} \in BV_n\,.$$

It is easy to show that $\boldsymbol{P}, \boldsymbol{Q}$ are bounded projection operators. Further, evidently $R(\boldsymbol{P}) = N(\boldsymbol{I} - \boldsymbol{K})$ and by Theorem 6 also

$$N(\boldsymbol{Q}) = \{\boldsymbol{x} \in X;\ \langle \boldsymbol{x}, \varphi \rangle = 0 \text{ for every } \varphi \in N(\boldsymbol{I} - \boldsymbol{K}')\} = R(\boldsymbol{I} - \boldsymbol{K})\,.$$

The projections $\boldsymbol{P}, \boldsymbol{Q}$ generate decompositions of the Banach space BV_n into direct sums

$$BV_n = R(\boldsymbol{P}) \oplus N(\boldsymbol{P}) = N(\boldsymbol{I} - \boldsymbol{K}) \oplus N(\boldsymbol{P})\,, \tag{23}$$

$$BV_n = R(\boldsymbol{Q}) \oplus N(\boldsymbol{Q}) = R(\boldsymbol{Q}) \oplus R(\boldsymbol{I} - \boldsymbol{K})\,. \tag{24}$$

Let us now define the linear operator

$$(25)\qquad \mathbf{L}x = \sum_{i=1}^{r} \langle x, \eta^i \rangle y^i = \sum_{i=1}^{r} y^i(t) \int_0^1 x^*(s)\, d\eta^i(s) = $$

$$= \int_0^1 d_s[\sum_{i=1}^{r} y^i(t)\, \eta^{i*}(s)]\, x(s)\,.$$

$\mathbf{L}$ is evidently a bounded finite-dimensional (and consequently completely continuous) operator on BV_n and

$$N(\mathbf{L}) = \{x \in BV_n;\ \langle x, \eta^i \rangle = 0 \text{ for every } i = 1, \ldots, r\} = N(\mathbf{P})\,,$$

$$R(\mathbf{L}) \subset R(\mathbf{Q})\,.$$

Let us set

$$(26)\qquad \mathbf{K}^\circ = \mathbf{K} + \mathbf{L}$$

where $\mathbf{K}$ is the operator corresponding to the kernel $K : J \to L(R_n)$ via the relation (4). $\mathbf{K}^\circ$ is evidently a completely continuous operator on BV_n and $\operatorname{ind}(\mathbf{I} - \mathbf{K}^\circ) = 0$. Let us assume that $x \in N(\mathbf{I} - \mathbf{K}^\circ)$. Then

$$(\mathbf{I} - \mathbf{K}^\circ)\, x = (\mathbf{I} - \mathbf{K})\, x - \mathbf{L}x = 0$$

and by (24) necessarily $(\mathbf{I} - \mathbf{K})\, x = 0$ and $\mathbf{L}x = 0$ because $R(\mathbf{L}) \subset R(\mathbf{Q})$. Hence $x \in N(\mathbf{I} - \mathbf{K}) \cap N(\mathbf{L}) = N(\mathbf{I} - \mathbf{K}) \cap N(\mathbf{P})$ and, consequently, by (23) we obtain $x = 0$. This yields $N(\mathbf{I} - \mathbf{K}^\circ) = \{0\}$ and $\dim N(\mathbf{I} - \mathbf{K}^\circ) = 0$. Using the complete continuity of the operator $\mathbf{K}^\circ$ we obtain $R(\mathbf{I} - \mathbf{K}^\circ) = BV_n$ and by the Bounded Inverse Theorem also the existence of a bounded inverse operator $(\mathbf{I} - \mathbf{K}^\circ)^{-1}$. Since $x^i \in N(\mathbf{I} - \mathbf{K})$ we have $(\mathbf{I} - \mathbf{K})\, \mathbf{P}x = \sum_{i=1}^{r} \langle x, \eta^i \rangle (\mathbf{I} - \mathbf{K})\, x^i = 0$ for all $x \in BV_n$ and

$$(27)\qquad (\mathbf{I} - \mathbf{K})\, x = (\mathbf{I} - \mathbf{K})(\mathbf{I} - \mathbf{P})\, x\,.$$

Since $\mathbf{P}$ is a projection we have $R(\mathbf{I} - \mathbf{P}) = N(\mathbf{P}) = N(\mathbf{L})$. Hence $\mathbf{L}(\mathbf{I} - \mathbf{P})\, x = 0$ for every $x \in BV_n$ and also

$$(\mathbf{I} - \mathbf{K})\, x = (\mathbf{I} - \mathbf{K})(\mathbf{I} - \mathbf{P})\, x - \mathbf{L}(\mathbf{I} - \mathbf{P})\, x = (\mathbf{I} - \mathbf{K}^\circ)(\mathbf{I} - \mathbf{P})\, x$$

for every $x \in BV_n$. Multiplying from the left by $(\mathbf{I} - \mathbf{K})(\mathbf{I} - \mathbf{K}^\circ)^{-1}$ and using (27) we obtain further

$$(28)\qquad (\mathbf{I} - \mathbf{K})(\mathbf{I} - \mathbf{K}^\circ)^{-1}(\mathbf{I} - \mathbf{K})\, x = (\mathbf{I} - \mathbf{K})(\mathbf{I} - \mathbf{K}^\circ)^{-1}(\mathbf{I} - \mathbf{K}^\circ)(\mathbf{I} - \mathbf{P})\, x =$$

$$= (\mathbf{I} - \mathbf{K})(\mathbf{I} - \mathbf{P})\, x = (\mathbf{I} - \mathbf{K})\, x$$

for every $\boldsymbol{x} \in BV_n$. Hence

$$(\boldsymbol{I} - \boldsymbol{K})(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} \boldsymbol{f} = \boldsymbol{f}$$

for every $\boldsymbol{f} \in R(\boldsymbol{I} - \boldsymbol{K})$, i.e. $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} \boldsymbol{f}$ is a solution of the equation $(\boldsymbol{I} - \boldsymbol{K}) \boldsymbol{x} = \boldsymbol{f}$. It is easy to see that if we set

$$\boldsymbol{K}^\circ(t, s) = \boldsymbol{K}(t, s) + \sum_{i=1}^{r} \boldsymbol{y}^i(t) \boldsymbol{\eta}^{i*}(s)$$

then for the operator $\boldsymbol{K}^\circ$ given by (26) we have

$$\boldsymbol{K}^\circ \boldsymbol{x} = \int_0^1 \mathrm{d}_s[\boldsymbol{K}^\circ(t, s)] \, \boldsymbol{x}(s)$$

and $v_J(\boldsymbol{K}^\circ) < v_J(\boldsymbol{K}) + \sum_{i=1}^{r} \operatorname{var}_0^1 \boldsymbol{y}^i \, . \operatorname{var}_0^1 \boldsymbol{\eta}^i < \infty$, $\operatorname{var}_0^1 \boldsymbol{K}^\circ(0, \cdot) \leqq \operatorname{var}_0^1 \boldsymbol{K}(0, \cdot) + \sum_{i=1}^{r} \|\boldsymbol{y}^i(0)\| \operatorname{var}_0^1 \boldsymbol{\eta}^i < \infty$. Hence the kernel $\boldsymbol{K}^\circ(t, s) : J \to L(R_n)$ satisfies all assumptions of Theorem 8 and, consequently, by this theorem there exists a $\boldsymbol{\Gamma}^\circ(t, s) : J \to L(R_n)$ which satisfies the equation

$$\boldsymbol{\Gamma}^\circ(t, s) = \boldsymbol{K}^\circ(t, s) - \boldsymbol{K}^\circ(t, 0) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}^\circ(t, r)] \, \boldsymbol{\Gamma}^\circ(r, s) \,, \quad t, s \in [0, 1] \tag{29}$$

and $\boldsymbol{\Gamma}^\circ(t, 0) = \boldsymbol{0}$ for every $t \in [0, 1]$, $\operatorname{var}_0^1 \boldsymbol{\Gamma}^\circ(0, \cdot) < \infty$, $v_J(\boldsymbol{\Gamma}^\circ) < \infty$. Moreover, for every $\boldsymbol{f} \in BV_n$ the unique solution $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} \boldsymbol{f}$ of the equation

$$\boldsymbol{x} - \boldsymbol{K}^\circ \boldsymbol{x} = \boldsymbol{f}$$

is given by the relation

$$\boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^0(t, s)] \, \boldsymbol{f}(s) \,,$$

i.e. $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} = \boldsymbol{I} + \boldsymbol{\Gamma}^\circ$ where $\boldsymbol{\Gamma}^\circ \boldsymbol{x} = \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)] \, \boldsymbol{x}(s)$ for $\boldsymbol{x} \in BV_n$.

Let us now summarize the above results.

9. Theorem. *Let $\boldsymbol{K} : J \to L(R_n)$ satisfy (2) and (3). Then there exists an $n \times n$-matrix valued function $\boldsymbol{\Gamma}^\circ(t, s) : J \to L(R_n)$ such that* $\operatorname{var}_0^1 \boldsymbol{\Gamma}^\circ(0, \cdot) < \infty$, $v_J(\boldsymbol{\Gamma}^\circ) < \infty$, $\boldsymbol{\Gamma}^\circ(t, 0) = \boldsymbol{0}$ *for all* $t \in [0, 1]$, $\boldsymbol{\Gamma}^\circ(t, s)$ *satisfies* (29) *for all* $t, s \in [0, 1]$ *and the relation*

$$\boldsymbol{x}(t) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)] \, \boldsymbol{f}(s) \,, \quad t \in [0, 1] \tag{30}$$

defines a solution of the Fredholm-Stieltjes integral equation (13) *provided* $\boldsymbol{f} \in BV_n$ *belongs to* $R(\boldsymbol{I} - \boldsymbol{K})$ (*i.e. when the equation* (13) *has a solution for the given* $\boldsymbol{f} \in BV_n$).

If $\mathbf{f} \in R(\mathbf{I} - \mathbf{K})$ then the general form of solutions of the equation (13) *is given by*

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)]\, \mathrm{f}(s) + \sum_{i=1}^{r} \alpha_i\, \mathbf{x}^i(t)$$

where $\mathbf{x}^i \in BV_n$, $i = 1, \ldots, r$ are all the linearly independent solutions of the homogeneous Fredholm-Stieltjes integral equation 16) *and $\alpha_1, \ldots, \alpha_r$ are arbitrary real constants.*

Remark. The last part of the theorem follows from the well-known properties of linear equations. The theorem includes also the statement of the previous Theorem 8 and gives in the general situation the desired "solving kernel result". Naturally, for the case $\dim N(\mathbf{I} - \mathbf{K}) > 0$ the construction of the solving kernel $\boldsymbol{\Gamma}^\circ$ depends upon the knowledge of the structure of the null-spaces of the operators $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^\circ$.

References

[1] *Heuser H.:* Funktionalanalysis. B. G. Teubner, Stuttgart, 1975.

[2] *Schechter M.:* Principles of functional analysis. Academic Press, New York, London, 1973.

[3] *Schwabik Š.:* On an integral operator in the space of functions with bounded variation. Časopis pěst. mat., *97*, 1972, 297—330.

[4] *Schwabik Š.:* On the relation between Young's and Kurzweil's concept of Stieltjes integral. Časopis pěst. mat., *98*, 1973, 237—251.

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