Časopis pro pěstování matematiky

Miroslav Sova

On inversion of Laplace transform. I.

Časopis pro pěstování matematiky, Vol. 102 (1977), No. 2, 166--172

Persistent URL: `http://dml.cz/dmlcz/117954`

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON INVERSION OF LAPLACE TRANSFORM (I)

MIROSLAV SOVA, Praha

(Received January 21, 1976)

The aim of this note is to show how a complex inversion theorem may be deduced from the general Post-Widder inversion theorem.

1. We denote by R and C respectively the real and complex number fields and by R^+ the set of all positive numbers. Further, if M_1, M_2 are two arbitrary sets, then $M_1 \to M_2$ will denote the set of all mappings of the set M_1 into the set M_2.

2. Lemma. *For every $\alpha \geqq 0$ and $r \in \{1, 2, \ldots\}$ such that $r > \alpha$, we have*

$$\left(\frac{r}{r-\alpha}\right)^r \leqq e^{(r/r-\alpha)\alpha} = e^{\alpha} e^{\alpha^2/(r-\alpha)} .$$

Proof. Under our assumptions we have

$$\log\left(\frac{r}{r-\alpha}\right)^r = \log\left(\frac{1}{1-\frac{\alpha}{r}}\right)^r = -r\log\left(1-\frac{\alpha}{r}\right) = r\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha}{r}\right)k \leqq$$

$$\leqq r\sum_{k=1}^{\infty}\left(\frac{\alpha}{r}\right)^k = r\frac{\alpha}{r}\frac{1}{1-\frac{\alpha}{r}} = \alpha\frac{r}{r-\alpha} = \alpha + \frac{\alpha^2}{r-\alpha}$$

and our result follows.

3. Lemma. *For every $z \in C$, $(1 + z/q)^q \to e^z (q \to \infty)$.*

Proof. According to the binomial theorem, we can write

$$(1)\qquad \left(1+\frac{z}{q}\right)^q=\sum_{k=0}^{q}\binom{q}{k}\frac{z^k}{q^k}=1+z+\sum_{k=2}^{q}\frac{q(q-1)\dots(q-k+1)}{k!}\frac{z^k}{q^k}=$$

$$=1+z+\sum_{k=2}^{q}\frac{q(q-1)\dots(q-k+1)}{q^k}\frac{z^k}{k!}=$$

$$=1+z+\sum_{k=2}^{q}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z^k}{k!}$$

for every $q\in\{2,3,\dots\}$.

Let now $z\in C$ and $\varepsilon>0$. Then there exists a $k_0\in\{2,3,\dots\}$ such that

$$(2)\qquad \sum_{k=k_0+1}^{\infty}\frac{|z|^k}{k!}\leqq\frac{\varepsilon}{3}.$$

It follows from (2) that

$$(3)\qquad \left|e^z-\sum_{k=0}^{k_0}\frac{z^k}{k!}\right|\leqq\frac{\varepsilon}{3}.$$

Further by (1) and (2),

$$(4)\qquad \left|\left(1+\frac{z}{q}\right)^q-\left[1+z+\sum_{k=2}^{k_0}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right]\right|=$$

$$=\left|\sum_{k=k_0+1}^{q}\left(1-\frac{1}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right|\leqq\sum_{k=k_0+1}^{k}\frac{|z|^k}{k!}\leqq\frac{\varepsilon}{3}$$

for every $q\geqq k_0+1$.

Finally it is easy to see that there exists a $q_0\in\{k_0+1,k_0+2,\dots\}$ such that

$$(5)\qquad \left|\sum_{k=0}^{k_0}\frac{z^k}{k!}-\left[1+z+\sum_{k=2}^{k_0}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right]\right|\leqq\frac{\varepsilon}{3}$$

for every $q\geqq q_0$.

Now we have immediately from (3), (4) and (5)

$$\left|e^z-\left(1-\frac{z}{q}\right)^q\right|\leqq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

for every $q\geqq q_0$ and this gives the assertion.

4. Theorem (Post-Widder). *Let $f\in R^+\to E$ and let M,ω be two nonnegative constants. If*

(α) *the function f is measurable over R^+,*

(β) $|f(t)| \leqq Me^{\omega t}$ *for almost all $t \in R^+$, then*

$$\text{(a)} \qquad \int_0^\infty e^{-((p+1)/t)\tau}\tau^p f(\tau)\,\mathrm{d}\tau \textit{ exists for every } t \in R^+ \textit{ and } p + 1 > \omega t,$$

$$\text{(b)} \qquad \left|\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1} \int_0^\infty e^{-(p+1)/\tau}\tau^p f(\tau)\,\mathrm{d}\tau\right| \leqq Me^{\omega t}e^{\omega^2 t^2/(p+1-\omega t)}$$

for every $t \in R^+$ and $p + 1 > \omega t$,

$$\text{(c)} \qquad \frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1} \int_0^\infty e^{-((p+1)/t)\tau}\tau^p f(\tau)\,\mathrm{d}\tau \to f(t) \quad (p \to \infty,\ p + 1 > \omega t)$$

for almost all $t \in R^+$.

5. Lemma. *Let α be a nonnegative constant and $J \in \{z : \operatorname{Re} z \geqq \alpha\} \to C$. If*

(α) *J is continuous on $\{z : \operatorname{Re} z \geqq \alpha\}$,*

(β) *J is analytic on $\{z : \operatorname{Re} z > \alpha\}$,*

(γ) $J(\lambda) \to 0$ $(\lambda \geqq \alpha,\ \lambda \to \infty)$,

(δ) *there exist a constant K and a number $k \in \{0, 1, \ldots\}$ so that for every* $\operatorname{Re} z \geqq \alpha$, *we have* $|J(z)| \leqq K(1 + |z|)^k$,

$$(\varepsilon) \qquad \int_{-\infty}^{\infty} \frac{|J(\alpha + i\beta)|}{1 + |\beta|}\,\mathrm{d}\beta < \infty\,,$$

then for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$,

$$\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} J(\lambda) = (-1)^p \frac{p!}{2\pi i} \int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\lambda - \alpha - i\beta)^{p+1}}\,\mathrm{d}\beta\,.$$

Proof. Let us first fix a $\lambda > \alpha$.

Moreover, we choose fixed numbers K, k so that the assumption (δ) holds.

By virtue of Cauchy's integral theorem, we obtain from (α), (β) that

$$\text{(1)} \qquad \begin{aligned} \frac{2\pi}{p!} J^{(p)}(\lambda) = &-\int_{-N}^{N} \frac{1}{(\alpha + i\beta - \lambda)^{p+1}} J(\alpha + i\beta)\,\mathrm{d}\beta + \\ &+ \int_{-N}^{N} \frac{1}{(\alpha + 2N + i\beta - \lambda)^{p+1}} J(\alpha + 2N + i\beta)\,\mathrm{d}\beta - \\ &- i\int_0^{2N} \frac{1}{(\alpha + \eta + iN - \lambda)^{p+1}} J(\alpha + \eta + iN)\,\mathrm{d}\eta + \\ &+ i\int_0^{2N} \frac{1}{(\alpha + \eta - iN - \lambda)^{p+1}} J(\alpha + \eta - iN)\,\mathrm{d}\eta \end{aligned}$$

for every $p \in \{0, 1, \ldots\}$ and $N > \frac{1}{2}\lambda$.

Using (δ), we obtain

$$(2)\qquad \left|\int_{-N}^{N}\frac{1}{(\alpha+2N+i\beta-\lambda)^{p+1}}\,J(\alpha+2N+i\beta)\,\mathrm{d}\beta\right| \leqq$$

$$=\int_{-N}^{N}\frac{1}{((\lambda-\alpha+2N)^2+\beta^2)^{(p+1)/2}}\left[1+((\alpha+2N)^2+\beta^2)^{1/2}\right]^k\mathrm{d}\beta=$$

$$=\int_{-N}^{N}\frac{1}{(\lambda-\alpha+2N)^{p+1}}\left[1+((\alpha+2N)^2+N^2)^{1/2}\right]^k=$$

$$=\frac{2N}{(\lambda-\alpha+2N)^{p+1}}\left[1+((\alpha+2N)^2+N^2)^{1/2}\right]^k,$$

$$\left|\int_0^{2N}\frac{1}{(\alpha+\eta+iN-\lambda)^{p+1}}\,J(\alpha+\eta+iN)\,\mathrm{d}\eta\right|\leqq$$

$$\leqq\int_0^{2N}\frac{1}{((\lambda+\eta-\alpha)^2+N^2)^{(p+1)/2}}\left[1+(\alpha^2+N^2)^{1/2}\right]^k\mathrm{d}\eta\leqq$$

$$\leqq\int_0^{2N}\frac{1}{N^{p+1}}\left[1+(\alpha^2+N^2)^{1/2}\right]^k\leqq\frac{2}{N^p}\left[1+(\alpha^2+N^2)^{1/2}\right]^k,$$

$$\int_0^{2N}\frac{1}{(\alpha+\eta+iN-\lambda)^{p+1}}\,J(\alpha+\eta-iN)\,\mathrm{d}\eta\leqq\frac{2}{N^p}\left[1+(\alpha^2+N^2)^{1/2}\right]^k$$

for every $p\in\{0,1,\ldots\}$ and $N>\frac{1}{2}\lambda$.

Letting $N\to\infty$, we see from (2) that

$$(3)\qquad \int_{-N}^{N}\frac{1}{(\alpha+2N+i\beta-\lambda)^{p+1}}\,J(\alpha+2N+i\beta)\,\mathrm{d}\beta\to_{N\to\infty}0,$$

$$\int_0^{2N}\frac{1}{(\alpha+\eta+iN-\lambda)^{p+1}}\,J(\alpha+\eta+iN)\,\mathrm{d}\eta\to_{N\to\infty}0,$$

$$\int_0^{2N}\frac{1}{(\alpha+\eta-iN-\lambda)^{p+1}}\,J(\alpha+\eta-iN)\,\mathrm{d}\eta\to_{N\to\infty}0$$

for every $p\in\{k+1,k+2,\ldots\}$.

Now we conclude from (1) and (3) by means of (ε) that

$$(4)\qquad J^{(p)}(\lambda)=-\frac{p!}{2\pi}\int_{-\infty}^{\infty}\frac{J(\alpha+i\beta)}{(\alpha+i\beta-\lambda)^{p+1}}\,\mathrm{d}\beta$$

for every $\lambda>\alpha$ and $p\in\{k+1,k+2,\ldots\}$.

On the other hand, let us define, on the basis of (ε)

$$J_0(\lambda) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)}\,\mathrm{d}\beta \quad \text{for every} \quad \lambda > \alpha\,.$$

It is easy to verify that

(6) the function J_0 is infinitely differentiable on (α, ∞),

(7) $$J_0^{(p)}(\lambda) = -\frac{p!}{2\pi}\int_{-\infty}^{\infty}\frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)^{p+1}}\,\mathrm{d}\beta$$

for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$,

(8) $$J_0(\lambda) \to 0 \quad (\lambda > \alpha,\ \lambda \to \infty)\,.$$

By (4)–(7),

(9) $$J^{(k+1)}(\lambda) = J_0^{(k+1)}(\lambda)$$

for every $\lambda > \alpha$. Consequently, by (9)

(10) $$J - J_0 \quad \text{is a polynomial}\,.$$

Taking (γ) and (8) into account, we see that

(11) $$J(\lambda) - J_0(\lambda) \to 0 \quad (\lambda > \alpha,\ \lambda \to \infty)\,.$$

Hence (10) and (11) imply $J = J_0$ and the conclusion of Lemma 5 follows immediately from (7).

6. Theorem. *Let $f \in R^+ \to C$ and $\alpha > 0$. If*

(α) *the function f is measurable,*

(β) *there exist two nonnegative constants M, ω so that $\omega < \alpha$ and $|f(t)| \leqq Me^{\omega t}$ for almost all $t \in R^+$,*

(γ) $$\int_{-\infty}^{\infty}\left|\int_0^{\infty} e^{-(\alpha+i\beta)\tau} f(\tau)\,\mathrm{d}\tau\right|\mathrm{d}\beta < \infty\,,$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t}\left(\int_0^{\infty} e^{-(\alpha+i\beta)\tau} f(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\beta\,.$$

for almost all $t \in R^+$.

Proof. Let us first fix the constants M, ω so that the assumption (β) holds.

Further let us define a function $F \in (\omega, \infty) \to C$ by

(1) $$F(\lambda) = \int_0^{\infty} e^{-\lambda\tau} f(\tau)\,\mathrm{d}\tau \quad \text{for} \quad \lambda > \omega\,.$$

By Theorem 4 we have

$$(2)\qquad \frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \to f(t) \quad (p \to \infty,\ p+1 > \alpha t)$$

for almost all $t \in R^+$.

On the other hand, let J be the function defined by

$$(3)\qquad J(z) = \int_0^\infty e^{-z\tau} f(\tau)\,\mathrm{d}\tau$$

for every $\operatorname{Re} z \geqq \alpha$.

It is easy to deduce from our assumptions that

(4) the function J has the properties 5 $(\alpha)-(\varepsilon)$.

Hence by (4), we obtain from Lemma 5 that

$$(5)\qquad J^{(p)}(\lambda) = (-1)^p \frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\lambda - \alpha - i\beta)^{p+1}}\,\mathrm{d}\beta$$

for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$.

Now it follows from (1), (3) and (5) that

(6)

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left(\frac{p+1}{t}\right)^{p+1}}{\left(\frac{p+1}{t} - \alpha - i\beta\right)^{p+1}}\, J(\alpha + i\beta)\,\mathrm{d}\beta =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\left(1 - \frac{(\alpha + i\beta)\,t}{p+1}\right)^{p+1}}\, J(\alpha + i\beta)\,\mathrm{d}\beta$$

for every $t \in R^+$ and $p + 1 > \alpha t$.

By Lemma 2 we see that

$$(7)\qquad \left|\frac{1}{\left(1 - \frac{(\alpha + i\beta)\,t}{p+1}\right)^{p+1}}\right| = \frac{1}{\left[\left(1 - \frac{\alpha t}{p+1}\right)^2 + \left(\frac{\beta t}{p+1}\right)^2\right]^{(p+1)/2}} \leqq$$

$$\leqq \frac{1}{\left(1 - \frac{\alpha t}{p+1}\right)^{p+1}} \leqq e^{\alpha t} e^{\alpha t/(p+1-\alpha t)}$$

for every $t \in R^+$ and $p + 1 > \alpha t$.

By Lemma 3,

$$\frac{1}{\left(1-\frac{(\alpha+i\beta)\,t}{p+1}\right)^{p+1}} \to e^{(\alpha+i\beta)t} \quad (p\to\infty,\ p+1>\alpha t)\,. \tag{8}$$

Now we get from (6)–(8)

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t}\, J(\alpha+i\beta)\,\mathrm{d}\beta \tag{9}$$

$$(p\to\infty,\ p+1>\alpha t)$$

for every $t \in R^+$.

We see from (2) and (9) that the assertion of our theorem is fulfilled and this completes the proof.

Note. In the continuation of this paper, we shall study different complex inversion formulas for the Laplace transform as relatively simple consequences of Theorem 6 from a new unified point of view.

References

[1] *Widder, D. V.:* The Laplace transform, 1946.
[2] *Widder, D. V.:* An introduction to transform theory, 1971.
[3] *Doetsch, G.:* Introduction to the theory and application of the Laplace transformation, 1974.
[4] *Doetsch, G.:* Theorie und Anwendung der Laplace-Transformation, 1937.
[5] *Ditkin, V. A.:* Operacionnoe isčislenije. Uspechi mat. nauk, 2/6, 1947, 72–158.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).