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ON DISTRIBUTIONAL SOLUTIONS OF SOME SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Let $A(t) = (a_{ij}(t))$ be a matrix such that $a_{ij}(t)$ is a measure in the interval $(a, b) \subseteq \subseteq R^1$ for $i, j = 1, \ldots, n$, and let $f(t)$ be a vector whose all components $f_i(t)$ are also measures (defined in (a, b)). In this note we consider the system of equations

$$(*) \qquad y'(t) = A(t)\, y(t) + f(t)\,,$$

where y is an unknown vector. The derivative is understood in the distributional sense. Our result generalizes some theorems for linear differential equations (see [1], [2], [3], [6]).

2. THE PRINCIPAL RESULT

First we introduce some notations.

Let $A(t) = (a_{ij}(t))$ be a matrix whose all elements $a_{ij}(t)$ are measures defined in the interval (a, b) $(i, j = 1, \ldots, n)$, and let $\hat{A}(t) = (\hat{a}_{ij}(t))$ be a matrix whose all elements $\hat{a}_{ij}$ are functions such that $a_{ij} = [\hat{a}_{ij}]'$. We put $\Delta\, \hat{A}(t) = \hat{A}(t+) - \hat{A}(t-)$, $\|A\|\,(t) = \sum_{i,j=1}^{n} |a_{ij}|\,(t)$, $y_0 = (y_1^0, \ldots, y_n^0)$, $|y_0| = \sum_{i=1}^{n} |y_i^0|$, $y^*(t) = (y_1^*(t), \ldots, y_n^*(t))$, $|y|^*\,(t) = \sum_{i=1}^{n} |y_i|^*\,(t)$, where $t \in (a, b)$, $\hat{A}(t+) = (\hat{a}_{ij}(t+))$, $\hat{A}(t-) = (\hat{a}_{ij}(t-))$ and $y_i^0 \in R^1$. The remaining notations in this paper are taken from [5].

Theorem 2.1. *We assume that $a_{ij}(t)$ and $f_i(t)$ are measures defined in the interval (a, b) for $i, j = 1, \ldots, n$. Moreover, for every $t \in (a, b)$*

$$(2.1) \qquad \det(2I - \Delta\, \hat{A}(t)) \neq 0 \quad \textit{and} \quad \det(2I + \Delta\, \hat{A}(t)) \neq 0\,,$$

where I denotes the identity matrix. Then the problem

$$(2.2)\qquad \begin{cases} y'(t) &= A(t)\, y(t) + f(t) \\ y^*(t_0) &= y_0 \end{cases}$$

has exactly one solution in the class $V^n_{(a,b)}$ *for every* $t_0 \in (a, b)$.

Remark 1. The assumption 2.1 in Theorem 2.1 is essential. This can be observed from the following

Example .

$$(2.3)\qquad \begin{cases} y'(t) &= 2\,\delta(t)\, y(t) \\ y^*(-1) &= 0\,, \end{cases}$$

$$(2.4)\qquad \begin{cases} z'(t) &= -2\,\delta(t)\, z(t) \\ z^*(1) &= 0 \end{cases}$$

(δ denotes Dirac's delta distribution). In fact, let H denote Heaviside's function and let c be a constant. From the equality

$$(2.5)\qquad H\delta = \tfrac{1}{2}\delta \quad \text{(see [4])}$$

it is not difficult to show that the distributions $y = cH$ and $z = c(H - 1)$ are solutions of the problem (2.3) and (2.4), respectively.

3. PROOFS

Before giving the proof of Theorem 2.1 we shall prove some lemmas.

Lemma 3.1. *We assume that* $P_1, P_2 \in V^1_{(a,b)}$ *and* $\lim_{t\to r} P_1^*(t) = \lim_{t\to r} P_2^*(t)$. *Moreover, let*

$$(3.1)\qquad p_1(t) + c_1\,\delta(t - r) = p_2(t) + c_2\,\delta(t - r)\,,$$

where $P_1' = p_1$, $P_2' = p_2$, $r \in (a, b)$ *and* c_1, c_2 *are constants. Then* $c_1 = c_2$.

This fact follows easily from the equality

$$(3.2)\qquad \int_r^t p_1(s)\,\mathrm{d}s + c_1\int_r^t \delta(s - r)\,\mathrm{d}s = \int_r^t p_2(s)\,\mathrm{d}s + c_2\int_r^t \delta(s - r)\,\mathrm{d}s\,.$$

Lemma 3.2. *Let* $a_{ij}(t)$ *and* $f_i(t)$ *be measures defined in the interval* (a, b) $(i, j = 1, \ldots, n)$, $t_0 \in (a, b)$. *Then there exists a number* $r > 0$ *such that:*

1. $a < t_0 - r < t_0 + r < b$,
2. *the problem* (2.2) *has exactly one solution* $y(t)$ *in the class*

$$V^n_{(t_0 - r, t_0 + r)}\,,$$

3. *there exist finite limits* $\lim_{t\to t_0 - r+} y^*(t)$, $\lim_{t\to t_0 + r-} y^*(t)$.

Proof. If there exists a number $r > 0$ such that

$$\int_{t_0-r}^{t_0+r} \|A\| (t) \, dt < 1 , \tag{3.3}$$

then in view of [5] the problem (2.2) has exactly one solution in the class $V^n_{(t_0-r,t_0+r)}$. In the opposite case we consider matrices $\hat{A}_1(t, t_0)$ and $\hat{A}_2(t, t_0)$ defined as follows:

$$\hat{A}_1(t, t_0) = \begin{cases} \hat{A}(t), & \text{for} \quad a < t < t_0 \\ \hat{A}(t_0-), & \text{for} \quad t_0 \leqq t < b , \end{cases} \tag{3.4}$$

$$\hat{A}_2(t, t_0) = \begin{cases} \hat{A}(t_0+), & \text{for} \quad a < t \leqq t_0 \\ \hat{A}(t), & \text{for} \quad t_0 < t < b . \end{cases} \tag{3.5}$$

Hence, we have

$$\hat{A}(t) = \hat{A}_1(t, t_0) + \hat{A}_2(t, t_0) - H(t - t_0) \hat{A}(t_0-) - H(t_0 - t) \hat{A}(t_0+) \tag{3.6}$$

and

$$y'(t) = U(t, t_0) y(t) + \delta(t - t_0) (\Delta \hat{A}(t_0)) y^*(t_0) + f(t) , \tag{3.7}$$

where $U(t, t_0) = A_1(t, t_0) + A_2(t, t_0)$, $A_1(t, t_0) = (\hat{A}_1(t, t_0))'$ and $A_2(t, t_0) = (\hat{A}_2(t, t_0))'$. Moreover, there exists a number $r_1 > 0$ such that

$$\int_{t_0-r_1}^{t_0+r_1} \|U\| (t, t_0) \, dt < 1 . \tag{3.8}$$

Taking into account [5], we infer that the system (2.2) has exactly one solution $y(t)$ in the class V^n_I, where $I = (t_0 - r_1, t_0 + r_1)$. We claim that $\sup_{t\in I} |y|^* (t) < \infty$. Indeed, let us put

$$\bar{f} = \delta(t - t_0) (\Delta \hat{A}(t_0)) y^*(t_0) + f , \quad K = 1 - \int_{t_0-r_1}^{t_0+r_1} \|U\| (t, t_0) \, dt , \tag{3.9}$$

$$M_i = \left| \int_{t_0-r_1}^{t_0+r_1} f_i(t) \, dt \right| , \quad M = \sum_{i=1}^{n} M_i , \quad \varepsilon > 0 , \quad J = [t_0 - r_1 + \varepsilon, t_0 + r_1 - \varepsilon]$$

$$(a < t_0 - r_1 + \varepsilon < t_0 + r_1 - \varepsilon < b) .$$

Then the relation (2.2) and [5] imply

$$\sup_{t\in J} |y|^* (t) \leqq |y_0| + \sup_{t\in J} |y|^* (t) \int_J \|U\| (t, t_0) \, dt + M \tag{3.10}$$

and

$$\sup_{t\in I} |y|^* (t) \leqq K^{-1}(|y_0| + M) . \tag{3.11}$$

Now we consider an arbitrary sequence $\{t_k\}$ such that $t_k \in I$ $(k = 1, 2, \ldots)$ and $t_k \to t_0 + r_1 -$. Using (2.2), [5] and (3.11), we have

$$(3.12)\quad |y_i^*(t_k) - y_i^*(t_m)| \leqq \sup_{t\in I} |y|^*(t) \left|\int_{t_m}^{t_k} \|U\|(t, t_0)\,\mathrm{d}t\right| + \left|\int_{t_m}^{t_k} \bar{f}_i(t)\,\mathrm{d}t\right| \leqq$$
$$\leqq K^{-1}(|y_0| + M)|\,Z^*(t_k) - Z^*(t_m)| + |G_i^*(t_k) - G_i^*(t_m)|\,,$$

where $Z' = \|U\|$, $G_i' = \bar{f}_i$ and $i = 1, \ldots, n$. Similarly we prove that there exists a finite limit $\lim_{t\to t_0 - r_1 +} y^*(t)$. Thus our assertion follows.

Lemma 3.3. *Let the assumptions of Theorem* 2.1 *be fulfilled and let* $y(t)$ *be a solution of the problem* (2.2) *in the class* $V^n_{(a,b)}$. *Then for every* $t \in (a, b)$

$$(3.13)\qquad y(t+) = (2I - \Delta\,\hat{A}(t))^{-1}\,[(2I + \Delta\,\hat{A}(t))\,y(t-) + 2\Delta\,F(t)]$$

and

$$(3.14)\qquad y(t-) = (2I + \Delta\,\hat{A}(t))^{-1}\,[(2I - \Delta\,\hat{A}(t))\,y(t+) - 2\Delta\,F(t)]\,,$$

where $F' = f$ and $\Delta\,F(t) = F(t+) - F(t-)$.

Proof. Let $y(t)$ be a solution of the problem (2.2). We consider vectors $\tilde{y}(t)$ and $\bar{y}(t)$ defined as follows:

$$(3.15)\qquad \tilde{y}(t) = \begin{cases} y(t)\,, & \text{for } a < t < t_1 \\ y(t_1-)\,, & \text{for } t_1 \leqq t < b\,, \end{cases}$$

$$(3.16)\qquad \bar{y}(t) = \begin{cases} y(t_1+)\,, & \text{for } a < t \leqq t_1 \\ y(t)\,, & \text{for } t_1 < t < b\,. \end{cases}$$

Then

$$(3.17)\qquad y(t) = \tilde{y}(t) + \bar{y}(t) - H(t - t_1)\,y(t_1-) - H(t_1 - t)\,y(t_1+)$$

and

$$(3.18)\qquad y'(t) = \tilde{y}'(t) + \bar{y}'(t) + \delta(t - t_1)\,(\Delta\,y(t_1))\,.$$

On the other hand,

$$(3.19)\quad y'(t) = U(t, t_1)\,y(t) + \delta(t - t_1)\,(\Delta\,\hat{A}(t_1))\,y^*(t_1) + \tilde{f}(t) + \bar{f}(t) +$$
$$+ \delta(t - t_1)\,(\Delta\,F(t_1))\,,$$

where $\tilde{f} = \tilde{F}'$, $\bar{f} = \bar{F}'$ and

$$(3.20)\qquad \tilde{F}(t) = \begin{cases} F(t)\,, & \text{for } a < t < t_1 \\ F(t_1-)\,, & \text{for } t_1 \leqq t < b\,, \end{cases}$$

$$(3.21)\qquad \bar{F}(t) = \begin{cases} F(t_1+)\,, & \text{for } a < t \leqq t_1 \\ F(t)\,, & \text{for } t_1 < t < b\,. \end{cases}$$

Applying (3.17), (3.18), (3.19), (3.20) and (3.21), we obtain

$$\tilde{y}'(t) + \bar{y}'(t) + \delta(t - t_1)(\Delta\, y(t_1)) = U(t, t_1)(\tilde{y}(t) + \bar{y}(t)) + \tag{3.22}$$

$$+ U(t, t_1)[-H(t - t_1)\, y(t_1-) - H(t_1 - t)\, y(t_1+)] +$$

$$+ \delta(t - t_1)(\Delta\, \hat{A}(t_1))\, y^*(t_1) + \tilde{f}(t) + \bar{f}(t) + \delta(t - t_1)(\Delta\, F(t_1))\,.$$

Using Lemma 3.1 and (3.22), we infer that

$$\Delta\, y(t_1) = (\Delta\, \hat{A}(t_1))\, y^*(t_1) + \Delta\, F(t_1)\,. \tag{3.23}$$

An application of condition (2.1) completes the proof of Lemma 3.3.

Proof of Theorem 2.1. We consider an arbitrary interval $[c, d]$ such that $c, d \in (a, b)$. Let $r_0 = \min(t_1 - a,\ b - t_1)$, where $t_1 \in [c, d]$. Then the properties of functions of locally bounded variation yield that the set of all points t_1 such that

$$\int_{t_1 - r}^{t_1 + r} \|A\|\,(t)\,\mathrm{d}t > 1\,, \tag{3.24}$$

for every $0 < r < r_0$ is finite. Thus, applying (3.13), (3.14) and Lemma 3.2 we can extend uniquely the local solution in the whole interval (a, b) and this completes the proof of the theorem.

Remark 2. Let the assumptions of Theorem 2.1 be satisfied. Then by Lemma 3.3 and Theorem 2.1 it is not difficult to show that the system $(*)$ with the initial condition $y(t_0+) = y_0$ $(y(t_0-) = y_0)$ has exactly one solution in the class $V^n_{(a,b)}$.

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