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# CERTAIN RELATION BETWEEN VECTOR FIELDS AND DISTRIBUTIONS ON A DIFFERENTIABLE MANIFOLD

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Let $M$ be a differentiable manifold. Let $X$ be a vector field on $M$ and let $\Delta$ be a distribution of $h$-dimensional tangent subspaces on $M$.

In this paper we shall study some relations between the fields $X$ and distribution $\Delta$. In the following computations we use the calculus of jets, see [2].

**1.** Let $T_h^1(M)$ be the vector space of all $h^1$-velocities on the manifold $M$, i.e. the space of all 1-jets of local mappings from $R^h$ into $M$ with the source $0 \in R^h$. Let $X$ be a vector field on $M$ and ${}^t\Phi$ its 1-parametric local group. Let us remind that the vector field $X$ can be naturally prolonged to $T_h^1(M)$ according to the rule

$$ {}^1X(u) = j_0^1[j_0^1({}^t\Phi \, . \, \varphi)] \, , \quad \text{for} \quad u \in T_h^1(M) \, , \quad \text{where} \quad u = j_0^1\varphi \, . $$

Let $(x^i)$, $i = 1, 2, \ldots, n$, be local coordinates on $M$, $\dim M = n$; let $(x^i, y_j^i)$, $j = 1, 2, \ldots, h$, $h \leqq n$, be corresponding local coordinates on $T_h^1(M)$ and let

$$ {}^1X \equiv a^i(x) \, \partial/\partial x_i + b_j^i \, \partial/\partial y_i^j \tag{1} $$

be the prolongation of the vector field $X$ to the manifold $T_h^1(M)$. To determine the components $b_j^i$ let us remind that

$$ T_h^1(M) \equiv (x^i, y_j^i) \equiv j_0^1\varphi \, , \quad \text{where} \quad \varphi : R^h \to M \, , \quad \text{i.e.} \quad \varphi : x^i = \varphi^i(u^j) \, . $$

The vector field $X$ determines a local 1-parametric group ${}^t\Phi$ of transformations on $M$

$$ {}^t\Phi : \bar{x}^k = f^k(x^i, t) \quad \text{for} \quad k = 1, 2, \ldots, n \, . $$

Then

$$ {}^t\Phi \, . \, \varphi : \bar{x}^k = f^k(\varphi^i(u^j), t) \quad \text{and} \quad j_0^1({}^t\Phi \, . \, \varphi) \equiv \frac{\partial f^k}{\partial x^i} \cdot \frac{\partial x^i}{\partial u^j} \, . $$

Hence

$$ j_{0t}^1[j_{0x}^1({}^t\Phi \, . \, \varphi)] \equiv \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial f^k}{\partial x^i} \cdot \frac{\partial x^i}{\partial u^j} \right) = \frac{\partial}{\partial x^i} \left( \frac{\mathrm{d}f^k}{\mathrm{d}t} \right) y_j^i = \frac{\partial a^k}{\partial x^i} \, y_j^i \equiv b_j^k \, . $$

Consequently the vector field ${}^1X$ from (1) can be expressed in the form

$$(1') \qquad {}^1X \equiv a^i(x)\,\partial/\partial x_i + \frac{\partial a^i}{\partial x^k}\, y^k_j\, \partial/\partial y^j_i\,.$$

**2.** Let us consider a global cross-section

$$(2) \qquad \bar{\Gamma} : M \to T^1_h(M)\,.$$

In local coordinates

$$\bar{\Gamma} : x^i = x^i\,, \quad y^i_j = f^i_j(x^k)\,.$$

The restriction of the prolonged vector field ${}^1X$ to the submanifold $\bar{\Gamma}(M) \subset T^1_h(M)$ is given by

$$(3) \qquad {}^1X/\bar{\Gamma}(M) \equiv a^i(x)\,\partial/\partial x_i + \frac{\partial a^i}{\partial x^\lambda} f^\lambda_j\,(x^k)\,\partial/\partial y^j_i$$

for $\lambda = 1, 2, \ldots, n$.

**Definition 1.** A vector field $X$ on $M$ is said *to be conjugate to the map* $\bar{\Gamma}$ if

$$(4) \qquad \bar{\Gamma}_*(X) \equiv {}^1X/\bar{\Gamma}(M)\,.$$

$\bar{\Gamma}_*(X)$ is a vector field on $\bar{\Gamma}(M)$. Therefore, according to (2) and (3), its local expression is given by

$$(5) \qquad \bar{\Gamma}_*(X) \equiv a^i(x)\,\partial/\partial x_i + a^k \frac{\partial f^i_j}{\partial x^k}\,\partial/\partial y^j_i\,.$$

Substituing from (3) and (5) into (4) we obtain

$$(6) \qquad \frac{\partial a^i}{\partial x^\lambda} f^\lambda(x) = a^k \frac{\partial f^i(x)}{\partial x^k}\,.$$

The map (2) determines $h$ vector fields $X_r$ on $M$ with the local expressions

$$(7) \qquad X_r = f^i_r(x)\,\partial/\partial x_i \quad \text{for} \quad r = 1, 2, \ldots, h\,.$$

When $X$ is an arbitrary vector field conjugate to the map $\bar{\Gamma}$, we get

$$(8) \qquad [X, X_r] = \left\{\frac{\partial f^k_r(x)}{\partial x^i}\,.\,a^i(x) - \frac{\partial a^k(x)}{\partial x^i} f^i_r(x)\right\}\partial/\partial x_k\,,$$

where $[X, X_r]$ are Lie brackets.

By comparison of (6) and (8) we obtain

**Lemma 1.** *A necessary and sufficient condition for a vector field $X$ to be conjugate to the map $\bar{\Gamma}$ is that*

$$[X, X_r] = 0 \quad \textit{for each} \quad r = 1, 2, \ldots, h\,.$$

3. Let $R\,T_h^1(M)$ denote the set of all regular $h^1$-velocities on $M$. Obviously, $RT_h^1(M)$ is an open submanifold of the manifold $T_h^1(M)$. If $X$ is a vector field on $M$ then ${}^1X/R\,T_h^1(M)$ is a vector field on $R\,T_h^1(M)$. Let $K_h^1(M)$ denote the factor space of $RT_h^1(M)$ consisting of all classes of the form $Y\,.\,L_h^1$, where $Y$ is a regular $h^1$-velocity on $M$ and $L_h^1$ is the full linear transformation group of $R^h$. Let $\Delta$ be the distribution of $h$-dimensional tangent subspaces determined by a map $\Gamma : M \to K_h^1(M)$. We have the canonical projection $\varrho : R\,T_h^1(M) \to K_h^1(M)$. If ${}^t\Phi$ is the 1-parametric local group on $M$ generated by the vector field $X$ and if ${}^t\Phi_h^1$ is the 1-parametric local group on $R\,T_h^1(M)$ generated by the vector field ${}^1X/R\,T_h^1(M)$ then

$$(9) \qquad {}^t\Phi_h^1(u) = {}^t\Phi\,.\,u\,, \quad u \in RT_h^1(M)\,.$$

Here the dot denotes the composition of jets. It follows from (9) that the map ${}^t\Phi_h^1$ preserves the classes $Y\,.\,L_h^1$. Thus we can define a 1-parametric local group on $K_h^1(M)$ by the formula

$$(10) \qquad {}^t\bar{\Phi}_h^1 : [v] \mapsto [{}^t\Phi\,.\,v]$$

where $[v] \in K_h^1(M)$ denotes the class of $v \in R\,T_h^1(M)$. The vector field ${}^1\bar{X}$ on $K_h^1(M)$ induced by ${}^t\bar{\Phi}_h^1$ will be called the *$h^1$-tangent prolongation of the vector field $X$*. Obviously we have

$$(11) \qquad \varrho_*({}^1X) = {}^1\bar{X}\,.$$

The map $\Gamma : M \to \Gamma(M)$ is a diffeomorfism and $\Gamma_*(X)$ is a vector field on $\Gamma(M) \subset$ $\subset K_h^1(M)$.

**Definition 2.** A vector field on $M$ is said *to be conjugate to the distribution $\Delta$* if $\Gamma_*(X) = {}^1\bar{X}/\Gamma(M)$.

Let us remark that A. DEKRÉT in [1] investigates the conjugacy of special vector fields and special distributions on the manifold $T(M)$.

**Definition 3.** A vector field on $M$ is called *a subfield of the distribution $\Delta$* if $X(m) \in \Gamma_m$ for all $m \in M$.

**Lemma 2.** *Suppose that, for each subfield $Y$ of $\Delta$, the Lie bracket $[X, Y]$ is also a subfield of $\Delta$. Then for each point $u \in M$ there is a neighbourhood $U \subset M$ and vector fields $X \equiv X_1, X_2, \ldots, X_{h+1}$ on $U$ such that $\Delta$ is generated by $X_2, \ldots, X_{h+1}$ and $[X, X_s] = 0$ holds for $s = 2, 3, \ldots, h + 1$.*

Proof. Let $u \in M$. Let $(U, x^1, x^2, \ldots, x^n)$ is a chart on $M$ such that $X \equiv \partial/\partial x_1$ in the neighbourhood $U$ of $u$. On $U$ there are vector fields $X \equiv \partial/\partial x_1 \equiv Y_1, Y_2, \ldots$ $\ldots, Y_{h+1}$, where $Y_2, \ldots, Y_{h+1}$ generate the distribution $\Delta$ on $U$. Because $[X, Y]$ is a subfield of $\Delta$ for each subfield $Y$ of $\Delta$ we can see that $[Y_1, Y_\alpha]$ are subfields of $\Delta$ on $U$ for $\alpha = 2, \ldots, h + 1$. Let

$$Y_1 \equiv X \equiv \partial/\partial x_1\,, \quad Y_\alpha = a_\alpha^i\,\partial/\partial x_i\,,$$

where $a^i_\alpha$ are real functions on $U$. The matrix $\|a^i_\alpha\|$ has rank $h$ for each point $x \in U$. Suppose e.g. that det. $\|a^s_\alpha\| \neq 0$. Let $\|b^\sigma_s\|$ denote the inverse if $\|a^s_\alpha\|$. Put

$$X_s = b^\alpha_s Y_\alpha = b^\alpha_s a^i_\alpha \, \partial/\partial x_i .$$

Then

$$X_s = \partial/\partial x_s + b^\alpha_s a^\beta_\alpha \, \partial/\partial x_\beta , \quad \text{where} \quad \beta = 1,\ h + 2, \ldots, n ,$$

i.e.

$$X_s = \partial/\partial x_s + c^\beta_s \, \partial/\partial x_\beta . \tag{12}$$

Because $[X, Y]$ is a subfield of $\Delta$ and the vector fields $X_s$ generate the distribution $\Delta$ on $U$, we get

$$[X_1, X_\alpha] = \lambda^s_\alpha X_s = \lambda^s_\alpha \, \partial/\partial x_s + \lambda^s_\alpha c^\beta_s \, \partial/\partial x_\beta . \tag{13}$$

From (12) and (13) we derive

$$[X_1, X_s] = \frac{\partial c^\beta_s}{\partial x_1} \, \partial/\partial x_\beta . \tag{14}$$

By comparing (13) and (14) we obtain

$$\lambda^s_\alpha = 0 . \tag{15}$$

Finally, substituting (15) into (13) we get $[X, X_s] = 0$, q.e.d.

**Definition 4.** The map $\bar{\Gamma} : M \to RT^1_h(M)$ is called *related to the distribution* $\Delta$ given by the map $\Gamma : M \to K^1_h(M)$ if $\Gamma = \varrho \, . \, \bar{\Gamma}$.

**Lemma 3.** *Let the map $\bar{\Gamma}$ be related to the distribution $\Delta$, let a vector field $X$ be conjugate to the map $\Gamma$. Then $X$ is conjugate to the distribution $\Delta$.*

The proof follows from (11).

**Lemma 4.** *If the distribution $\Delta$ is conjugate to the field $X$ then there is locally a map $\bar{\Gamma}$ from $M$ into $RT^1_h(M)$ which is releated to the distribution $\Delta$ and conjugate to the field $X$.*

The proof follows from the definition of $K^1_h(M)$.

**Theorem.** *Let $\Delta$ be a distribution on a manifold $M$ given by a map $\Gamma : M \to$ $\to K^1_h(M)$. Let $X$ be a vector field on $M$ and $Y$ a vector subfield of $\Delta$. Then a necessary and sufficient condition for $X$ to be conjugate with $\Delta$ is that $[X, Y]$ is a subfield of $\Delta$ for each subfield $Y$.*

Proof. Let $X$ satisfy the condition that $[X, Y]$ is a subfield of $\Delta$ for each subfield $Y$. According to Lemma 2 there are vector fields $X_s$ such that $[X, X_s] = 0$. The vector fields $X_2, \ldots, X_{h+1}$ determine uniquely the map $\bar{\Gamma} : M \to RT^1_h(M)$. From Lemma 1 it follows that $\bar{\Gamma}$ is conjugate with $X$. Lemma 3 implies that $\Delta$ is conjugate with $X$, too.

Conversely: Let $\Delta$ be conjugate with $X$. According to Lemma 4 there is a map $\bar{\Gamma}$ which is conjugate with $X$. The map $\bar{\Gamma}$ determines vector fields $X_2, \ldots, X_{h+1}$ such that $[X, X_\alpha] = 0$, on account of Lemma 1. Let $Y = \mu^\alpha X_\alpha$, the $[X, Y]$ is of the form $\nu^\alpha X_\alpha$, as well, and hence $[X, Y]$ is a subfield of $\Delta$, q.e.d.

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*Literature*

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