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BEHAVIOUR OF SOLUTIONS OF AN INTEGRAL EQUATION

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1. INTRODUCTION

Let B be a Banach space, $I = [0, \infty)$, $\Omega = \{(t, s) \in R^2 : t \in I,\ s \in [0, t]\}$, $\Omega_T = \{(t, s) \in \Omega,\ t \in [0, T]\}$ for $T > 0$. We shall consider two functions p and W with the following properties:

A 1
$$p \in C(I \times B, B)\,, \quad (\forall u \in B)\, p(0, u) = u\,,$$
$$(\exists M, k = \text{const})\, (\forall t \in I)\, (\forall u, v \in B) \quad \|p(t, u) - p(t, v)\| \leqq M e^{-kt} \|u - v\|\,.$$

A 2
$$W \in C(\Omega \times B, B)\,, \quad W(t, s, 0) = 0\,,$$
$$(\exists L \in C(I, R))\, (\forall (t, s) \in \Omega)\, (\forall u, v \in B) \quad \|W(t, s, u) - W(t, s, v)\| \leqq$$
$$\leqq L(s)\, e^{-k(t-s)} \|u - v\|\,,$$

where k is the same as in A 1.

For any $u_0 \in B$ we shall consider the equation

$$u = p(t, u_0) + \int_0^t W(t, s, u)\, \mathrm{d}s \tag{1}$$

and the problems of existence, uniqueness and asymptotic behaviour of its solutions under the assumptions A 1, A 2 and some others. The equation (1) in particular describes the mild solutions of the Cauchy problem: $\dot{u} = A(t)\, u + f(t, u)$, $u(0) = u_0$. In this case $W(t, s, u) = U(t, s)\, u$, $p(t, u_0) = U(t, 0)\, u_0$, $U(t, s)$ is Green's function, the assumptions A 1 and A 2 are satisfied if f is Lipschitz continuous in u, $\|U(t, s)\| \leqq M e^{-k(t-s)}$ and so on. The equation (1) is connected with the problem $\dot{u} = A(t, u) + f(t, u)$, where

$$p(t, u_0) = x(t, 0, u_0)\,, \quad W(t, s, u) = \frac{\partial x(t, t_0, u)}{\partial u_0} f(t, u)\,,$$

$x(t, t_0, u_0)$ is the solution of $\dot{x} = A(t, x)$ [2], and also with similar problems. The results obtained in this paper are a generalization of those already known, see for example [1, 3].

2. EXISTENCE AND UNIQUENESS

Theorem 1. *If the assumptions* A 1, A 2 *are satisfied then for any* $u_0 \in B$ *the equation* (1) *has a unique solution on* I (*and on every interval* $[0, T]$, $T > 0$).

Proof. Consider for any $T > 0$ the interval $[0, T]$ and the Banach space $C_T = C([0, T], B)$ of the continuous functions from $[0, T]$ to B with the sup-norm. Proving that for any $T > 0$ the solution (1) exists on $[0, T]$ and is unique, we also prove the same on I.

Let us fix any $u_0 \in B$ and consider on C_T an operator K which is defined by the formula

$$(K\varphi)(t) = p(t, u_0) + \int_0^t W(t, s, \varphi(s))\, ds, \quad t \in [0, T].$$

From A 1, A 2 it follows that if $\varphi \in C_T$ then $K\varphi \in C_T$, $KC_T \subset C_T$. Now we want to prove the existence of a positive integer n such that the operator K^n is a contraction. Let $\varphi, \psi \in C_T$, then for $t \in [0, T]$

$$\|(K\varphi)(t) - (K\psi)(t)\| = \left\| \int_0^t W(t, s, \varphi(s))\, ds - \int_0^t W(t, s, \psi(s))\, ds \right\| \leqq$$

$$\leqq \int_0^t L(s)\, e^{-k(t-s)} \|(\varphi(s) - \psi(s)\|\, ds.$$

Since $L(s)$, $e^{-k(t-s)}$ are continuous on Ω_T there exists such $L = \text{const}$ that for all $(t, s) \in \Omega_T$ we have $L(s)\, e^{-k(t-s)} \leqq L$, then for $t \in [0, T]$

$$\|(K\varphi)(t) - (K\psi)(t)\| \leqq \int_0^t L\|\varphi(s) - \psi(s)\|\, ds \leqq Lt\|\varphi - \psi\|_{C_T},$$

and for $n \in N$,

$$\|(K^n\varphi)(t) - (K^n\psi)(t)\| \leqq \int_0^t L\|(K^{n-1}\varphi)(s) - (K^{n-1}\psi)(s)\|\, ds.$$

It can be shown by induction that for $n \in N$ and $t \in [0, T]$

$$\|(K^n\varphi)(t) - (K^n\psi)(t)\| \leqq \frac{L^n t^n}{n!} \|\varphi - \psi\|_{C_T},$$

hence

$$\|K^n\varphi - K^n\psi\|_{C_T} \leqq \frac{(LT)^n}{n!} \|\varphi - \psi\|_{C_T}.$$

Since $(LT)^n/n! \to 0$ as $n \to \infty$, there exists such $n \in N$ that $(LT)^n/n! < 1$. For this n the operator K^n is a contraction. From a corollary to the Banach contraction theorem [4] we conclude that there exists one and only one point $u \in C_T$ such that $Ku = u$. This point $u \in C_T$ is a (unique) continuous solution of the problem considered.

3. PROPOSITIONS

Let

$$\Phi(t) =^{\mathrm{df}} -kt + \int_0^t L(s)\,\mathrm{d}s\,, \quad q(t) =^{\mathrm{df}} \|p(t, 0)\|\,,$$

$$\Phi, q \in C(I, R)\,, \quad \Phi(0) = 0\,, \quad q(0) = 0\,.$$

Proposition 1. *If the assumptions* A 1, A 2 *are satisfied then for any solution u of* (1) *and for any $t \in I$,*

$$\|u(t)\| \leqq q(t) + \left[M\|u_0\| + \int_0^t q(s)\,L(s)\,e^{-\Phi(s)}\,\mathrm{d}s\right] e^{\Phi(t)}\,.$$

Proof. For a solution u of the equation (1) we have

$$\|u(t)\| \leqq \|p(t, u_0) - p(t, 0)\| + \|p(t, 0)\| + \int_0^t \|W(t, s, u(s))\|\,\mathrm{d}s \leqq$$

$$\leqq Me^{-kt}\|u_0\| + q(t) + \int_0^t L(s)\,e^{-k(t-s)}\|u(s)\|\,\mathrm{d}s\,,$$

hence

$$\|u(t)\|\,e^{kt} \leqq M\|u_0\| + q(t)\,e^{kt} + \int_0^t L(s)\,\|u(s)\|\,e^{ks}\,\mathrm{d}s\,.$$

From this inequality we have [3]

$$\|u(t)\|\,e^{kt} \leqq M\|u_0\| + q(t)\,e^{kt} + \int_0^t L(s)\,(M\|u_0\| + q(s)\,e^{ks})\,e^{\int_s^t L(\tau)\mathrm{d}\tau}\,\mathrm{d}s =$$

$$= q(t)\,e^{kt} + M\|u_0\|\,e^{\int_0^t L(s)\mathrm{d}s} + \int_0^t L(s)\,q(s)\,e^{ks}\,e^{-\int_0^s L(\tau)\mathrm{d}\tau}\,e^{\int_0^t L(\tau)\mathrm{d}\tau}\,\mathrm{d}s$$

and hence

$$\|u(t)\| \leqq q(t) + M\|u_0\|\,e^{\Phi(t)} + \int_0^t L(s)\,q(s)\,e^{-\Phi(s)}\,\mathrm{d}s\,e^{\Phi(t)}\,.$$

Proposition 2. *If the assumptions* A 1, A 2 *are satisfied then for any solutions u, v of the equation* (1), $u(0) = u_0$, $v(0) = v_0$, *and for any $t \in I$*

$$\|u(t) - v(t)\| \leqq Me^{\Phi(t)}\|u_0 - v_0\|\,.$$

Proof. From (1), A 1, A 2 we have

$$\|u(t) - v(t)\| \leqq \|p(t, u_0) - p(t, v_0)\| + \int_0^t \|W(t, s, u(s)) - W(t, s, v(s))\|\,\mathrm{d}s \leqq$$

$$\leqq Me^{-kt}\|u_0 - v_0\| + \int_0^t L(s)\, e^{-k(t-s)}\|u(s) - v(s)\|\, ds$$

and

$$\|u(t) - v(t)\|\, e^{kt} \leqq M\|u_0 - v_0\| + \int_0^t L(s)\, \|u(s) - v(s)\|\, e^{ks}\, ds\, .$$

From Bellman-Gronwall's lemma we have

$$\|u(t) - v(t)\|\, e^{kt} \leqq M\|u_0 - v_0\|\, e^{\int_0^t L(s)ds}$$

and this proves the above proposition.

Let us introduce another assumption

A 3 $$(\exists P \in R)\,(\forall t \in I)\;\; q(t) = \|p(t, 0)\| \leqq P\, .$$

Then we have

Proposition 3. *If the assumptions* A 1, A 2, A 3 *are satisfied then for any solution* u *of* (1) *and for any* $t \in I$,

$$\|u(t)\| \leqq \left[M\|u_0\| + P + kP\int_0^t e^{-\Phi(s)}\, ds\right] e^{\Phi(t)}\, .$$

Proof. From Proposition 1 and A 3 we have

(*) $$\|u(t)\| \leqq P + \left[M\|u_0\| + P\int_0^t L(s)\, e^{ks} e^{-\int_0^s L(\tau)d\tau}\, ds\right] e^{\Phi(t)}\, .$$

Integration by parts gives

$$\int_0^t e^{ks}\, L(s)\, e^{-\int_0^s L(\tau)d\tau}\, ds = -e^{ks} e^{-\int_0^s L(\tau)d\tau}\Big|_0^t + k\int_0^t e^{ks} e^{-\int_0^s L(\tau)d\tau}\, ds =$$

$$= -e^{-\Phi(t)} + 1 + k\int_0^t e^{-\Phi(s)}\, ds$$

and the required inequality is obtained from (*).

Let us introduce an assumption

A 4 $$(\exists t_0 \in I)\,(\exists \varepsilon \in R)\,(\forall t \geqq t_0)\;\; \Phi'(t) = -k + L(t) \leqq -\varepsilon\, .$$

Notice that $k - \varepsilon \geqq L(t) \geqq 0$. We have now

Proposition 4. *If the assumptions* A 1, A 2, A 4 *are satisfied then for any solution* u *of* (1) *and for any* $t \in I$,

$$\|u(t)\| \leqq q(t) + \left[M\|u_0\| + \int_0^{t_0} q(s)\, L(s)\, e^{-\Phi(s)}\, ds\right] e^{\Phi(t)} + (k - \varepsilon)\, e^{-\varepsilon t}\int_{t_0}^t q(s)\, e^{\varepsilon s}\, ds\, .$$

Proof. From Proposition 1 we have

$$\|u(t)\| \leqq q(t) + \left[M\|u_0\| + \int_0^{t_0} q(s)\, L(s)\, e^{-\Phi(s)}\, ds\right] e^{\Phi(t)} + \int_{t_0}^t q(s)\, L(s)\, e^{\Phi(t)-\Phi(s)}\, ds\,.$$

Since for $s \in [t_0, t]$ we have $\Phi(t) - \Phi(s) = \Phi'(\theta)(t - s) \leqq -\varepsilon(t - s)$ for some $\theta \in (s, t)$, it is

$$\int_{t_0}^t q(s)\, L(s)\, e^{\Phi(t)-\Phi(s)}\, ds \leqq \int_{t_0}^t q(s)\, L(s)\, e^{-\varepsilon(t-s)}\, ds\,.$$

Taking in account that $L(s) \leqq k - \varepsilon$ for $s \geqq t_0$ we obtain the desired result.

4. BOUNDNESS AND STABILITY

Let

A 5 $$\varepsilon > 0\,,$$

where ε is from A 4.

Theorem 2. *If the assumptions* A 1 – A 5 *are satisfied then*

(i) *every solution of* (1) *is bounded,*

(ii) $(\exists N \in I)(\forall u_0 \in B)(\exists \bar{t}_0 \in I)(\forall t > \bar{t}_0)\ \|u(t)\| \leqq N$, *where u is the solution of* (1) *such that* $u(0) = u_0$,

(iii) $(\forall u_0, v_0 \in B) \lim_{t \to \infty} \|u(t) - v(t)\| = 0$, *where u, v are the solutions of* (1) *with the initial data* u_0, v_0,

(iv) *every solution of* (1) *is asymptotically stable.*

Proof. (i) Using Proposition 4 and the assumptions A 4, A 5 we have for $t \geqq t_0$

$$\|u(t)\| \leqq P + \left[M\|u_0\| + \int_0^{t_0} q(s)\, L(s)\, e^{-\Phi(s)}\, ds\right] e^{\Phi(t)} + P(k - \varepsilon)\frac{1}{\varepsilon}(1 - e^{-\varepsilon(t-t_0)}) \leqq$$

$$\leqq \frac{Pk}{\varepsilon} + \left[M\|u_0\| + \int_0^{t_0} q(s)\, L(s)\, e^{-\Phi(s)}\, ds\right] e^{\Phi(t)}\,.$$

From A 4 we have for $t \geqq t_0$ and some $\theta \in (t_0, t)$ that $\Phi(t) = \Phi(t) - \Phi(t_0) + \Phi(t_0) = \Phi'(\theta)(t - t_0) + \Phi(t_0) \leqq -\varepsilon(t - t_0) + \Phi(t_0)$. Then

$$(*) \qquad \|u(t)\| \leqq \frac{Pk}{\varepsilon} + \left[M\|u_0\| + \int_0^{t_0} q(s)\, L(s)\, e^{-\Phi(s)}\, ds\right] e^{\Phi(t_0)+\varepsilon t_0} e^{-\varepsilon t}\,.$$

Since $e^{-\varepsilon t} \leqq 1$, $(*)$ implies that $u(t)$ is bounded for $t \geqq t_0$. Since $u(t)$ is a continuous function, it is bounded on I.

(ii) Since $e^{-\varepsilon t} \to 0$ as $t \to \infty$, then for every $N = \text{const}$, $N > Pk/\varepsilon$, and every u_0 there exists $\bar{t}_0$ such that for $t > \bar{t}_0$ we have $\|u(t)\| \leqq N$.

(iii) In part (i) we obtained that $\Phi(t) \leqq -\varepsilon(t - t_0) + \Phi(t_0)$, hence A 5: $\varepsilon > 0$ implies $\Phi(t) \to -\infty$ as $t \to \infty$. We have $e^{\Phi(t)} \to 0$ when $t \to \infty$. The following assertion results from Proposition 2.

(iv) Since $e^{\Phi(t)} \to 0$ as $t \to \infty$, there exists S such that $e^{\Phi(t)} \leqq S$ for $t \in I$. For any γ denote $\delta = \gamma/SM$. If $\|u_0 - v_0\| < \delta$ then by Proposition 2 $\|u(t) - v(t)\| \leqq \leqq MS\delta = \gamma$. Every solution is stable. The asymptotical stability may be obtained from (iii).

Remarks. 1. Some properties of solutions of (1) do not require assumptions so strong as A 4, A 5. It is easy to see that for stability it is sufficient that $\Phi(t)$ be bounded (see proof of (iv)), to prove the properties (iii) and (iv) it is sufficient that $\Phi(t) \to -\infty$ as $t \to \infty$. However, this is not sufficient for boundedness. Consider for example the scalar equation

$$u = 1 - e^{-t} + u_0 e^{-t} + \int_0^t \frac{s}{s+1} e^{-(t-s)} u \, ds$$

possessing solutions of the form

$$u(t) = \frac{2u_0 - 1}{2} \frac{1}{t+1} + \tfrac{1}{2}(t + 1).$$

All these solutions tend to infinity as $t \to \infty$. In this case we have

$$p(t, u_0) = 1 - e^{-t} + u_0 e^{-t}, \quad W(t, s, u) = \frac{s}{s+1} e^{-(t-s)} u,$$

the assumptions A 1, A 2, A 3 are satisfied with $k = 1$, $M = 1$, $L(s) = s/(s + 1)$, $P = 1$. The assumption A 4 is satisfied with any $\varepsilon \leqq 0$ since $\Phi'(s) = -1 + [s/(s + 1)] = -[1/(s + 1)]$, the assumption A 5 is not satisfied; however, $\Phi(t) = -\ln(t + 1) \to -\infty$ as $t \to \infty$. Some weaker assumptions than A 4, A 5 will be given below (in parts 5 and 6).

2. If the equation (1) has at least one bounded solution and if A 1, A 2, A 3 are satisfied then: if $\Phi(t)$ is bounded then all solutions are bounded and stable, if $\Phi(t) \to -\infty$ as $t \to \infty$ then all solutions of (1) have all properties mentioned in Theorem 1. This is implied by Proposition 2. In particular, we have

Corollary 1. *If* A 1, A 2 *are satisfied*; $p(t, 0) = 0$ *and* $\Phi(t)$ *is bounded then all solutions of the equation* (1) *are bounded and stable, if moreover* $\Phi(t) \to -\infty$ *as* $t \to \infty$ *then all solutions are asymptotically stable and tend to zero as* $t \to \infty$.

Indeed, in the case considered the equation (1) has the solution $u = 0$.

Let us change the assumption A 3 to

A 3' $$q(t) = \|p(t, 0)\| \to 0 \quad \text{as} \quad t \to \infty .$$

Theorem 3. *If* A 1, A 2, A 3′, A 4, A 5 *are satisfied then the solutions of* (1) *have all properties mentioned in Theorem* 2 *and, moreover: every solution of* (1) *tends to zero as* $t \to \infty$.

Proof. By A 1 we have $p(t, 0) \in C(I, B)$. Hence A 3′ ⇒ A 3 and the solutions have the properties of Theorem 2. Applying Proposition 4 we have for $t \geqq t_0$

$$\|u(t)\| \leqq q(t) + \left[M\|u_0\| + \int_0^t q(s)\, L(s)\, e^{-\Phi(s)}\, \mathrm{d}s\right] e^{\Phi(t)} + (k - \varepsilon)\, e^{-\varepsilon t} \int_{t_0}^t q(s)\, e^{\varepsilon s}\, \mathrm{d}s\,.$$

Similarly to the proof of Theorem 2 we conclude that $\Phi(t) \to -\infty$. Two first terms on the right side of the inequality tend to zero when $t \to \infty$. Using the rule of de l'Hospital in the form of Stolz we obtain

$$\lim_{t\to\infty} \frac{\int_{t_0}^t q(s)\, e^{\varepsilon s}\, \mathrm{d}s}{e^{\varepsilon t}} = \lim_{t\to\infty} \frac{q(t)\, e^{\varepsilon t}}{\varepsilon e^{\varepsilon t}} = 0\,,$$

hence the last term tends also to zero (by A′ 3).

Remark. The scalar equation

$$u = q(t) + e^{-t} u_0 + \int_0^t \frac{s}{s+1}\, e^{-(t-s)} u\, \mathrm{d}s\,,$$

$$q(t) = \begin{cases} t & \text{for } t \in [0, 1)\,, \\ \dfrac{1}{\sqrt{t}} & \text{for } t \in [1, \infty]\,, \end{cases}$$

with the functions $p(t, u_0) = q(t) + e^{-t}u_0$, $W(t, s, u) = [s/(s + 1)]\, e^{-(t-s)}u$ fulfils the assumptions A 1, A 2 (with $M = 1$, $k = 1$, $L(s) = [s/(s + 1)]$), A 3, A 4 (with any $\varepsilon \leqq 0$), but does not satisfy A 5. In this case we have $\Phi(t) \to -\infty$ as $t \to -\infty$, but the solutions of the equation are of the type

$$u(t) = \begin{cases} \left(u_0 - \dfrac{1}{3}\right)\dfrac{1}{t+1} + \dfrac{1}{3}(t+1)^2 & \text{for } t \in [0, 1)\,, \\ \left(u_0 - \dfrac{1}{3}\right)\dfrac{1}{t+1} + \dfrac{2}{3}\dfrac{t\sqrt{t}}{t+1} + \dfrac{1}{\sqrt{t}} & \text{for } t \in [1, \infty) \end{cases}$$

and tend to infinity as $t \to \infty$.

5. PARTICULAR PERIODIC CASE

In this part we shall consider a "linear" periodic case of the problem. Let

B 1 $\qquad U \in C(\Omega, L(B, B))$,

U is an evolution operator: $(\forall t, s, \tau \in I,\ t \geqq s \geqq \tau \geqq 0)\ U(t, s)\, U(s, \tau) = U(t, \tau)$.

B 2 $$f \in C(I \times B, B)$$

and consider "the linear" case of the problem (1)

$$u = U(t, 0)\, u_0 + \int_0^t U(t, s) f(s, u)\, ds\,. \tag{1'}$$

Proposition 5. *If* B 1, B 2 *are satisfied,* $\{t_n\}$ *is a sequence from* I, $0 = t_0 < t_1 < t_2 < \ldots$, $\{u_n\}$ *is a sequence of functions* $u_n \in C([t_{n-1}, t_n], B)$, $n = 1, 2, \ldots$, *such that for* $t \in [t_{n-1}, t_n]$

$$u_n(t) = U(t, t_{n-1})\, u_{n-1}(t_{n-1}) + \int_{t_{n-1}}^t U(t, s) f(s, u_n(s))\, ds\,,$$

then the function u *composed from the functions* u_n($u = u_n$ *for* $t \in [t_{n-1}, t_n]$) *is the solution of the equation* (1') *with with* $u_0(t_0) = u_0$.

Proof. It is easy to see that the function u is continuous and that for $t \in [0, t_1]$ the assertion is satisfied. Let it be satisfied for n. We have now for $t \in [0, t_n]$

$$u(t) = U(t, 0)\, u_0 + \int_0^t U(t, s) f(s, u(s))\, ds \tag{*}$$

and for $t \in [t_n, t_{n+1}]$

$$u(t) = u_{n+1}(t) = U(t, t_n)\, u(t_n) + \int_{t_n}^t U(t, s) f(s, u(s))\, ds\,. \tag{**}$$

From the equation (*) we have

$$u(t_n) = U(t_n, 0)\, u_0 + \int_0^{t_n} U(t_n, s) f(s, u(s))\, ds\,;$$

putting $u(t_n)$ into the equation (**) for any $t \in [t_n, t_{n+1}]$ we conclude

$$u(t) = U(t, t_n)\, U(t_n, 0)\, u_0 + U(t, t_n) \int_0^{t_n} U(t_n, s) f(s, u(s))\, ds\, +$$

$$+ \int_{t_n}^t U(t, s) f(s, u(s))\, ds = U(t, 0)\, u_0 + \int_0^{t_n} U(t, s) f(s, u(s))\, ds\, +$$

$$+ \int_{t_n}^t U(t, s) f(s, u(s))\, ds = U(t, 0)\, u_0 + \int_0^t U(t, s) f(s, u(s))\, ds\,.$$

The proof is complete.

Consider the following assumption:

B 3 $$(\exists T > 0)(\forall (s, t) \in \Omega)(\forall \varphi \in B) \quad U(t + T, s + T) = U(t, s),$$
$$f(t + T, \varphi) = f(t, \varphi)$$

Notice that U has this periodicity property in particular when $U(t, s) = V(t - s)$.

Proposition 6. *If* B 1, B 2 *and* B 3 *are satisfied, if u is a solution of* (1′) *on* $[0, T]$ *then the function v defined on* $[pT, (p + 1) T]$, $p = 1, 2, 3, \ldots$, *by the formula* $v(t) = u(t - pT)$ *is a solution of the equation*

$$u = U(t, pT) u_0 + \int_{pT}^{t} U(t, s) f(s, u) \, ds, \quad t \in [pT, (p + 1) T].$$

Proof. As u is a solution of (1′) for $t \in [0, T]$,

$$u(t) = U(t, 0) u_0 + \int_0^t U(t, s) f(s, u(s)) \, ds$$

and hence for $t \in [pT, (p + 1) T]$

$$v(t) = u(t - pT) = U(t - pT, 0) u_0 + \int_0^{t-pT} U(t - pT), s) f(s, u(s)) \, ds.$$

After changing the integration variable ($s = \tau - pT$) we have

$$v(t) = U(t - pT, 0) u_0 + \int_{pT}^{t} U(t - pT, \tau - pT) f(\tau - pt, u(\tau - pT)) \, d\tau,$$

and taking into account B 3 and the definition of v we obtain

$$v(t) = U(t, pT) u_0 + \int_{pT}^{t} U(t, \tau) f(\tau, v(\tau)) \, d\tau.$$

Corollary 2. *If* B 1, B 2 *and* B 3 *are satisfied, if u is a solution of* (1′) *on* $[0, T]$ *such that* $u(T) = u(0) = u_0$, *then the periodic prolongation v of u on I is a solution of* (1′).

Indeed, v is continuous, $v(pT) = u_0$ for every $p = 1, 2, \ldots$, by Proposition 6 v is a solution of

$$v = U(t, pT) v(pT) + \int_{pT}^{t} U(t, \tau) f(\tau, v) \, d\tau$$

on $[pT, (p + 1) T]$, and Proposition 5 completes the proof.

Proposition 7. *If* B 1, B 2 *and* B 3 *are satisfied, if u is a solution of* (1′) *on I, then the function v defined on* $[0, T]$ *by the formulae* $v(t) = u(t + pT)$, $p \in N$ *is the solution of* (1′) *with initial value* $v(0) = u(pT)$.

Proof. Since u is a solution of $(1')$ we have

$$u(pT) = U(pT, 0)\, u_0 + \int_0^{pT} U(pT, s) f(s, u(s))\, \mathrm{d}s$$

and B 3, B 1 imply that

$$U(t, 0)\, u(pT) = U(t + pT, pT)\, u(pT) = U(t + pT, 0)\, u_0 + \\ + \int_0^{pT} U(t + pT, s) f(s, u(s))\, \mathrm{d}s\,.$$

Since u is a solution of $(1')$ we have

$$v(t) = u(t + pT) = U(t + pT, 0)\, u_0 + \int_0^{t+pT} U(t + pT, s) f(s, u(s))\, \mathrm{d}s = \\ = U(t + pT, 0)\, u_0 + \int_0^{pT} U(t + pT, s) f(s, u(s))\, \mathrm{d}s + \\ + \int_{pT}^{t+pT} U(T + pT, s) f(s, u(s))\, \mathrm{d}s = \\ = U(t, 0)\, u(pT) + \int_0^t U(t + pT, \tau + pT) f(\tau + pT, u(\tau + pT))\, \mathrm{d}\tau$$

and

$$v(t) = U(t, 0)\, u(pT) + \int_0^t U(t, \tau) f(\tau, v(\tau))\, \mathrm{d}\tau$$

which proves the proposition.

Introduce the following assumptions:

B 4 $(\exists M, k = \text{const})\, (\forall (t, s) \in \Omega) \quad \|U(t, s)\| \leqq M e^{-k(t-s)}$;

B 5 $(\exists R \in C(I, I))\, (\forall t \in I)\, (\forall \varphi, \psi \in B) \quad \|f(t, \varphi) - f(t, \psi)\| \leqq R(t)\, \|\varphi - \psi\|$,
$R(t + T) = R(t)$;

B 6 $$-kT + \int_0^T M\, R(s)\, \mathrm{d}s = \int_0^T (-k + MR(s))\, \mathrm{d}s < 0\,.$$

Theorem 4. *If B 1 – B 6 are satisfied, then the equation $(1')$ has a unique periodic solution, its period is T, and all solutions of $(1')$ have the properties mentioned in Theorem 2.*

Notice that in this case all solutions of $(1')$ tend to the periodic one as $t \to \infty$.

Proof. 1. The equation (1′) can be written in the form of the equation (1) after the following transformations

$$u = U(t, 0)\, u_0 + \int_0^t U(t, s) f(s, 0)\, ds + \int_0^t U(t, s) [f(s, u) - f(s, 0)]\, ds$$

and definitions:

$$p(t, u_0) =^{df} U(t, 0)\, u_0 + \int_0^t U(t, s) f(s, 0)\, ds\,,$$

$$W(t, s, u) =^{df} U(t, s) [f(s, u) - f(s, 0)]\,.$$

In this case we have the following implications: B 1, B 2, B 4 $\Rightarrow$ A 1, B 1, B 2, B 4, B 5 $\Rightarrow$ A 2 with $L(s) = MR(s)$. Then in the case considered, Propositions 1 and 2 hold.

2. Since $R(s)$ is periodic, $-k + MR(s)$ is also periodic and for every positive integer p we have $\int_{pT}^{(p+1)T} (-k + MR(s))\, ds = \int_0^T (-k + MR(s))\, ds$, thus $\int_0^{pT} (-k + MR(s))\, ds = p \int_0^T (-k + MR(s))\, ds$. Defining $\Phi(t) = \int_0^t (-k + MR(s))\, ds$ we have $\Phi(pT) = p\, \Phi(T)$. In virtue of $\Phi(T) < 0$ (by B 6) there exists a positive integer p such that $\Phi(pT) = p\, \Phi(t) < -\ln M$; let us fix this p.

3. Consider in the space B the operator K of translation along the solution of (1′) from $t = 0$ to $t = T$. It seems that if $u_0 \in B$, u is a solution of (1′) such that $u(0) = u_0$, then $Ku_0 = u(T)$. It is evident from Theorem 1 that the domain of the operator K is B and that $KB \subset B$. Consider the iterations $K^2, K^3, \ldots, K^p$ of the operator K. Let $v_0 \in B$, v is such solution of (1′) that $v(0) = v_0$; then $Kv_0 = v(T)$, $K^2 v_0 = K(Kv_0) = K\, v(T)$, which is the value at $t = T$ of the solution of (1′) starting from $v(T)$ at $t = 0$. Proposition 7 implies that this solution can be obtained from the solution v by its translation from $[T, 2T]$ to $[0, T]$. Then $K^2 v_0 = K\, v(T) = v(2T)$ and so on. By induction we have that $K^p v_0 = v(pT)$. We want to prove that the operator K^p is a contraction.

4. From Proposition 2 we have for any solutions u, v with the initial data u_0, v_0 that

$$\|u(t) - v(t)\| \leqq M \|u_0 - v_0\|\, e^{\Phi(t)}$$

and

$$\|K^p u_0 - K^p v_0\| = \|u(pT) - v(pT)\| \leqq M \|u_0 - v_0\|\, e^{\Phi(pT)} = \alpha \|u_0 - v_0\|\,,$$

where $\alpha = Me^{\Phi(pT)} < Me^{-\ln M} = 1$, K^p is a contraction. Hence there exists a unique point $w_0 \in B$ such that $Kw_0 = w_0$. Denoting the corresponding solution of (1′) by w ($w(0) = w_0$) we have $w(T) = w(0)$. In virtue of the uniqueness of the point w_0 and the uniqueness of solutions of (1′) it follows that the equation (1′) has at most one periodic solution with period T. Existence of that solution follows immediately from Corollary 1 (it has the initial point w_0).

5. Consider the behaviour of the function $\Phi(t) = -kt + \int_0^t MR(s)\,ds$ as $t \to \infty$. Defining

$$A = \frac{M}{T}\int_0^T R(s)\,ds\,, \quad \psi(t) = MR(t) - A$$

we obtain that $\psi(t)$ is a T-periodic function and $\int_0^T \psi(t)\,dt = 0$. Then $\Phi(t) = (-k + A)\,t + \int_0^t \psi(s)\,ds$, $\Phi(T) = (-k + A)\,T$. B 6 implies that $-k + A < 0$. Since ψ is continuous on $[0, T]$, there exists a constant C that for $t \in [0, T]$ we have $\int_0^t \psi(s)\,ds \leqq C$; as ψ is continuous on I and $\int_0^T \psi(t)\,dt = 0$, it holds $\int_0^t \psi(s)\,ds \leqq C$ for all $t \in I$. Finally $\Phi(t) \leqq (-k + A)\,t + C$, $-k + A < 0$ and $\Phi(t) \to -\infty$ as $t \to \infty$.

6. Let u be any solution of (1′), let w be the periodic one. Proposition 2 yields for $t \in I$:

$$\|u(t) - w(t)\| \leqq M\|u_0 - w_0\|\, e^{\Phi(t)}$$

and

$$(*) \qquad \|u(t)\| \leqq \|u(t) - w(t)\| + \|w(t)\| \leqq M\|u_0 - w_0\|\, e^{\Phi(t)} + \|w(t)\|\,.$$

Since w is a periodic solution it is bounded and since $\Phi(t) \to -\infty$ as $t \to \infty$ (hence $e^{-\Phi(t)}$ is bounded) we obtain that u is bounded.

7. Let $w = \max_I \|w(t)\|$, and let R be any constant such that $R > w$. From $(*)$ it results that for this R and any u_0 there exists such $t_0 \geqq 0$ that for any $t \geqq t_0$ we have $\|u(t)\| \leqq R$ (because $e^{\Phi(t)} \to 0$ as $t \to \infty$).

8. Proposition 2 implies that for any two solutions u, v we have $\|u(t) - v(t)\| \to 0$ as $t \to \infty$. If we take $v = w$ – the periodic solution, we obtain that all solutions tend to the periodic one as $t \to \infty$. Hence the equation (1′) has only one periodic solution (with period T).

9. For any $\varepsilon > 0$ and $\delta = \varepsilon/M \max_I e^{\Phi(t)}$ (this $\max_I$ exists because $\Phi(t) \to -\infty$ as $t \to \infty$ and Φ is continuous) let $u_0, v_0 \in B$ be such that $\|u_0 - v_0\| < \delta$. Then Proposition 2 yields for $t \geqq 0$ that $\|u(t) - v(t)\| \leqq M\|u_0 - v_0\|\, e^{\Phi(t)} < M \max_I e^{\Phi(t)}\delta = \varepsilon$. Any solution of (1′) is stable. Asymptotic stability results from 8.

6. THE CASE $L \in \mathscr{L}^p(0, \infty)$, $p \geqq 1$

Now we turn back to the general "nonlinear" case. Assume that

A 6 $$k > 0\,,$$

A 7 $$(\exists p \geqq 1) \quad \int_0^\infty L^p(s)\,ds < \infty\,.$$

Define (for such p)

$$N = \left(\int_0^\infty L^p(s)\,\mathrm{d}s\right)^{1/p}.$$

Theorem 5. *If the assumptions* A 1, A 2, A 3, A 6, A 7 *are satisfied then the solutions of* (1) *have all properties mentioned in Theorem* 2.

Proof. Let $\tau \in [0, t]$, consider $\Phi(t) - \Phi(\tau) = -k(t - \tau) + \int_\tau^t ML(s)\,\mathrm{d}s$. If $p = 1$ then $\Phi(t) - \Phi(\tau) \leqq -k(t - \tau) + MN$, if $p > 1$ then for $\varrho = (p - 1)/p$ we have

$$\Phi(t) - \Phi(\tau) \leqq -k(t - \tau) + \left(\int_\tau^t M^{1/\varrho}\,\mathrm{d}s\right)^\varrho \left(\int_\tau^t L^p(s)\,\mathrm{d}s\right)^{1/p} \leqq$$

$$\leqq -k(t - \tau) + MN(t - \tau)^\varrho .$$

Then for any $p \geqq 1$ and $\tau \in [0, t]$

$$(*) \qquad \Phi(t) - \Phi(\tau) \leqq -k(t - \tau) + MN(t - \tau)^\varrho, \quad \varrho = 1 - \frac{1}{p} \in [0, 1)$$

and in particular (for $\tau = 0$)

$$(**) \qquad \Phi(t) \leqq -kt + MNt^\varrho .$$

Let $t_0 =^{\mathrm{df}} (2NM/k)^p$, then for $t \geqq t_0$ we have $t^{1/p} \geqq 2NM/k$, $NMt^\varrho \leqq (k/2)\, t^{\varrho + 1/p} = (k/2)\, t$. From $(**)$ we obtain

$$(***) \qquad t \geqq t_0 \Rightarrow \Phi(t) \leqq -kt + MNt^\varrho \leqq -\frac{k}{2}\, t$$

and then $\Phi(t) \to -\infty$ as $t \to \infty$ (from A 6 $k > 0$). From Proposition 3 and $(*)$ we have for $t \in I$

$$\|u(t)\| \leqq [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^t e^{\Phi(t) - \Phi(s)}\,\mathrm{d}s \leqq$$

$$\leqq [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^t \exp\left(-k(t - s) + MN(t - s)^\varrho\right) \mathrm{d}s .$$

Change the variable in the last integral ($\tau = t - s$). Then we have for $t \in I$

$$\|u(t)\| \leqq [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^t \exp\left(-k\tau + MN\tau^\varrho\right) \mathrm{d}\tau$$

and for $t \geqq t_0$

$$\|u(t)\| \leqq [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^{t_0} \exp\left(-k\tau + MN\tau^\varrho\right) \mathrm{d}\tau +$$

$$+ kP \int_{t_0}^t \exp\left(-k\tau + MN\tau^\varrho\right) \mathrm{d}\tau .$$

From (***) we have

$$\|u(t)\| \leqq [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^{t_0} \exp\left(-k\tau + MN\tau^{\varrho}\right) \mathrm{d}\tau +$$

$$+ kP \int_{t_0}^{t} \exp\left(-\frac{k}{2}\tau\right) \mathrm{d}\tau =$$

$$= [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^{t_0} \exp\left(-k\tau + MN\tau^{\varrho}\right) \mathrm{d}\tau + 2P\left(e^{-(k/2)t_0} - e^{-(k/2)t}\right)$$

and

$$\|u(t)\| \leqq [M\|u_0\| + P]\, e^{\Phi(t)} + kP \int_0^{t_0} \exp\left(-k\tau + MN\tau^{\varrho}\right) \mathrm{d}\tau + 2Pe^{-(k/2)t_0}$$

for $t \geqq t_0$. It is evident that this inequality holds also for $t \in [0, t_0]$. Hence it is satisfied for $t \in I$. Since $\Phi(t) \to -\infty$ as $t \to +\infty$, the last inequality implies that every solution of (1) is bounded and that for any $\tilde{N} = \text{const} > kP \int_0^{t_0} \exp\left(-k\tau + MN\tau^{\varrho}\right) \mathrm{d}\tau + 2Pe^{-(k/2)t_0}$ there exists $\bar{t}_0$ such that for $t \geqq \bar{t}_0$ we have $\|u(t)\| \leqq \tilde{N}$ ($\bar{t}_0$ depends on u_0). The other properties follow from Proposition 2 (similarly as in the proof of Theorem 2).

Theorem 6. *If the assumptions* A 1, A 2, A 3′, A 6, A 7 *are satisfied then the solutions of* (1) *have the properties described in Theorem* 5 *and tend to zero as* $t \to \infty$.

Proof. The first part of the assertion results immediately from Theorem 5 because A 1, A 3′ ⇒ A 3. From Proposition 1 we have

$$\|u(t)\| \leqq q(t) + M\|u_0\|\, e^{\Phi(t)} + \int_0^t q(s)\, L(s)\, e^{\Phi(t)-\Phi(s)}\, \mathrm{d}s\,.$$

As in the previous proof $\Phi(t) \to -\infty$ as $t \to \infty$, $q(t) \to 0$ as $t \to \infty$ by A 3′, to complete the proof of the theorem we have to show that

$$I = \int_0^t q(s)\, L(s)\, e^{\Phi(t)-\Phi(s)}\, \mathrm{d}s$$

tends to zero as $t \to \infty$. We have

$$I \leqq \left(\int_0^t [q(s)\, e^{\Phi(t)-\Phi(s)}]^{1/\varrho}\, \mathrm{d}s\right)^{\varrho} \left(\int_0^t L^p(s)\, \mathrm{d}s\right)^{1/p} \leqq$$

$$\leqq N \left(\int_0^t (q(s))^{1/\varrho}\, e^{(1/\varrho)(\Phi(t)-\Phi(s))}\, \mathrm{d}s\right)^{\varrho}$$

and as in the previous proof

$$\left(\frac{I}{N}\right)^{1/\varrho} \leqq \int_0^t (q(s))^{1/\varrho} \exp\left(-(k/\varrho)(t-s) + (MN/\varrho)(t-s)^\varrho\right) \mathrm{d}s\,.$$

After changing the integration variable $(s = t - \tau)$ we obtain

$$\left(\frac{I}{N}\right)^{1/\varrho} \leqq \int_0^t [q(t-\tau)]^{1/\varrho} \exp\left[(1/\varrho)(-k\tau + MN\tau^\varrho)\right] \mathrm{d}\tau =$$

$$= \int_0^{t_0} [q(t-\tau)]^{1/\varrho} \exp\left[(1/\varrho)(-k\tau + MN\tau^\varrho)\right] \mathrm{d}\tau +$$

$$+ \int_{t_0}^t [q(t-\tau)]^{1/\varrho} \exp\left[(1/\varrho)(-k\tau + MN\tau^\varrho)\right] \mathrm{d}\tau\,,$$

where t_0 is the same as in the proof of Theorem 5. In the first integral we have $(k > 0)$

$$I_1 = \int_0^{t_0} [q(t-\tau)]^{1/\varrho} \exp\left[(1/\varrho)(-k\tau + MN\tau^\varrho)\right] \mathrm{d}\tau \leqq$$

$$\leqq \exp\left[(1/\varrho)\, MNt_0^\varrho\right] \int_0^{t_0} [q(t-\tau)]^{1/\varrho} \mathrm{d}\tau\,.$$

Take any $\varepsilon > 0$ and $\eta = \varepsilon^\varrho / t_0^\varrho \exp MNt_0^\varrho$. Then to this $\eta > 0$ exists $T \geqq t_0$ such that $q(t) < \eta$ for $t > T - t_0$ ($q(t) \to 0$ as $t \to \infty$). Then for $t > T$ and $s \in [0, t_0]$ we have $t - s > T - t_0$ and $q(t-s) < \eta$. Finally, for any $\varepsilon > 0$ there exists such T that $I_1 \leqq \exp\left[(1/\varrho)\, MNt_0^\varrho\right] \eta^{(1/\varrho)} \int_0^{t_0} \mathrm{d}\tau = \varepsilon$ for $t > T$. Hence $I_1 \to 0$ as $t \to \infty$. Consider the other integral satisfies

$$I_2 = \int_{t_0}^t [q(t-\tau)]^{1/\varrho} \exp\left[\frac{1}{\varrho}(-k\tau + MN\tau^\varrho)\right] \mathrm{d}\tau \leqq \int_{t_0}^t [q(t-\tau)]^{1/\varrho} e^{(-k/2\varrho)\tau} \mathrm{d}\tau$$

and hence, for $s = t - \tau + t_0$

$$I_2 \leqq \int_{t_0}^t [q(s-t_0)]^{1/\varrho} e^{-(k/2\varrho)(t_0+t-s)} \mathrm{d}s = e^{-(k/2\varrho)t_0} \frac{\int_{t_0}^t [q(s-t_0)]^{1/\varrho} e^{(k/2\varrho)s} \mathrm{d}s}{e^{(k/2\varrho)t}}\,.$$

From the rule of de l'Hospital (in Stolz's form, $k > 0$) we have

$$\lim_{t\to\infty} I_2 \leqq e^{-(k/2\varrho)t_0} \lim_{t\to\infty} \frac{[q(t-t_0)]^{1/\varrho} e^{(k/2\varrho)t}}{\dfrac{k}{2\varrho} e^{(k/2\varrho)t}} = 0$$

ebcause $q(t) \to 0$ as $t \to \infty$ Then $I_1, I_2 \to 0$ as $t \to \infty$ and $(I/N)^{1/\varrho} \leqq I_1 + I_2$ tends to zero as $t \to \infty$. Finally, $I \to 0$ as $t \to \infty$ and the proof is complete.

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