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ON THE EXISTENCE OF PERIODIC BOUNDARY CONDITIONS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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In this note, we are concerned with the following non-linear second order differential equation

$$(1) \qquad x'' + Kx = f(t, x, x'),$$

$$(2) \qquad x(0) = x(\omega), \quad x'(0) = x'(\omega).$$

It shall be shown that if certain conditions are imposed on the function f, then there exists a unique solution of (1) satisfying the boundary conditions (2). In so doing we shall use a result reported by Ďurikovič in [1].

If in addition to our assumptions given below, the function f is ω-periodic function of t, i.e.

$$f(t + \omega, x, x') = f(t, x, x')$$

then the result of our paper gives an ω-periodic solution (see [3]).

In the sequel it is assumed that:

(A_1) $f(t, x, x')$ is a real-valued, continuous, bounded function with the domain $E = [0, T] \times R^2$, $T > 0$,

(A_2) $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqq K_2^r\{|x_1 - x_2| + 1/\sqrt{|K|}\,|y_1 - y_2|\}$, $K_2 > 0$,

$$\frac{2K_2}{|K|} < 1.$$

For $\omega \in (0, \pi/\sqrt{K})$, $K > 0$, let $G(t, s)$ be the Green's function

$$
(3) \qquad G(t,s) = \begin{cases} \dfrac{1}{2\sqrt{K}} \cdot \dfrac{\cos \sqrt{K}\left(\dfrac{\omega}{2} + s - t\right)}{\sin \sqrt{K}\,\dfrac{\omega}{2}} & \text{for} \quad 0 \leqq s \leqq t \leqq \omega, \\[5mm] \dfrac{1}{2\sqrt{K}} \cdot \dfrac{\cos \sqrt{K}\left(\dfrac{\omega}{2} + t - s\right)}{\sin \sqrt{K}\,\dfrac{\omega}{2}} & \text{for} \quad 0 \leqq t \leqq s \leqq \omega, \end{cases}
$$

and, $\omega \in (0, +\infty)$, $K < 0$, let $G(t,s)$ be defined as

$$
(4) \qquad G(t,s) = \begin{cases} \dfrac{\exp\left[-\sqrt{|K|}\,(t-s)\right]\cdot\exp\left[\sqrt{|K|}\,w\right] + \exp\left[\sqrt{|K|}\,(t-s)\right]}{2\sqrt{|K|}\left[1 - \exp\sqrt{|K|}\,\omega\right]} & \text{for} \quad s \leqq t, \\[5mm] \dfrac{\exp\left[-\sqrt{|K|}\,(s-t)\right]\cdot\exp\left[\sqrt{|K|}\,\omega\right] + \exp\left[\sqrt{|K|}\,(s-t)\right]}{2\sqrt{|K|}\left[1 - \exp\sqrt{|K|}\,\omega\right]} & \text{for} \quad s \geqq t \end{cases}
$$

and $K > 0$,

$$
(5) \qquad G_t(t,s) = \begin{cases} \dfrac{1}{2} \dfrac{\sin \sqrt{K}\left(\dfrac{\omega}{2} + s - t\right)}{\sin \sqrt{K}\,\dfrac{\omega}{2}} & \text{for} \quad 0 \leqq s \leqq t \leqq \omega, \\[5mm] -\dfrac{1}{2} \dfrac{\sin \sqrt{K}\left(\dfrac{\omega}{2} + t - s\right)}{\sin \sqrt{K}\,\dfrac{\omega}{2}} & \text{for} \quad 0 \leqq t \leqq s \leqq \omega, \end{cases}
$$

$K < 0$,

$$
(6) \qquad G_t(t,s) = \begin{cases} \dfrac{\exp\left[\sqrt{|K|}\,(t-s)\right] - \exp\left[-\sqrt{|K|}\,(t-s)\right]\cdot\exp\left[\sqrt{|K|}\,\omega\right]}{2\left[1 - \exp\sqrt{|K|}\,\omega\right]} & \text{for} \quad s \leqq t, \\[5mm] \dfrac{-\exp\left[\sqrt{|K|}\,(s-t)\right] + \exp\left[-\sqrt{|K|}\,(s-t)\right]\cdot\exp\left[\sqrt{|K|}\,\omega\right]}{2\left[1 - \exp\sqrt{|K|}\,\omega\right]} & \text{for} \quad s \geqq t. \end{cases}
$$

It is easy to see that

$$
\int_0^\omega |G(t,s)|\, ds \leqq \frac{1}{|K|},
$$

$$
\int_0^\omega |G_t(t,s)|\, ds \leqq \frac{1}{\sqrt{|K|}}.
$$

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Then, the equation (1) together with boundary conditions (2) is equivalent to solving the integrodifferential equation

$$(7) \qquad x(t) = \int_0^\omega G(t, s) f(s, x(s), x'(s)) \, ds \, .$$

With respect to (7), the sequence of Piccard's successive approximations $\{x_n(t)\}_1^\infty$ is defined by the equation

$$(8) \qquad x_n(t) = \int_0^\omega G(t, s) f(s, x_{n-1}(s), x'_{n-1}(s)) \, ds$$

and the sequence of derivatives is determined by

$$(9) \qquad x'_n(t) = \int_0^\omega G_t(t, s) f(s, x_{n-1}(s), x'_{n-1}(s)) \, ds \, .$$

Theorem 1. *Let $f(t, x, x')$ satisfy the assumptions $(A_1)-(A_2)$. Then there exists one and only one solution $x(t)$ of the problem (1) satisfying boundary conditions (2). Moreover Piccard's sequence of successive appriximations which is defined by (8) for any function $x_0(t)$ specified below converges uniformly to $x(t)$.*

Proof. We consider the space S of all continuous function with continuous first derivative on $[0, T]$, $[0, \omega] \subset [0, T]$. Let us define the distance

$$(10) \qquad \max_{t \in [0,T]} \frac{|x_1(t) - x_2(t)| + \dfrac{1}{\sqrt{|K|}} |x'_1(t) - x'_2(t)|}{|K|^p} \, , \quad p \in R$$

for arbitrary pair of elements $x_1(t), x_2(t) \in S$. It is obvious that the set S, on which the distance is defined by equality (10) is a complete metric space X (see [1]). Let us define an operator U on X as

$$(11) \qquad U x(t) = \int_0^\omega G(t, s) f(s, x(s), x'(s)) \, ds \, .$$

The operator U maps the space X into itself. Moreover

$$\frac{d}{dt} U x(t) = \int_0^\omega G_t(t, s) f(s, x(s), x'(s)) \, ds \, .$$

To complete the proof, we have to show that all the hypothesis of Luxemburg's Theorem 1, [1] are satisfied. Proof of the property 1°. Let $x_1(t), x_2(t)$ be two arbitrary functions from the space X, then

$$|U x_1(t) - U x_2(t)| \leq \int_0^\omega |G(t, s)| \, |f(s, x_1(s), x'_1(s)) - f(s, x_2(s), x'_2(s))| \, ds \leq$$

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$$\leq K_2 \int_0^\omega |G(t, s)| \frac{|x_1(s) - x_2(s)| + \frac{1}{\sqrt{|K|}} |x_1'(s) - x_2'(s)|}{|K|^p} |K|^p \, ds \leq$$

$$\leq \frac{K_2}{|K|} d(x_1, x_2) |K|^p \, ,$$

and

$$\frac{1}{\sqrt{|K|}} \left| \frac{d}{dt} U x_1(t) - \frac{d}{dt} U x_2(t) \right| \leq$$

$$\leq \frac{K_2}{\sqrt{|K|}} \int_0^\omega |G_t(t, s)| \frac{|x_1(s) - x_2(s)| + \frac{1}{\sqrt{|K|}} |x_1'(s) - x_2'(s)|}{|K|^p} \leq$$

$$\leq \frac{K_2}{|K|} d(x_1(t), x_2(t)) \cdot |K|^p \, .$$

From the given inequalities, it follows

$$d(U x_1(t), U x_2(t)) \leq \frac{2K_2}{|K|} d(x_1(t), x_2(t)) \, .$$

Thus, the first condition of Theorem 1 is proved. It follows from the definition of the metric (10) that the arbitrary two elements of S have a finite distance. So conditions 2° and 3° of Theorem 1, [1] is obvious. Thereby, we have proved the existence and the uniqueness of the solution of the integro-differential equation (3), and the uniform convergence of successive approximation (8) to this solution for any function $x_0(t) \in S$.

In the following two theorems, we shall assume $K > 0$. It is easy to see that $G(t, s)$ is nonnegative in $0 \leq s \leq t \leq \omega$ and in $0 \leq t \leq s \leq \omega$. Since $\omega \in (0, \pi/\sqrt{K})$, and hence

$$\sin \sqrt{K} \frac{\omega}{2} \geq \frac{2}{\pi} \sqrt{K} \frac{\omega}{2}$$

which implies

$$G(t, s) \leq \frac{\pi}{2K\omega}$$

and

$$|G_t(t, s)| \leq \frac{\pi}{2\sqrt{(K)} \, \omega} \, .$$

Now, let us assume that U is the operator defined by (11) and US is the set of all images S under the mapping U. If we denote the complete metric space which was

obtained by the completion of the metric space $[US, d_2]$ in the sence of the distance

$$d_2(z_1(t), z_2(t)) = \underset{[0,T]}{\text{Max}} \left(|z_1(t) - z_2(t)| + \frac{1}{\sqrt{K}} |z_1'(t) - z_1'(t)| \right)$$

by $[S^*, d_2]$, then the following theorems hold.

Theorem 2. *Let $f(t, x, y)$ be a function defined and continuous on E, and let it fulfil the following conditions*

(A_3) $\qquad\qquad |f(t, x, y)| \leqq \dfrac{K}{2\pi} t^p, \quad p \geqq 0, \quad (t, x, y) \in E,$

(A_4) $\quad |f(t, x_1, y_1) - f(t, x_2, y_2)| < \dfrac{K}{2\pi t^r} \left(|x_1 - x_2|^q + \left[\dfrac{1}{\sqrt{K}} |y_1 - y_2| \right]^q \right),$

for $(t, x_i, y_i) \in E,$ $i = 1, 2$ *where* $q \geqq 1, 0 < r < 1, r = p(q - 1)$
and

$$\frac{1}{(1 - r)} \left(\frac{1}{p + 1} \right)^{q-1} < 1 .$$

Then there exists one and only one solution $x(t) \in S^*$ of problem (1), (2). And moreover the sequence of Piccord's approximations defined by (8) for any $x_0(t) \in S$ in $[0, T]$ converges uniformly on $[0, T]$ to this unique solution.

Proof. The proof will be given similarly as that of Theorem 1. The set S^* is a subset of the set S. On the set we can define the distance

(12) $\qquad d(z_1(t), z_2(t)) = \underset{t \in [0,T]}{\max} \dfrac{|z_1(t) - z_2(t)| + \dfrac{1}{K} |z_1'(t) - z_2'(t)|}{\omega^p} .$

Again the set S^*, on which the distance is defined by equality (12) is a complete metric space X (see [1]). The operator U by (11) maps X into itself. To complete the proof we have to show all conditions of Luxemburg's theorem are satisfied. The proof of condition 1°. Let $z_1(t), z_2(t)$ be two arbitrary elements of X. Then from (11) and (A_3), we obtain

$$|z_1(t) - z_2(t)| \leqq \frac{K}{\pi} \int_0^\omega |G(t, s)| s^p \, ds < \frac{1}{2(p + 1)} \omega^p$$

and

$$\frac{1}{\sqrt{K}} |z_1'(t) - z_2'(t)| \leqq \frac{K}{\pi \sqrt{K}} \int_0^\omega |G_t(t, s)| s^p \, ds < \frac{1}{2(p + 1)} \omega^p .$$

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Moreover by (A_4) and (12), we obtain

$$|U z_1(t) - U z_2(t)| \leqq \frac{K}{\pi} \int_0^\omega \frac{|z_1(s) - z_2(s)|^q + \left[\frac{1}{\sqrt{K}}|z_1'(s) - z_2'(s)|\right]^q}{s^r} G(t, s)\, ds \leqq$$

$$\leqq \frac{K}{\pi} \omega^p \left(\frac{\omega^p}{p + 1}\right)^{q-1} \cdot \frac{\pi}{2K\omega} \int_0^\omega \frac{|z_1(s) - z_2(s)| + \left|\frac{1}{\sqrt{K}}\right||z_1'(s) - z_2'(s)|}{\omega^p} \frac{ds}{s^r} \leqq$$

$$\leqq \frac{\omega^p}{2} \cdot \frac{1}{(p + 1)^{q-1}} \cdot \frac{d(z_1(t), z_2(t))}{1 - r}$$

and similarly

$$\frac{1}{\sqrt{K}} \left|\frac{d}{dt} U z_1(t) - \frac{d}{dt} U z_2(t)\right| \leqq \frac{\omega^p}{2} \cdot \frac{1}{(p + 1)^{q-1}} \cdot \frac{d(z_1(t), z_2(t))}{1 - r} .$$

From the last two inequalities it follows that

$$d(U z_1(t), U z_2(t)) \leqq \frac{1}{(p + 1)^{q-1}} \cdot \frac{1}{1 - r} d(z_1(t), z_2(t)) .$$

Hence Condition $1°$ is proven. Condition $2°$ follows directly from (12) because we have $d(x_n, x_{n+1}) \leqq (p + 1)^{-1} < \infty$, for $n = 1, 2, \ldots$. From (12) we also obtain Condition $(3°)$ too.

Remark. The assumption (A_3) of Theorem (2) guarantees the boundedness of the function $f(t, x, y)$ in E. In the following theorem we shall show that this assumption is not necessary.

Theorem 3. *Let the function $f(t, x, y)$ be continuous in E and let it satisfy the following conditions*

(A_5) $\qquad |f(t, x, y)| \leqq \frac{K}{\pi} t^{-p}, \quad 0 < p < 1 \quad \text{for all} \quad (t, x, y) \in E .$

(A_6) $\quad |f(t, x_1, y_1) - f(t, x_2, y_2)| \leqq \frac{K}{\pi} t^{p(q-1)} \left\{z_1 - x_2|^q + \left[\frac{1}{\sqrt{K}}|y_1 - y_2|\right]^q\right\}$

and

$$\left(\frac{1}{1 - p}\right)^{q-1} \cdot \frac{1}{p(q - 1) + 1} < 1 .$$

Then, there exists one and only one solution of problem (1) *satisfying boundary conditions* (2), *and moreover the sequence of Piccard's approximations defined by* (8) *for arbitrary function $x_0(t) \in S$ in $t \in [0, T]$, converges uniformly on $[0, T]$ to this unique solution.*

Proof. The operator U is defined by relation (11) as in proceeding theorems.

Anologically to Theorem 2, it can be shown that the set S^* with the distance

$$d(z_1(t), z_2(t)) = \max_{t \in [0,T]} \frac{|z_1(t) - z_2(t)| + \dfrac{1}{\sqrt{K}} |z_1'(t) - z_2'(t)|}{\omega^{-p}}$$

is a complete metric space. Let us denote it by X, it follows

$$|z_1(t) - z_2(t)| \leq \frac{2K}{\pi} \int_0^\omega G(t, s) \, s^{-p} \, ds \leq \frac{1}{1-p} \, \omega^{-p} \,,$$

$$\frac{1}{\sqrt{K}} |z_1'(t) - z_2'(t)| \leq \frac{2K}{\pi \sqrt{K}} \int_0^\omega |G_t(t, s)| \, s^{-p} \, ds \leq \frac{1}{1-p} \, \omega^{-p} \,.$$

From the obove inequalities and assumption (A_6) we obtain the estimates

$$|U z_1(t) - U z_2(t)| \leq \frac{1}{2(1-p)^{q-1}} \cdot \frac{1}{p(q-1)+1} \, \omega^{-p} \, d(z_1(t), z_2(t)) \,,$$

and

$$\frac{1}{\sqrt{K}} \left| \frac{d}{dt} U z_1(t) - \frac{d}{dt} U z_2(t) \right| < \frac{1}{2(1-p)^{q-1}} \cdot \frac{1}{p(q-1)+1} \, \omega^{-p} \, d(z_1(t), z_2(t))$$

for $t \in [0, T]$, hence we have

$$d(U z_1(t), U z_2(t)) \leq \frac{1}{(1-p)^{q-1}} \cdot \frac{1}{p(q-1)+1} \, d(z_1(t), z_2(t)) \,.$$

Condition 1° of Theorem 1 of [1] is proved. With a similar estimations we can show that Conditions (2°) and (3°) are satisfied, hence the theorem is proven.

References

[1] *V. Ďurikovič:* On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem for certain differential equations of the type $U_{xy} = f(x, y, u, U_x, U_y)$. Spisy přírodov. fak. Univ. J. E. Purkyně v Brně, *4* (1968), 223−236.

[2] *B. Mehri:* Boundary value problem for certain nonlinear third order differential equations Rev. Roum. Math. Pures Et Appl. *XIX* (1974), 773−776.

[3] *J. Cronin:* Fixed points and Topological degree in Non-linear Analysis, Mathematical surveys. No. *11*, 1964.

[4] *W. A. J. Luxemburg:* On the convergence of successive approximations in the theory of ordinary differential equations 111, Nieuw Archief voor Wiskunde (*3*), VI (1958), 93−98.

[5] *W. A. J. Luxemburg:* On the convergence of successive approximations in the theory of ordinary differential equations 11, Indag. Math. *20* (1958), 540−546.

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