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BOUNDARY VALUE PROBLEMS FOR LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS NON-HOMOGENEOUS EQUATION IN THE HALF-PLANE

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We shall deal with the differential operator $p_n(-iD_\xi)\,D_t^n + \ldots + p_0(-iD_\xi)$, where in contradistinction to the usual conception we do not suppose $p_n \equiv 1$. We shall pursue a somewhat unusual and rather wide space of distributions. It will surprise us that neither the conditions of compatibility nor any unnatural restrictions on the right hand side of our equations will occur. These results are quite different from [2], [3]. All these restrictions appear to be a consequence of an unsuitable choice of the space in which the solution is considered. It would be interesting to find whether a similar phenomenon also occurs in other cases, for example in elliptical boundary problems. It would mean that their indices would be zero on a sufficiently wide space.

Until now, the case mostly studied in literature has been that of $p_n \equiv 1$ (exceptions are few, e.g. [4]), therefore many questions arise. Some of them are collected at the end of the paper. It would be interesting to study the case of ξ being a vector. The disadvantage of the approach used consists in the fact that the variable t plays an exceptional role.

The work is a continuation of [1] but the necessary definitions and results are given at the corresponding places in a somewhat modified form which I believe to be far better. In this sense it is self-contained.

1. Distribution spaces (compare [1], 1–3). The symbol $\mathscr{R}$ will mean in the sequel a rational function continuous for $-\infty < x < \infty$ which is not identically zero. It means that $\mathscr{R}^{-1} = 1/\mathscr{R}$ is a function of the type $\mathscr{R}^{-1} = P(x)/Q(x)$ where $P(x) \neq 0$ $(-\infty < x < \infty)$ and Q is not a zero polynomial.

Let L be the space of all functions $v(x)$ $(-\infty < x < \infty)$ such that $|v| = \int |v(x)|\,dx < \infty$. Let $\mathscr{K}$ be the space of all distributions u for which $\mathscr{R}$ exists such that $v = \mathscr{R}u \in L$. It means that $\mathscr{K} = \bigcup_{\mathscr{R}} \mathscr{R}^{-1}L$. In the space $\mathscr{K}$, the topology of inductive limit of all subspaces $\mathscr{R}^{-1}L$ ($\mathscr{R}$ fixed) is defined where the topology of every subspace $\mathscr{R}^{-1}L$ is the weakest separated topology induced by the map $u \to v = \mathscr{R}u \in L$ $(u \in \mathscr{R}^{-1}L)$.

For the sake of definiteness let $\mathscr{R} = (x - a)^s\, Q(x)/P(x)$, where $Q(a) \neq 0$. Let $\varphi \in C_0^\infty$, $\alpha \in C_0^\infty$, where the support of the function φ is in a small neighbourhood of the point $x = a$ and $\alpha(x) \equiv 1$ in a neighbourhood of the point $x = a$. Then

$$(1) \qquad \langle a, \varphi\rangle = \int \mathscr{R}u\, \psi(x)\,\mathrm{d}x + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, u^j_{a,\alpha}\,,$$

where we denoted

$$(2) \qquad \psi = \mathscr{R}^{-1}\left(\varphi - \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\,(x - a)^j\,\alpha\right), \quad u^j_{a,\alpha} = \langle u, (x - a)^j\,\alpha\rangle\,.$$

The function $U(x) = \mathscr{R}^{-1}\, v(x) = \mathscr{R}^{-1}(\mathscr{R}u)$ is called a *regular component* of the distribution u. It is clear that for $s' \geqq s$ it holds

$$(1)' \qquad \langle u, \varphi\rangle = \int U(x)\left(\varphi(x) - \sum_{j=0}^{s'-1} \frac{\varphi^{(j)}(a)}{j!}\,(x - a)^j\,\alpha(x)\right)\mathrm{d}x + \sum_{j=0}^{s'-1} \frac{\varphi^{(j)}(a)}{j!}\, u^j_{a,\alpha}\,,$$

where

$$(2)' \qquad u^j_{a,\alpha} = \langle u, (x - a)^j\,\alpha\rangle = \int U(x)\,(x - a)^j\,\alpha(x)\,\mathrm{d}x \quad (j = s, s + 1, \ldots, s')\,.$$

The least non-negative number s which can appear in the equation (1) is called the *order* of the distribution u at the point $x = a$. If this order is equal to zero, then we say that the distribution u is *regular* at the point $x = a$.

By means of the Fourier transform we obtain the spaces $L^\wedge$, $\mathscr{K}^\wedge$. They are the distribution spaces where the independent variable is ξ $(-\infty < \xi < \infty)$. For example, $\hat{v}(\xi) = \int v(x)\, e^{-i\xi x}\,\mathrm{d}x \in L^\wedge$ if $v \in L$ while in the general case $u \in \mathscr{K}$ the Fourier transform $\hat{u}$ is defined in the theory of distributions. One introduces a topology in the spaces $L^\wedge$, $\mathscr{K}^\wedge$ by transferring the corresponding topology from the spaces L, $\mathscr{K}$.

2. Distribution depending on a parameter (compare [1], 4—5). We shall deal with the distributions $u(t)$ depending on the parameter t $(0 \leqq t < \infty)$. Derivatives in the sense of topology of the corresponding space will be denoted by $\mathrm{d}^j/\mathrm{d}t^j$. Point-wise derivatives with respect to the parameter t are denoted by "$^{(j)}$" or "$'$". The following rules hold: $\mathrm{d}^j/\mathrm{d}t^j \,.\, P(x)\, u(t) \equiv P(x)\, \mathrm{d}^j/\mathrm{d}t^j \,.\, u(t)$, $\mathrm{d}^j/\mathrm{d}t^j \,.\, \varphi(x)\, u(t) \equiv \varphi(x)\, \mathrm{d}^j/\mathrm{d}t^j \,.\, u(t)$, $\langle \mathrm{d}^j/\mathrm{d}t^j \,.\, u(t), \varphi\rangle \equiv \langle u(t), \varphi\rangle^{(j)}$, $(\mathrm{d}^j/\mathrm{d}t^j \,.\, u(t))^\wedge \equiv \mathrm{d}^j/\mathrm{d}t^j \,.\, u(t)$. Symbol $D = D_\xi$ will mean the derivatives in the sense of the distribution theory: $P(-iD)\, \mathrm{d}^j/\mathrm{d}t^j \,.\, \hat{u}(t) \equiv$ $\equiv \mathrm{d}^j/\mathrm{d}t^j\, P(-iD)\, \hat{u}(t) \equiv \mathrm{d}^j/\mathrm{d}t^j\, (P(x)\, u(t))^\wedge = (P(x)\, \mathrm{d}^j/\mathrm{d}t^j \,.\, u(t))^\wedge$.

The distribution $u(t)$ $(0 \leqq t < \infty)$ is called *tempered* in a given space of distributions if it depends continuously on the parameter t and for a suitable N, $\lim_{t\to\infty} u(t)\, t^{-N} =$ $= 0$ (both in the sense of the corresponding topology).

From the definition of the inductive topology it follows that if $u(t)$ is tempered in the space $\mathscr{K}$, then there exists $\mathscr{R}$ such that $v(\cdot, t) = \mathscr{R}\, u(t) = \mathscr{R}\, U(t) \in L(0 \leqq t < \infty)$

and the distribution $v(\cdot, t)$ is tempered in the space L. It means that $\lim_{t \to \infty} \int |v(x, t)| \, t^{-N} \, . \, dx = 0$ (in L), consequently $\int dx \int_0^\infty |v(x, t)| (1 + t)^{-N-2} \, dt < \infty$ and therefore $\int_0^\infty v(\cdot, t) (1 + t)^{-N-2} \, dt < \infty$,

$$(3) \qquad \int_0^\infty \mathcal{R} \, U(\cdot, t) (1 + t)^{N'} \, dt \in L \quad (N' = N + 2) .$$

3. Non-homogeneous boundary problem. We come to the problem which will be dealt with. Let $p_0, \ldots, p_n$ $(n \geqq 1)$ be polynomials where $p_n \neq 0$. Let $\hat{f}(t) \in \mathcal{K}^\wedge$ be a tempered distribution. Let us consider the conditions

$$(4)^\wedge \quad p_n(-iD) \, d^n \hat{u}/dt^n + \ldots + p_0(-iD) \, \hat{u} = \hat{f}(t) , \quad u = \hat{u}(t) \text{ is tempered in } \mathcal{K}^\wedge .$$

They are equivalent to

$$(4) \qquad p_n(x) \, d^n u/dt^n + \ldots + p_0(x) \, u = f(t) , \quad u = u(t) \text{ is tempered in } \mathcal{K} .$$

4. Formulation of problem. Our aim is to describe all distributions for which (4)^ holds. It suffices to deal with the equivalent problem (4). A decisive step will be the proof of the fact that for the regular component $y = U(x, t)$ of the distribution $u(t)$ the following ordinary differential equation dependent on the parameter x and a point-wise growth condition hold:

$$(5) \qquad p_n(x) \, y^{(n)} + \ldots + p_0(x) \, y = F(x, t) \, (-\infty < x < \infty, \, 0 \leqq t < \infty) ,$$

$$\lim_{t \to \infty} y(x, t) \, t^{-N} \equiv 0 \quad \text{for suitable} \quad N .$$

At the same time $F(x, t)$ is a regular component of the distribution $f(t)$. Conditions (5) are required except sets of measure 0 with regard to the variable x.

It will be necessary to bring into relation different types of derivation and growth conditions which are in (4), (5). (These problems are generally passed over unnoticed.) We shall start studying the conditions (5), for the present without their relation to (4).

5. Definition (compare [1], 19). Let Ω_∞ be the set of those x for which $p_n(x) = \ldots = p_0(x) = 0$. If $x \notin \Omega_\infty$, then let $p_n(x) = \ldots = p_{r+1}(x) = 0$, $p_r(x) \neq 0$ $(r = r(x))$. Let us suppose that the roots $z_1(x), \ldots, z_r(x)$ of the polynomial $p_n(x) \, z^n + \ldots + p_0(x) = p_r(x) \, z^r + \ldots + p_0(x)$ are ordered so that $\operatorname{Re} z_1(x) \leqq \ldots \leqq \operatorname{Re} z_m(x) \leqq 0 < \operatorname{Re} z_{m+1}(x) \leqq \ldots \leqq \operatorname{Re} z_r(x)$ $(m = m(x))$. Denote by Ω_c the set of all x for which $m(x) = c = \text{const.}$

Further denote $p(z) = p_n(x) \, z^n + \ldots + p_0(x)$, $p_-(z) = (z - z_1(x)) \ldots (z - z_m(x))$, $p_+(z) = (z - z_{m+1}(x)) \ldots (z - z_r(x))$ $(r = r(x), \; m = m(x))$. Evidently $p(z) = p_r(x) \, p_-(z) \, p_+(z)$, the empty product being considered equal to one.

6. Lemma. *Let* $x \in \Omega_c$, $p_n(x) \neq 0$. *For any constants* $y_0, \ldots, y_{c-1}$ *there exists a unique function* $y(t)$ $(0 \leqq t < \infty)$ *such that* $p_n(x)\, y^{(n)}(t) + \ldots + p_0(x)\, y(t) \equiv 0$ $(0 \leqq t < \infty)$, $y(0) = y_0, \ldots, y^{(c-1)}(0) = y_{c-1}$, $\lim_{t\to\infty} y(t)\, c^{-\varepsilon t} = 0$ $(\varepsilon > 0, \varepsilon$ *arbitrary*). *This function satisfies the equation* $p_-(')\, y = 0$ *and the inequality* $|y^{(j)}(t)| \leqq$ $\leqq M_n^c\, C(x)^{c-1+j} (1 + t)^{c-1} (|y_0| + \ldots + |y_{c-1}|)$ $(0 \leqq t < \infty,\ j = 0, 1, \ldots)$ *where* M_n^c *are absolute constants*, $C(x) = 1 + |p_{n-1}(x)/p_n(x)| + \ldots + |p_0(x)/p_n(x)|$.

For a proof see [1], 12–13.

7. Particular solution of (5). Because the sets of measure 0 are inessential, we can limit ourselves to those x's for which $p_n(x) \neq 0$. By 6 there exists at most one function y for which (5) and the initial conditions

$$(6) \qquad y(x, 0) = 0, \ldots, y^{(m-1)}(x, 0) = 0 \quad (m = m(x))$$

hold. This solution will be now constructed.

Let $y = y_+(x, t)$ be the solution of the equation $p_+(')\, y = 0$ defined by the conditions $y(0) = \ldots = y^{(n-m-2)}(0) = 0$, $y^{(n-m-1)}(0) = 1$. Denote

$$(7) \qquad g(x, t) = \int_\infty^t y_+(x, t - \tau)\, F(x, \tau)\, d\tau\,.$$

Let $y = y_-(x, t)$ be the solution of the equation defined by the conditions $y(0) = \ldots$ $\ldots = y^{(m-2)}(0) = 0$, $y^{(m-1)}(0) = 1$. Denote

$$(8) \qquad h(x, t) = \int_0^t y_-(x, t - \tau)\, g(x, \tau)\, d\tau\,.$$

It can be verified that $p_+(')\, g(x, t) \equiv F(x, t)$, $p_-(')\, h(x, t) \equiv g(x, t)$. Hence $p(')\, h(x, t) = p_n(x)\, p_+(')\, p_-(')\, h(x, t) \equiv p_n(x)\, F(x, t)$ and the function

$$(9) \qquad y(x, t) = h(x, t)/p_n(x)$$

satisfies therefore the differential equation (5). (6) can be easily verified. The validity of the growth condition in (5) and the convergence of the integral (7) will follow from the estimates of Sec. 8–11.

8. Estimate of the function $y_+(x, t)$. It can be verified that

$$(10) \qquad y_+(x, t) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{e^{zt}}{p_+(z)}\, dz\,,$$

where the curve $\mathscr{C}$ encircles the roots of the polynomial p_+. Let us denote $\varepsilon(x) =$ $= \frac{1}{2} \operatorname{Re} z_{m+1}(x)$. We can suppose that $\mathscr{C}$ is in the domain $\operatorname{Re} z \geqq \varepsilon(x)$ and at the same time the distance of $\mathscr{C}$ from the roots $z_{m+1}(x), \ldots, z_n(x)$ of the polynomial p_+ is at least $\varepsilon(x)$. Then

$$|e^{zt}/p_+(z)| \leqq \varepsilon(x)^{m-n}\, e^{t\varepsilon(x)} \quad (z \in \mathscr{C},\ -\infty < t \leqq 0)\,.$$

It is known that $|z_i(x)| \leqq C(x)$. Therefore we can suppose that the length of $\mathscr{C} \leqq$ $\leqq 2\pi(C(x) + \varepsilon(x))$. Then (10) yields the estimate

$$|y_+(x, t)| \leqq 2\pi(C(x) + \varepsilon(x))\, \varepsilon(x)^{m-n}\, e^{t\varepsilon(x)} \quad (-\infty < t \leqq 0)\,.$$

Now let us assume that in a suitable neighbourhood of every point $x = a$, and also in a suitable neighbourhood of the point ∞ inequalities of the type

(11) $$\varepsilon(x) \geqq A|x - a|^\alpha \,(|x - a| < \delta)\,, \quad \varepsilon(x) \geqq A|x|^\alpha \,(|x| > N)$$

with $A > 0$ hold. These inequalities imply

$$|y_+(x, t)| \leqq \mathscr{R}^{-1} e^{t\varepsilon(x)}\,.$$

Analogously, by differentiating (10) we obtain estimates for the derivatives:

(12) $$|y_+(x, t)| \leqq \mathscr{R}^{-1} e^{t\varepsilon(x)} \quad (i = 0, 1, \ldots)$$

where $\mathscr{R}$ depends on i.

9. Estimate of the function $g(x, t)$. We know by (3) that the regular component F of the distribution f satisfies $\Phi(\cdot) = \int_0^\infty \mathscr{R}\, F(\cdot, t)\,(1 + t)^{-N}\, dt \in L$. Hence by (7)

$$|g(x, t)| \leqq \int_\infty^t |\mathscr{R}^{-1}\, y_+(x, t - \tau)\,(1 + \tau)^N\, \mathscr{R}\, F(x, t)\,(1 + \tau)^{-N}|\, d\tau \leqq$$

$$\leqq |\mathscr{R}^{-1} \max_{t \leqq \tau < \infty} |y_+(x, t - \tau)\,(1 + \tau)^N|\, |\Phi(x)|\,.$$

Now, the estimate $|g(x, t)| \leqq |\mathscr{R}^{-1} . (1 + t)^N\, \Phi(x)|$ is a consequence of the inequality $\max |y_+(x, -\tau)\,(1 + \tau)^N| \leqq \mathscr{R}^{-1}(1 + t)^N$, which can be deduced as follows:

It suffices to obtain this inequality locally, i.e., in a neighbourhood of every finite point $x = a$ and in a neighbourhood of ∞. But, by $(12)^0$, (11) it holds

$$\max_{t \leqq \tau < \infty} |y_+(x, t - \tau)\,(1 + \tau)^N| \leqq \mathscr{R}^{-1} \max_{t \leqq \tau < \infty} e^{(t-\tau)A|x-a|^\alpha}(1 + \tau)^N$$

and this is majorized by a function of the form $\mathscr{R}^{-1}(1 + t)^N$ by means of an elementary calculation. The case of the point ∞ is similar.

Analogously, by differentiating (7) we obtain inequalities of the type

(13) $$|g^{(i)}(x, t)| \leqq \mathscr{R}^{-1}\, \Phi(x)\,(1 + t)^N$$

$$(i = 1, \ldots, n - m - 1,\ \Phi \in L,\ 0 \leqq t < \infty)\,.$$

10. Estimate of the function $y_-(x, t)$. It concerns a special case of Lemma 6 when $y_0 = \ldots = y_{m-2} = 0$, $y_{m-1} = 1$.

Let us write the corresponding estimate more concisely:

$$(14)^i \qquad \left|y_-^{(i)}(x, t)\right| \leqq \mathscr{R}^{-1}(1 + t)^N \quad (i = 0, 1, \ldots).$$

11. Estimate of the solution $y(x, t)$. From the relations (9), (8), $(13)^i$, $(14)^i$ we can easily obtain estimates of the type

$$(15)^i \qquad \left|y^{(i)}(x, t)\right| \leqq \mathscr{R}^{-1}\, \Phi(x)\,(1 + t)^N$$

$$(i = 0, 1, \ldots, n,\ \Phi \in L,\ 0 \leqq t < \infty).$$

Thus we have verified even the validity of the growth conditions from (5).

12. Non-homogeneous initial conditions. Let $y_0(x), \ldots, y_{n-1}(x)$ be given functions. By Lemmas 6 and 7 there exists exactly one function $y = y(x, t)$ for which (5) and the initial conditions

$$(16) \qquad y(x, 0) = y_0(x), \ldots, y^{(m-1)}(x, 0) = y_{m-1}(x) \quad (m = m(x))$$

hold.

According to the inequality in Lemma 6 and by $(15)^i$, this function satisfies inequalities of the type

$$(17)^i \qquad \left|y^{(i)}(x, t)\right| \leqq \mathscr{R}(1 + t)^N\,(\Phi(x) + |y_0(x)| + \ldots + |y_{m-1}(x)|)$$

$$(i = 0, 1, \ldots, n,\ \Phi \in L,\ 0 \leqq t < \infty).$$

This completes the study of conditions (5) and we shall resume our main task, i.e., the conditions (4), $(4)^\wedge$. In the course of solution we shall essentially make use of all results achieved.

13. Theorem. *Let (4) hold. Then* $\mathscr{R}$ *exists such that* $v(\cdot, t) = \mathscr{R}\, u(t) = \mathscr{R}\, U(\cdot, t) \in L$ *is a function for which* $\mathrm{d}^j/\mathrm{d}t^j \,.\, v(\cdot, t) = v^{(j)}(\cdot, t) \in L$ $(j = 0, 1, \ldots, n;\ 0 \leqq t < \infty)$.

Remark. In the case $f(t) \equiv 0$, this theorem is identical except for its formulation with Lemma 16 in [1] and we consider it known.

Proof. Let the function $y = y(x, t)$ be defined by the conditions (5), (16) where the regular components of the distributions $\mathrm{d}^j/\mathrm{d}t^j \,.\, u(0)$ are chosen to be the functions $y_j(x)$ $(j = 0, \ldots, n - 1)$. The estimates $(15)^i$ imply that $\mathscr{R}\, y^{(j)}(\cdot, t) \equiv \mathrm{d}^j/\mathrm{d}t^j \,.$ $.\, \mathscr{R}\, y(\cdot, t) \in L$ are tempered distributions satisfying by (5) the identity

$$p_n\, \mathrm{d}^n/\mathrm{d}t^n \,.\, \mathscr{R}y + \ldots + p_0\, \mathscr{R}y = \mathscr{R}f.$$

Then by (4)

$$p_n\, \mathrm{d}^n/\mathrm{d}t^n \,.\, \mathscr{R}(u - y) + \ldots + p_0\, \mathscr{R}(u - y) = 0$$

and according to the remark after Theorem 13, $\mathrm{d}^j/\mathrm{d}t^j \,.\, \mathscr{R}(u - y) = \mathscr{R}(u - y)^{(j)}$ for a suitable $\mathscr{R}$. Hence it follows that $\mathrm{d}^j/\mathrm{d}t^j \,.\, \mathscr{R}u = \mathscr{R}u^{(j)}$.

14. Corollary. *Let* (4) *hold. Then for the regular component* $y = U(x, t)$ *of the distribution* $u(t)$ *the conditions* (5) *hold. The functions* $u^j_{a,\alpha}(t) = \langle u, (x - a)^j\, \alpha\rangle$ *have continuous derivatives up to the order n and it is* $\lim_{t\to\infty} u^j_{a,\alpha}(t)\, t^{-N} = 0$ *(N suitable).*

Proof. The first part follows from 13, proof of the second part is easy.

15. Analysis of equations (4). Let us suppose that the sought distribution $u(t)$ and the known distribution $f(t)$ are defined in the neighbourhood of the point $x = a$ as follows:

$$\langle u(t), \varphi\rangle = \langle \mathscr{R}\, U(\cdot, t), \psi\rangle + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, u^j(t)\,,$$

$$\langle f(t), \varphi\rangle = \langle \mathscr{R}\, F(\cdot, t), \psi\rangle + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, f^j(t)\,,$$

where, for the sake of brevity, the indices "a, α" are omitted. Denote

$$p_k(x) = \sum_l p^l_k (x - a)^l\,.$$

By means of the equation (4) we obtain

$$p_n(x)\, U^{(n)}(x, t) + \ldots + p_0(x)\, U(x, t) \equiv F(x, t)\,, \tag{18}$$

$$\sum_l (p^l_n\, u^{j+l,(n)}(t) + \ldots + p^l_0\, u^{j+l}(t)) \equiv f^j(t) \quad (j = s - 1, \ldots, 0)\,. \tag{19}$$

Moreover, we have the growth conditions

$$\lim_{t\to\infty} U(x, t)\, t^{-N} = 0 \quad (N \text{ suitable})\,, \tag{18$'$}$$

$$\lim_{t\to\infty} u^j(t)\, t^{-N} = 0 \qquad (j = s - 1, \ldots, 0;\ N \text{ suitable})\,. \tag{19$'$}$$

In this reasoning we have used the results of 13, 14. The solution of the abstract problem (4) is now transferred to the solution of the system of ordinary differential equations (18)–(19)$'$. We can pass to the formulation of the main result.

16. Definition (compare [1], 25 which is unnecessarily restrictive). Let us say that a *distribution* $u \in \mathscr{K}$ *is equal to zero* on a set Ω if its regular component is almost everywhere equal to zero on Ω and if the distribution u is regular at every point $x \in \Omega$.

17. Theorem. *Let Ω_∞ be an empty set. For any distributions $u_0 \in \mathscr{K}, \ldots, u_{n-1} \in \mathscr{K}$ and for each distribution $f(t)$ tempered in the space $\mathscr{K}$ there exists exactly one distribution $u(t)$ such that (4) holds and at the same time, for every $j = 0, 1, \ldots, n$, the distributions $u(0) - u_0, \ldots, \mathrm{d}^{j-1}/\mathrm{d}t^{j-1} \,.\, u(0) - u_{j-1}$ are zero on the set Ω_j.*

Proof. Let the distributions $u_0, \ldots, u_{n-1}$ be defined in the neighbourhood of the point $x = a$ as follows:

$$\langle u_i, \varphi \rangle = \langle \mathscr{R} U_i, \psi \rangle + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!} u_i^j .$$

Then the regular component $U(x, t)$ of the distribution $u(t)$ fulfils the equations (18), (18)' and the initial conditions

$$U(x, 0) - U_0(x) = \ldots = U^{(j)}(x, 0) - U_j(x) \equiv 0 \quad (x \in \Omega_j,\ j = 0, \ldots, n)$$

by which it is uniquely defined in virtue of 12. The procedure until now has not required the knowledge of $\mathscr{R}$, s.

Now let us choose $\mathscr{R}$ such that $\mathscr{R}\, U^{(i)}(\cdot, t) \in L\ (i = 0, \ldots, n;\ 0 \leqq t < \infty)$, $\mathscr{R} f(t) \in \in L$. Then (19), (19)' hold for the function $u^i(t)$ and we have the initial conditions

(20) $$u^i(0) = u_0^i, \ldots, u^{i,(m)}(0) = u_m^i \quad (i = 0, \ldots, s - 1;\ m = m(a)) .$$

We shall show that these functions are uniquely defined.

Let us write the system (19) in more detail:

$$p_n^0\, u^{s-1,(n)}(t) + \ldots + p_0^0\, u^{s-1}(t) = \Phi_{s-1}(t) + f^{s-1}(t) ,$$

$$p_n^0\, u^{s-2,(n)}(t) + \ldots + p_0^0\, u^{s-2}(t) + p_n^1\, u^{s-1,(n)}(t) + \ldots$$
$$\ldots + p_0^1\, u^{s-1}(t) = \Phi_{s-2}(t) + f^{s-2}(t) ,$$

....

$$p_n^0\, u^{0,(n)}(t) + \ldots + p_0^0\, u^0(t) + p_n^1\, u^{1,(n)}(t) + \ldots$$
$$\ldots + p_0^1\, u^1(t) + \ldots = \Phi_0(t) + f^0(t) .$$

At the same time, the functions $f^0(t), \ldots, f^{s-1}(t)$ are given and the functions

$$\Phi_{s-1}(t) = -p_n^1\, u^{s,(n)}(t) - \ldots - p_0^1\, u^s(t) - \ldots$$
$$\Phi_0(t) \quad = -p_n^s\, u^{s,(n)}(t) - \ldots - p_0^s\, u^s(t) - \ldots$$

can be considered known by virtue of the relations $u^j(t) = \int U(x, t)\,(x - a)^s\, \alpha(x)\, \mathrm{d}x$ $(j = s, s + 1, \ldots)$ as well. Since by the supposition $a \notin \Omega_\infty$, it is not $p_j^0(=p_j(a)) \equiv 0$ $(j = 0, \ldots, n)$ and from the set (19) we can define successively the function $u^{s-1}(t), \ldots$ $\ldots, u^0(t)$ such that (19), (19)' hold. (This follows again from 12, even from the simplified version in which the parameter x is fixed.)

The above mentioned construction yields not only the existence but also the uniqueness of the solution.

18. Corollary. *Let Ω_∞ be an empty set. For every distributions $\hat{u}_{n-1} \in \mathscr{K}^\wedge, \ldots$ $\ldots, u_0 \in \mathscr{K}^\wedge$ and for every distribution $f(t)$ tempered in the space $\mathscr{K}^\wedge$ there exists exactly one distribution $u(t)$ such that $(4)^\wedge$ holds and at the same time, for every $j =$ $= 0, 1, \ldots, n,$ the distributions $u(0) - u_0, \ldots, \mathrm{d}^{j-1}/\mathrm{d}t^{j-1} \,.\, u(0) - u_{j-1}$ are zero on the set Ω_j.*

19. Remark on the set Ω_∞. If this set is nonempty, no principal difficulties arise. Theorems 17, 18, however, become formally more complicated. Let us indicate the essence of these changes.

The computation of the regular components does not change. The character of the system of equations (19), however, changes essentially. If, namely, $a \in \Omega_\infty$, then $p_n^0 = \ldots = p_0^0 = 0$ and the first equation of this system has the form

$$0 = \Phi_{s-1}(t) + f^{s-1}(t)\,.$$

It is the condition of compatibility for the functions already known. This condition means

$$p_n^1\, u^{s,(n)}(t) + \ldots + p_0^1\, u^s(t) + \ldots = f^{s-1}(t)$$

and therefore according to the definition of the functions $u^s(t), u^{s+1}(t), \ldots$ it means that

$$\langle p_n^1(x-a)\, \mathrm{d}^n/\mathrm{d}t^n \,.\, u(t) + \ldots + p_0^1(x-a)\, u(t) + \ldots, (x-a)^{s-1}\, \alpha \rangle =$$
$$= \langle f(t), (x-a)^{s-1}\, \alpha \rangle$$

and consequently.

$$\text{(20)} \qquad \langle p_n\, \mathrm{d}^n/\mathrm{d}t^n \,.\, u(t) + \ldots + p_0\, u(t), (x-a)^{s-1}\, \alpha \rangle = \langle f(t), (x-a)^{s-1}\, \alpha \rangle\,,$$

which is an identity.

One can prove quite analogously that if also $p_n^1 = \ldots = p_0^1 = 0$, then the second equation of the set (19) of the form $0 \equiv \Phi_{s-2}(t) + f^{s-2}(t)$ is satisfied identically, etc. Consequently one can see that there are no conditions of compatibility and if the first c equations of the set (19) are satisfied, then this system defined only the functions $u^{s-1}(t), \ldots, u^{s-c}(t)$ and for the remaining functions $u^{s-c-1}(t), \ldots, u^0(t)$ we have only the growth conditions (19)'.

Moreover, the equation (20) draws attention to the fact that in comparison with the preceding case it is necessary to increase the supposed order of the distribution $u(t)$ for it must be $s - 1 \geqq 0$. If the first c equations of the system (19) are satisfied identically, then $s \geqq c$.

20. Remark on regular equations. The initial conditions in 18 are not formulated by means of the sought solution $u(t)$ but by means of its Fourier transform $\hat{u}(t)$. If, however, such j' exists that the complement of the set $\Omega_{j'}$ is finite, the distribution $f(t)$ is regular on it and all real roots of the polynomial $p_n(x)$ lie in $\Omega_{j'}$, then the boundary conditions mean that $\hat{u}(0) = \hat{u}_0, \ldots, \mathrm{d}^{j'-1}/\mathrm{d}t^{j'} \,.\, \hat{u}(0) = \hat{u}_{j'-1}$. These conditions have the classical form and the problem mentioned can be called a *regular boundary value problem* (compare [2]).

21. Remark on transient conditions. Problems having a similar character as the regular boundary value problems just mentioned can be formulated even for the general types of equations. They appear as follows:

$$p_n(-iD)\,\mathrm{d}^n/\mathrm{d}t^n \,.\, \hat{u} + \ldots + p_0(-iD)\,\hat{u} = \hat{f}(t)$$

$$\hat{u} = \hat{u}(t) \in \mathscr{K}^{\wedge}(-\infty < t < \infty),\ \hat{u}(t) \text{ and } \hat{u}(-t) \text{ are tempered}$$

$$\mathrm{d}^j/\mathrm{d}t^j \,.\, u(0+) - \mathrm{d}^j/\mathrm{d}t^j \,.\, u(0-) = \hat{u}_j \quad (j = 0, 1, \ldots, n-1)$$

where $\hat{f}(t) \in \mathscr{K}^{\wedge}\ (-\infty < t < \infty)$, $\hat{u}_j \in \mathscr{K}^{\wedge}$ are given distributions, $\hat{f}(t)$ and $\hat{f}(-t)$ are tempered. Certain difficulties, however, can be caused by the transition of the roots of the polynomial $p_n(z)$ along the straight line $\operatorname{Re} z = 0$.

22. Remark on general boundary problems. The conditions $(4)^{\wedge}$ together with the generalized boundary conditions of the type $\Sigma a_{ij} D^i\, \mathrm{d}^n/\mathrm{d}t^n \,.\, \hat{u}(0) = \hat{f}_j$ may be studied by the same methods. Results are similar as before: It is possible to introduce c of these conditions on the set Ω_c for the corresponding Fourier transforms. If Ω_∞ is an empty set, there are no conditions of compatibility. (For example, Neumann problem has always a solution.)

23. Remark on Lemma 6. The inequality in this lemma can be strengthened. From the analysis of the proof it follows that for $x \in \Omega_m$ one can choose $C(x) = \max(|z_1(x)|, \ldots, |z_m(x)|)$. Further, the factor $(1 + t)^{m-1}$ can be replaced by the factor $(1 + t)^{m'-1}$ where m' is the maximal multiplicity of roots of the polynomial p_-. The factor $|y_0| + \ldots + |y_{m-1}|$ can be replaced by the factor $|y_0| + |y_1/C(x)| + \ldots + |y_{m-1}/C(x)^{m-1}|$. The inequality strengthened in this way is important for the study of the smoothness of solutions. On the other hand, the behaviour of $\operatorname{Re} z_m(x)$ $(|x| \to \infty)$ plays a role as well.

Investigating analogous problems for systems of equations, we can see that the solution does not depend so much on the multiplicity of roots as on the dimension if the submatrices in the Jordan canonical form. This shows the significance of the symmetric hyperbolical system where this canonical form is diagonal even if the characteristic equation has multiple roots.

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