Časopis pro pěstování matematiky

Svatopluk Fučík; Jean Mawhin

Periodic solutions of some nonlinear differential equations of higher order

Časopis pro pěstování matematiky, Vol. 100 (1975), No. 3, 276--283

Persistent URL: `http://dml.cz/dmlcz/117878`

Terms of use:

© Institute of Mathematics AS CR, 1975

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://project.dml.cz`

Časopis pro pěstování matematiky, roč. 100 (1975), Praha

PERIODIC SOLUTIONS OF SOME NONLINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

SVATOPLUK FUČÍK, Praha and JEAN MAWHIN, Louvain-la-Neuve

(Received February 5, 1974)

1. Introduction. This paper is devoted to the study of the vector differential equation

$$\text{(E)} \quad -(-1)^k x^{(2k)}(t) + A_1 x^{(2k-1)}(t) + \ldots + A_{2k-1} x'(t) + \frac{\mathrm{d}}{\mathrm{d}t}[\operatorname{grad} F(x(t))] + g(x(t)) = p(t)\,.$$

Under some conditions upon the matrices $A_1, \ldots, A_{2k-1}$ and under very general assumptions upon the functions F and g, the existence of a periodic solution of (E) is proved. In the scalar case (see Section 7), it is possible to give simple necessary and sufficient conditions for the existence of at least one periodic solution of (E).

The obtained results extend the ones from [2] and moreover illustrate some of the assumptions of [7] in that instead of assuming that some Brouwer's degree is non zero, we give explicit conditions upon the function g. Other connections with previous papers are discussed in [2, 7].

The method of proof is very close to the abstract investigations of nonlinear equations with noninvertible linear parts studied in [1] and [4]. As in [7], L_2-estimates and classical inequalities are used to obtain the C^1-a priori bounds (see Section 5) needed for applying the continuation theorem of coincidence degree theory [4] stated in Section 2 for reader's convenience. The main result is stated and proved in Section 6.

2. A continuation theorem. Let X, Z be normed vector spaces, L: dom $L \subset X \to Z$ be a Fredholm mapping of index zero, i.e. a linear mapping with closed range Im L having a finite codimension equal to the dimension of the null-space ker L of L. It is known that this implies the existence of continuous projectors $P : X \to X$, $Q : Z \to Z$ such that

$$\operatorname{Im} P = \ker L, \quad \operatorname{Im} L = \ker Q$$

and we shall assume that the inverse $K : \operatorname{Im} L \to X$ of the restriction $\tilde{L}$ of L to $\operatorname{dom} L \cap \cap \ker P$ is compact. Let now $\Omega \subset X$ be a bounded open set and $N : \bar{\Omega} \to Z$ be a continuous mapping such that $N(\bar{\Omega})$ is bounded. The following theorem is proved in [4] (see also [8], ch. XI, for the special case of periodic solutions of ordinary differential equations).

2.1. Proposition. *Let L and N be like above and suppose that the following conditions are satisfied.*

(1) *For each $\lambda \in (0, 1)$, every possible solution of equation*

$$Lx = \lambda N x$$

is such that $x \notin \partial\Omega \cap \operatorname{dom} L$.

(2) *For each $x \in \ker L \cap \partial\Omega$, $Nx \notin \operatorname{Im} L$ (or equivalently $QNx \neq 0$).*

(3) *The Brouwer degree* (see e.g. [3])

$$\mathrm{d}[\mathfrak{N}, \Omega \cap \ker L, 0] \neq 0 \,,$$

where $\mathfrak{N} : \Omega \cap \ker L \to \ker L$, $a \mapsto JQNa$ and $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.

Then, equation

$$Lx = Nx$$

has at least one solution $x \in \operatorname{dom} L \cap \bar{\Omega}$.

3. Linear differential operators. R^n being the n-dimensional Euclidian space, let us denote by $|\cdot|$ and by $(\cdot, \cdot)$ respectively its Euclidian norm and inner product.

If $l \geqq 0$ is an integer, we shall denote by C_T^l the (Banach) space of mappings $x : R^1 \to R^n$ which are continuous and T-periodic together with their first l derivatives with the norm

$$\|x\|_l = \sum_{j=0}^{l} [\sup_{t \in R} |x^{(j)}(t)|]$$

$(x^{(j)} = \mathrm{d}^j x / \mathrm{d}t^j)$.

Let us introduce the projector

$$P : C_T^l \to C_T^l \,, \quad x \mapsto T^{-1} \int_0^T x(t) \, \mathrm{d}t \,.$$

It is immediate that

$$\|Px\|_l = \|Px\|_0 \leqq \|x\|_0 \leqq \|x\|_l$$

for every $x \in C_T^l$ and that $\operatorname{Im} P$ is the subspace of C_T^l of constant functions.

If $k \geqq 1$ is an integer, let L be the differential operator defined on

$$\operatorname{dom} L = \{x \in C_T^1 : x^{(2k)} \text{ exists and is continuous}\}$$

by

$$Lx = -(-1)^k x^{(2k)} + A_1 x^{(2k-1)} + \ldots + A_{2k-1} x' \,,$$

where the A_i $(i = 1, \ldots, 2k - 1)$ are $(n \times n)$ constant symmetric matrices. Moreover let us write

$$\langle x, y\rangle = T^{-1} \int_0^T (x(t), y(t))\, \mathrm{d}t$$

for $x, y \in C_T^0$, and $A \leqq 0$ for a negative semi-definite matrix.

3.1. Lemma. *If*

$$\text{(A)} \qquad (-1)^j A_{2k-2j} \leqq 0\,, \quad (j = 1, \ldots, k - 1)$$

holds, then we have

$$\ker L = \operatorname{Im} P\,.$$

Proof. It is easy to see that constant mappings from R^1 to R^n belong to $\ker L$. Now, if $x_0 \in \ker L$, we have

$$-(-1)^k \langle x_0^{(2k)}, x_0\rangle + \langle A_1 x_0^{(2k-1)}, x_0\rangle + \ldots + \langle A_{2k-1} x_0', x_0\rangle = 0$$

and integrating by parts and using assumption (A) we obtain

$$0 \geqq \sum_{j=1}^{k-1} (-1)^j \langle A_{2k-2j} x_0^{(j)}, x_0^{(j)}\rangle = \langle x_0^{(k)}, x_0^{(k)}\rangle \geqq 0$$

which implies that x_0 is a constant mapping.

3.2. Lemma. *If assumption* (A) *holds, then*

$$\operatorname{Im} L = \{x \in C_T^0 : Px = 0\}.$$

Proof. Clearly the n linearly independant conditions $Px = 0$ are necessary for $x \in \operatorname{Im} L$ and the sufficiency follows from Lemma 3.1 and the fact that the Fredholm alternative holds for linear periodic ordinary differential equations. (See e.g. [8].)

If

$$\tilde{X} = \{x \in \operatorname{dom} L : Px = 0\},$$

then the restriction $\tilde{L}$ of the operator L to $\tilde{X}$ is a one-to-one mapping from $\tilde{X}$ onto $\operatorname{Im} L$ and we shall denote its inverse by K (K is called the right inverse of L).

3.3. Lemma. *The mapping $K : \operatorname{Im} L \to \tilde{X}$ is compact. (Note that $\tilde{X}$ is considered with the norm induced by C_T^1).*

Proof. It follows by a standard argument (see e.g. [7] or [8]) from Arzela-Ascoli theorem.

4. Necessary conditions. Except mentioning the contrary, we shall suppose from now that assumption (A) is satisfied. Suppose that conditions

(B) $F : R^n \to R^1$ is of class C^2;

(C) $g : R^n \to R^n$ is continuous

hold and, for $x, y \in R^n$ let us write $x \prec y$ when $x_i < y_i$ for $i = 1, \ldots, n$.

4.1. Lemma. *Let $p \in C_T^0$ and suppose that $a \prec g(s) \prec b$ for all $s \in R^n$. Then, a necessary condition for the existence of one $x \in \operatorname{dom} L$ satisfying*

$$(Lx)(t) + \frac{d}{dt}[\operatorname{grad} F(x(t))] + g(x(t)) = p(t) \tag{E}$$

is that

$$a \prec T^{-1} \int_0^T p(t)\, dt \prec b\,.$$

Proof. Suppose that $x_0 \in C_T^{2k}$ satisfied equation (E). Then

$$p - g(x_0(\cdot)) - \frac{d}{dt}[\operatorname{grad} F(x_0(\cdot))] \in \operatorname{Im} L$$

and, according to Lemma 3.2 we have

$$T^{-1} \int_0^T p(t)\, dt = T^{-1} \int_0^T g(x_0(t))\, dt + T^{-1} \int_0^T (d/dt)[\operatorname{grad} F(x_0(t))]\, dt =$$

$$= T^{-1} \int_0^T g(x_0(t))\, dt\,.$$

The assertion follows then immediately from this equality and our assumption.

5. C^1 – a priori estimates. Let $p \in C_T^0$, $Pp = 0$, be fixed and let $g : R^n \to R^n$ satisfying (C) and the following assumption:

(D) i) $\sup_{s \in R^n} |g(s)| = M < \infty$;

ii) There exists a strictly positive number r, a permutation $\{i_1, \ldots, i_n\}$ of of $\{1, \ldots, n\}$ and an integer $0 \leqq m \leqq n$ such that

$$g_{i_l}(x)\, x_{il} > 0 \quad \text{if} \quad |x_{i_l}| \geqq r \quad (l = 1, \ldots, m)\,,$$

$$g_{i_l}(x)\, x_{i_l} < 0 \quad \text{if} \quad |x_{i_l}| \geqq r \quad (l = m + 1, \ldots, n)\,.$$

5.1. Lemma. *If* $R_0 = n^{1/2}r$, *then, for each* $y \in C_T^0$ *such that* $\inf_{t\in R^1} |y(t)| \geqq R_0$, *one has*

$$T^{-1} \int_0^T g(y(t))\, dt \neq 0 .$$

Proof. It is obvious.

5.2. Lemma. *There* $\gamma_1 > R_0$ *such that for each* $\lambda \in (0, 1\rangle$ *and every possible solution* $x \in C_T^{2k}$ *of*

$$(\mathrm{E}_\lambda) \qquad (Lx)(t) + \lambda(d/dt)[\operatorname{grad} F(x(t))] + \lambda\, g(x(t)) = \lambda\, p(t)$$

one has

$$\sup_{t\in R^1} |x(t)| < \gamma_1 .$$

Proof. It is easy to see that

$$(1) \qquad \int_0^T g(x(t))\, dt = 0 .$$

From the equality

$$\langle Lx, x\rangle = \lambda[-\langle (d/dt)[\operatorname{grad} F(x)], x\rangle - \langle g(x), x\rangle + \langle p, x\rangle]$$

we deduce easily, using assumption (A) and simple computations, that

$$(2) \qquad \langle x^{(k)}, x^{(k)}\rangle \leqq \left|T^{-1} \int_0^T (g(x(t)) - p(t), (P^C x)(t))\, dt\right| \leqq$$
$$\leqq (M + \|p\|_0)\left(T^{-1}\int_0^T |(P^C x)(t)|^2\, dt\right)^{1/2}$$

where $P^C = I - P$, I the identity. If $\omega = 2\pi/T$, it is known (see e.g. [8], ch. XI) that

$$T^{-1}\int_0^T |(P^C x)(t)|^2\, dt \leqq \omega^{-2}T^{-1}\int_0^T |(P^C x)'(t)|^2\, dt = \omega^{-2}T^{-1}\int_0^T |x'(t)|^2\, dt$$

and

$$(3) \qquad T^{-1}\int_0^T |(P^C x)(t)|^2\, dt \leqq \omega^{-2k}T^{-1}\int_0^T |x^{(k)}(t)|^2\, dt .$$

Then using (2) and (3) we get

$$\left(T^{-1}\int_0^T |x^{(k)}(t)|^2\, dt\right)^{1/2} \leqq (M + \|p\|_0)\, \omega^{-k}$$

and hence, by well-known properties of periodic functions,

$$\left(T^{-1}\int_0^T |x'(t)|^2\, dt\right)^{1/2} \leqq (2\pi)^{1-k}\, \omega^{-1}(M + \|p\|_0) .$$

Since (see e.g. [8], chapter XI),

$$\sup_{t\in R^1} |(P^C x)(t)| \leqq 3^{-1/2}\pi\omega^{-1}\left(T^{-1}\int_0^T |x'(t)|^2\,\mathrm{d}t\right)^{1/2},$$

we obtain

$$\sup_{t\in R^1} |(P^C x)(t)| \leqq 3^{-1/2}\pi\omega^{-2}(2\pi)^{1-k}(M + \|p\|_0). \tag{4}$$

From (1) and Lemma 5.1 it follows that there will exist $\tau \in R^1$ such that

$$|x(\tau)| < R_0$$

which implies, using (4),

$$\|Px\|_0 = |(Px)(\tau)| \leqq |x(\tau)| + \|P^C x\|_0 < R_0 + 3^{-1/2}\pi\omega^{-2}(2\pi)^{1-k}(M + \|p\|_0). \tag{5}$$

Clearly the inequalities (4) and (5) imply the wanted assertion.

5.3. Lemma. *There exists $\gamma_2 > 0$ such that, for each $\lambda \in (0, 1\rangle$ and every possible T-periodic solution x of (E_λ) one has*

$$\sup_{t\in R^1} |x'(t)| \leqq \gamma_2 .$$

Proof. Let us first consider the case where $k \geqq 2$. Then (see e.g. [8], chapter XI),

$$\sup_{t\in R^1} |x'(t)| = \sup_{t\in R^1} |(\mathrm{d}/\mathrm{d}t)(P^C x)(t)| \leqq 3^{-1/2}\pi\omega^{-1}\left(T^{-1}\int_0^T |x''(t)|^2\,\mathrm{d}t\right)^{1/2} \leqq$$
$$\leqq 3^{-1/2}\pi\omega^{-3}(2\pi)^{1-k}(M + \|p\|_0).$$

Let now $k = 1$. Then,

$$x''(t) + A_1\, x'(t) = \lambda\{p(t) - (\mathrm{d}/\mathrm{d}t)[\operatorname{grad} F(x(t))] - g(x(t))\}$$

and hence

$$\langle x'', x''\rangle = \lambda\{\langle p, x''\rangle - \langle(\mathrm{d}/\mathrm{d}t)[\operatorname{grad} F(x)], x''\rangle - \langle g(x), x''\rangle\} \leqq$$
$$\leqq [\|p\|_0 + M + \omega^{-1}S(M + \|p\|_0)]\left(T^{-1}\int_0^T |x''(t)|^2\,\mathrm{d}t\right)^{1/2},$$

where

$$S = \sup_{|\xi|\leqq\gamma_1}\ \max_{i,j=1,\dots,n} \left|\frac{\partial^2 F(\xi)}{\partial\xi_i\,\partial\xi_j}\right|.$$

Therefore

$$\sup_{t\in R^1} |x'(t)| \leqq 3^{-1/2}\pi\omega^{-1}\left(T^{-1}\int_0^T |x''(t)|^2\,\mathrm{d}t\right)^{1/2} \leqq$$
$$\leqq 3^{-1/2}\pi\omega^{-1}(\|p\|_0 + M + \omega^{-1}S(M + \|p\|_0)) = \gamma_2 .$$

6. Sufficient condition. If we define $N : C_T^1 \to C_T^0$ by

$$(Nx)(t) = p(t) - (\mathrm{d}/\mathrm{d}t)[\operatorname{grad} F(x(t))] - g(x(t)),$$

it follows at once from assumptions (B) and (C) that N is continuous and takes bounded sets into bounded sets. Moreover, finding T-periodic solutions of equation (E) is clearly equivalent to solving the abstract equation

$$Lx = Nx \tag{6}$$

in $C_T^1 \cap \operatorname{dom} L$, with L defined in section 3. Clearly, using the results of Section 3, L and N verify the regularity assumptions needed for Proposition 2.1.

6.1. Theorem. *Suppose that $p \in C_T^0$ and $Pp = 0$ and that assumptions (A) to (D) hold. Then equation (E) has at least one T-periodic solution.*

Proof. We shall show that conditions (1) to (3) of Proposition 2.1 are satisfied for (6) and

$$\Omega = \{x \in C_T^1 : \|x\|_1 < \gamma_1 + \gamma_2\}$$

with γ_i $(i = 1, 2)$ defined in section 5. Condition (1) follows at once from Lemmas 5.2 and 5.3 and Condition (2) from Lemma 5.1 applied to constant mappings from R^1 into R^n. Moreover here, $P = Q$ and is the projector defined in Section 3 and we can take $J = -I$, which implies that $\mathfrak{N} : \ker L \to \ker L$ is defined by $\mathfrak{N}(a) = g(a)$, $a \in \ker L$. Now using assumption (D) and defining $\eta : \ker L \to \ker L$, with $\ker L$ naturally identified with R^n, by

$$\eta_{i_l}(a) = a_{i_l} \ (l = 1, \dots, m), \quad \eta_{i_l}(a) = -a_{i_l} \ (l = m + 1, \dots, n)$$

we have, when $|a| \geqq R_0$,

$$(g(a), \eta(a)) > 0$$

which implies, by the basic properties of Brouwer degree [3], that

$$\mathrm{d}[g, \Omega \cap \ker L, 0] = \mathrm{d}[\eta, \Omega \cap \ker L, 0] = \pm 1$$

and achieves the proof.

6.2. Remark. In the same way than in [5] p. 26 or [6], p. 598, it is possible to show that Theorem 6.1 remains true when the inequalities in (D-ii) are not strict.

7. Necessary and sufficient condition in the scalar case.

7.1. Theorem. *Let $a_1, \dots, a_{2k-1}$ be real numbers such that $(-1)^j a_{2k-2j} \leqq 0$ $(j = 1, \dots, k - 1)$. Let f and g be continuous real functions such that the limits*

$$g(-\infty) = \lim_{s \to -\infty} g(s), \quad g(+\infty) = \lim_{s \to +\infty} g(s)$$

exist and are finite and such that for all $s \in R^1$,

$$g(-\infty) < g(s) < g(+\infty).$$

Let $p \in C_T^0$. *Then the equation*

$$-(-1)^k x^{(2k)}(t) + a_1 x^{(2k-1)}(t) + \ldots + a_1 x'(t) + f(x(t))\, x'(t) + g(x(t)) = p(t)$$

has at least one T-periodic solution if and only if

$$g(-\infty) < T^{-1} \int_0^T p(t)\, \mathrm{d}t < g(+\infty)\,. \tag{7}$$

Proof. Necessity follows at once from Lemma 4.1. Now if (7) holds, the function

$$s \mapsto g(s) - T^{-1} \int_0^T p(t)\, \mathrm{d}t$$

satisfies assumptions (C) and (D) and the function

$$F : s \mapsto \int_0^s \int_0^y f(u)\, \mathrm{d}u\, \mathrm{d}y$$

satisfies assumption (B). Thus, by Theorem 6.1, there exists at least one T-periodic solution of equation

$$-(-1)^k x^{(2k)}(t) + a_1 x^{(2k-1)}(t) + \ldots + a_{2k-1} x'(t) + f(x(t))\, x'(t) +$$
$$+ g(x(t)) - T^{-1} \int_0^T p(t)\, \mathrm{d}t = p(t) - T^{-1} \int_0^T p(t)\, \mathrm{d}t\,,$$

and the proof is complete.

References

[1] *S. Fučík:* Nonlinear equations with noninvertible linear part, Czech. Math. J. *24 (99)*, (1974), 467—495.

[2] *S. Fučík:* Periodic solutions of generalized Liénard equation with forcing term, to appear.

[3] *S. Fučík, J. Nečas, J.* and *Vl. Souček:* Spectral Analysis of Nonlinear Operators, Lect. Notes in Math. No 346, Springer, Berlin, 1973.

[4] *J. Mawhin:* Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, J. Differential Equations *12* (1972), 610—636.

[5] *J. Mawhin:* An extension of a theorem of A. C. Lazer on forced nonlinear oscillations, J. Math. Anal. Appl. *40* (1972), 20—29.

[6] *J. Mawhin:* Periodic solutions of some vector retarded functional differential equations, J. Math. Anal. Appl. *45* (1974), 588—603.

[7] *J. Mawhin:* L_2-estimates and periodic solutions of some nonlinear differential equations, Boll. Un. Mat. Ital. *(4)* 10 (1974), 343—354.

[8] *N. Rouche* and *J. Mawhin:* Equations différentielles ordinaires, 2 volumes, Masson, Paris, 1973.

Authors' addresses: S. Fučík, 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU), J. Mawhin, B-1348 Louvain-la-Neuve, Belgium (Institut Mathématique, Université Catholique).