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QUASIGROUPS WHICH SATISFY CERTAIN GENERALIZED FORMS OF THE ABELIAN IDENTITY

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INTRODUCTION

This paper is devoted to an investigation of quasigroups satisfying some weak forms of the basic identity

$$ab \,.\, cd = ac \,.\, bd\,, \tag{α}$$

called Abelian identity (sometimes the medial law). If G is a groupoid then let

$$\mathscr{A}(G) = \{x \mid x \in G,\ ax \,.\, bc = ab \,.\, xc\ \forall a, b, c \in G\}\,,$$

$$\mathscr{B}_r(G) = \{x \mid x \in G,\ ab \,.\, cx = ac \,.\, bx\ \forall a, b, c \in G\}\,,$$

$$\mathscr{B}_l(G) = \{x \mid x \in G,\ xa \,.\, bc = xb \,.\, ac\ \forall a, b, c \in G\}\,.$$

In the first section of this paper we shall study quasigroups and division groupoids which have non-empty $\mathscr{A}(G)$. Some similar results for $\mathscr{B}_l(G)$ and $\mathscr{B}_r(G)$ are discussed in the second part. Another class of quasigroups satisfying a weak form of (α) is that of LWA-quasigroups (left weakly Abelian), i.e., of quasigroups in which the following law holds:

$$aa \,.\, bc = ab \,.\, ac\,. \tag{β}$$

Similarly, a quasigroup satisfying

$$bc \,.\, aa = ba \,.\, ca \tag{γ}$$

will be called an RWA-quasigroup. If a quasigroup Q is simultaneously an LWA- and RWA-quasigroup, we shall say that Q is a WA-quasigroup. Some structure theorems on WA-quasigroups are proved in the fourth part. Finally, in the third section we shall give some applications of the first one to F-quasigroups. Recall that F-quasigroups (introduced in [1]) are characterized by the following two laws

$$(\delta) \qquad a \,.\, bc = ab \,.\, e(a)\, c\,,$$

$$(\varepsilon) \qquad bc \,.\, a = b\, f(a) \,.\, ca$$

where $e(a)$ and $f(a)$ are the right and the left local unit of a, respectively.

Notation and basic definitions. If G is a groupoid and $a \in G$ then L_a will be the left and R_a the right translation by a (i.e. $L_a(x) = ax$, $R_a(x) = xa$ for each $x \in G$). The groupoid G will be called a division groupoid if the mappings L_a and R_a are mappings onto G for all $a \in G$. As in [2], we shall say that G is a μ-groupoid if there are two mappings α, β of the set G onto G and a groupoid $G(\circ)$ possessing a unit such that $ab = \alpha(a) \circ \beta(b)$ for all $a, b \in G$. Finally, if Q is a loop (i.e., a quasigroup with a unit) then the unit element of Q will be denoted by j, the nucleus of Q by $N(Q)$ and the center of Q by $C(Q)$.

1. DIVISION GROUPOIDS WITH NON-EMPTY $\mathscr{A}(G)$

Let G be a division groupoid. A four-tuple $(G(\circ), \alpha, \psi, g)$ is said to be a right linear form of G if $G(\circ)$ is a group, α a mapping of G onto G, ψ an endomorphism of $G(\circ)$ onto $G(\circ)$, $g \in G$ an element, and if $ab = \alpha(a) \circ g \circ \psi(b)$ for all $a, b \in G$. Similarly a left linear form of G is defined. Finally, a four-tuple $(G(\circ), \varphi, \psi, g)$ will be called a linear form of G if it is a right linear form of G and moreover φ is an endomorphism of $G(\circ)$.

1.1 Theorem. *Let G be a groupoid. Then the following statements are equivalent:*

(i) *G is a division μ-groupoid and $\mathscr{A}(G)$ is non-empty.*

(ii) *G has a linear form $(G(\circ), \varphi, \psi, g)$ such that $\varphi\, \psi(a) \circ g = g \circ \psi\, \varphi(a)$ for every $a \in G$.*

In this case, $C(G(\circ))$ and $\mathscr{A}(G)$ coincide.

Proof. (i) implies (ii). The assertion (ii) is an easy consequence of Theorem 15 from [2]. By this theorem we get the existence of a linear form $(G(\circ), \varphi, \psi, g)$ such that $\varphi\, \psi(a) \circ h = h \circ \psi\, \varphi(a)$ for all $a \in G$, where $h = \varphi\, \psi(x) \circ g$ for some $x \in \mathscr{A}(G)$. However, with respect to the proof of Theorem 15 and by Theorem 11 ([2]) we can suppose without loss of generality that the element x is the unit of $G(\circ)$. In this case, $h = \varphi\, \psi(x) \circ g = x \circ g = g$. (ii) implies (i). Since G possesses a linear form, G is a division μ-groupoid. Furthermore, $C(G(\circ)) \subseteq \mathscr{A}(G)$, as one may check easily. On the other hand, if $y \in \mathscr{A}(G)$ then $\varphi\, \psi(a) \circ g \circ \psi\, \varphi(y) = \varphi\, \psi(y) \circ g \circ \psi\, \varphi(a)$ for all

$a \in G$, and hence $g \circ \psi\, \varphi(a) \circ \psi\, \varphi(y) = g \circ \psi\, \varphi(y) \circ \psi\, \varphi(a)$. Therefore $\psi\, \varphi(y) \in$ $\in C(G(\circ))$ and consequently $y \in C(G(\circ))$. Thus $C(G(\circ)) = \mathscr{A}(G)$.

1.2 Proposition. *Let G be a division μ-groupoid with non-empty $\mathscr{A}(G)$. Then $\mathscr{A}(G) = \{a \mid \forall b \in G\; \exists c,\, d \in G$ such that $ca \,.\, bd = cb \,.\, ad\}$.*

Proof. Let $(G(\circ), \varphi, \psi, g)$ be a linear form of G from 1.1. We have, for all $x, y, u, v \in G$, $xy \,.\, uv = \varphi^2(x) \circ \varphi(g) \circ \varphi\, \psi(y) \circ g \circ \psi\, \varphi(u) \circ \psi(g) \circ \psi^2(v)$ and so the equality $xy \,.\, uv = xu \,.\, yv$ holds iff $\varphi\, \psi(y) \circ g \circ \psi\, \varphi(u) = \varphi\, \psi(u) \circ g \circ \psi\, \varphi(y)$. Then obviously $my \,.\, un = mu \,.\, yn$ for all $m, n \in G$.

1.3 Theorem. *Let G be a division μ-groupoid with non-empty $\mathscr{A}(G)$. Then the following statements are equivalent:*

(i) *There are mappings α, β of G onto G such that $\alpha(a) \,.\, \beta(b) = \alpha(b) \,.\, \beta(a)$ for all $a, b \in G$.*

(ii) *There is $x \in G$ such that for all $a, b, c, d \in G$, $ab \,.\, cd = x$ implies $ab \,.\, cd =$ $= ac \,.\, bd$.*

(iii) *The mapping $a \to aa$ is an endomorphism of G.*

(iv) *G is Abelian.*

(v) *G has a linear form $(G(+), \varphi, \psi, g)$ such that $G(+)$ is an Abelian group and $\varphi\psi = \psi\varphi$.*

Proof. (i) implies (v). This implication is an easy consequence of Theorem 8 ([2]) and of 1.1.

(v) implies (iv). By 1.1, since $\mathscr{A}(G) = C(G+)) = G$.

(iv) implies (ii) and (iii). Trivial.

(iv) implies (i). Let $c \in G$ be arbitrary. Then L_c, R_c are onto and $L_c(a) \,.\, R_c(b) =$ $= ca \,.\, bc = cb \,.\, ac = L_c(b) \,.\, R_c(a)$ for all $a, b \in G$.

(ii) implies (iv). By 1.2, using the fact that G is a division groupoid.

(iii) implies (iv). We have $aa \,.\, bb = ab \,.\, ab$ for all $a, b \in G$ and therefore $\mathscr{A}(G) =$ $= G$ by 1.2.

1.4 Corollary. *Let G be a division μ-groupoid with non-empty $\mathscr{A}(G)$. Then G is Abelian, provided at least one of the following conditions holds.*

(i) *G is commutative.*

(ii) *G is idempotent* (i.e. $aa = a\; \forall a \in G$).

(iii) *G is unipotent* (i.e. $aa = bb\; \forall a, b \in G$).

Proof. (i) If G is commutative, then we can use 1.3 (i) setting $\alpha = \beta = 1_G$.

(ii) Let G be idempotent. Then $aa \,.\, bb = ab = ab \,.\, ab$ for all $a, b \in G$ and 1.3 (iii) yields the result.

(iii) Let G be unipotent. Then there is $x \in G$ such that $aa = bb = x$ for all $a, b \in G$. Hence $xx = x$ and consequently $ab \,.\, ab = x = xx = aa \,.\, bb$. Thus the map $a \to a^2$ is an endomorphism of G and 1.3 (iii) may be applied.

1.5 Proposition. (i) *Any quasigroup is a division μ-groupoid.*

(ii) *Let $(Q(\circ), \alpha, \psi, g)$ be a right linear form of a quasigroup Q. Then α and ψ are permutations of the set Q.*

(iii) *Let $(Q(\circ), \varphi, \beta, g)$ be a left linear form of a quasigroup Q. Then φ and β are permutations of Q.*

Proof. The statement (i) is a well known fact.

(ii) We have $\alpha(a) = (a \,.\, j) \circ g^{-1} = R_j(a) \circ g^{-1}$ and $\psi(a) = Ly(a)$ with $y = \alpha^{-1}(g^{-1})$. So α, ψ are permutations of Q.

(iii) Similarly.

1.6 Corollary. *Let Q be a quasigroup. Then the following conditions are equivalent:*

(i) *$\mathscr{A}(Q)$ is non-empty.*

(ii) *Q has a linear form $(Q(\circ), \varphi, \psi, g)$ such that $\varphi\, \psi(a) \circ g = g \circ \psi\, \varphi(a)$ for every $a \in Q$.*

In this case, $C(Q(\circ)) = \mathscr{A}(Q)$.

1.7 Theorem. *Let Q be a quasigroup with non-empty $\mathscr{A}(Q)$. Then $\mathscr{A}(Q)$ is a subquasigroup of Q if and only if there exists $x \in \mathscr{A}(Q)$ such that $xx \in \mathscr{A}(Q)$. In this case, $\mathscr{A}(Q)$ is a normal subquasigroup.*

Proof. Let $x \in \mathscr{A}(Q)$ be such that $xx \in \mathscr{A}(Q)$ and let $(Q(\circ), \varphi, \psi, g)$ be the linear form of Q from 1.1. With respect to the proof of 1.1, we can assume that x is the unit in $Q(\circ)$. Then $g = xx \in \mathscr{A}(Q)$. However, $\mathscr{A}(Q) = C(Q(\circ))$ is a characteristic subgroup of $Q(\circ)$, and hence $\varphi \mid \mathscr{A}(Q)$ and $\psi \mid \mathscr{A}(Q)$ are automorphisms of $\mathscr{A}(Q)$. Now it is obvious that $\mathscr{A}(Q)$ is a subquasigroup of Q. Furthermore, the normal congruence relation of the group $Q(\circ)$ corresponding to $\mathscr{A}(Q)$ is also a normal congruence relation of the quasigroup Q and so $\mathscr{A}(Q)$ is normal in Q.

1.8 Example. Let M be a finite set with card $M \geqq 7$ and let Q be the group of all permutations of the set M. Then ([3], p. 82) Q is a perfect group, i.e., $C(Q) = \{j\}$ and every automorphism of Q is an inner automorphism. Hence Q is isomorphic to Aut Q and since Q is not commutative, there are $\varphi, \psi, \alpha \in \operatorname{Aut} Q$ such that $\varphi\psi = \alpha\psi\varphi$ and $\alpha \neq 1_Q$. However, α is an inner automorphism of Q and so $\alpha(x) = gxg^{-1}$ for all $x \in Q$, where $g \in Q$ is convenient. Consider $Q(*)$, the quasigroup which has the linear form $(Q, \varphi\, \psi, g)$. Clearly $\mathscr{A}(Q(*))$ is non-empty (the unit of Q lies in $\mathscr{A}(Q(*))$, and by 1.6, it is $\mathscr{A}(Q(*)) = C(Q) = \{j\}$. But $j * j = g$ and $g \neq j$ since $\alpha \neq 1_Q$. Thus $\mathscr{A}(Q(*))$ is not a subquasigroup in $Q(*)$.

2. DIVISION GROUPOIDS WITH NON-EMPTY $\mathscr{B}_l(G)$

2.1 Theorem. *Let G be a groupoid. Then the following conditions are equivalent:*

(i) *G is a division μ-groupoid and the set $\mathscr{B}_l(G)$ is non-empty.*

(ii) *G has a right linear form $(G(+), \sigma, \psi, g)$ such that $G(+)$ is an Abelian group, $\sigma(\psi(a) + g) = \sigma(g) + \psi\, \sigma(a)$ for all $a \in G$ and $\sigma(0) = 0$.*

In this case,

$$\mathscr{B}_l(G) = \{x \mid \sigma(\sigma(x) + g + \psi(a)) = \sigma(\sigma(x) + g) + \psi\, \sigma(a)\ \forall a \in G\}\,.$$

Proof. (i) implies (ii). Since G is a μ-groupoid, there are a groupoid $G(\circ)$ with a unit j and mappings α, β of G onto G such that

$$ab = \alpha(a) \circ \beta(b) \quad \text{for all} \quad a, b \in G\,. \tag{1}$$

Let $x \in \mathscr{B}_l(G)$ be an arbitrary but fixed element. Put $\gamma = \alpha L_x$ and $\delta_c = \beta R_c$ for each $c \in G$. Then, with respect to (1), we obtain

$$\begin{aligned}\gamma(a) \circ \delta_c(b) &= \alpha\, L_x(a) \circ \beta\, R_c(b) = \alpha(xa) \circ \beta(bc) = \\ &= xa\,.\,bc = xb\,.\,ac = \gamma(b) \circ \delta_c(a) \quad \text{for all} \quad a, b, c \in G\,.\end{aligned} \tag{2}$$

Since γ is a mapping onto G, there is $y \in G$ such that $\gamma(y) = j$. If we set $a = y$ in (2), we get $\gamma(y) \circ \delta_c(b) = \delta_c(b) = \gamma(b) \circ \delta_c(y)$ for all $b \in G$. Using this result we see from (2) that

$$\gamma(a) \circ (\gamma(b) \circ \delta_c(y)) = \gamma(b) \circ (\gamma(a) \circ \delta_c(y)) \quad \text{for all} \quad a, b \in G\,. \tag{3}$$

Now it is easy to show that $G(\circ)$ is an Abelian group. Indeed, let $u, v, z \in G$ be arbitrary. Since G is a division groupoid and α, β are onto G, there exist $a, b, c \in G$ such that $\gamma(a) = \alpha(xa) = u$, $\gamma(b) = v$ and $\delta_c(y) = \beta(yc) = z$. Then the equality (3) yields $u \circ (v \circ z) = v \circ (u \circ z)$. However, $G(\circ)$ possesses a unit and so $G(\circ)$ must be a commutative semigroup. Hence it is enough to prove that $G(\circ)$ is a division groupoid. Indeed, let $a, b \in G$ be arbitrary. There are $s, t, p \in G$ such that $\gamma(p) = a$, $\gamma(t) = b$ and $\delta_s(t) = j$. So $a = \gamma(p) = \gamma(p) \circ j = \gamma(p) \circ \delta_s(t) = \gamma(t) \circ \delta_s(p) = b \circ \delta_s(p)$ and we have proved that $G(\circ)$ is an Abelian group.

Let us proceed to the proof of (ii). The mappings $R_{e(x)}$, L_x are onto G, and hence there exist mappings φ, ξ of G into G such that $R_{e(x)}\varphi = L_x\xi = 1_G$ and $\varphi(x) = x$, $\xi(x) = e(x)$. We introduce a new binary operation $+$ on the set G as follows:

$$(4) \qquad a + b = \varphi(a) \,.\, \xi(b) \quad \text{for all} \quad a, b \in G\,.$$

The groupoid $G(+)$ possesses a zero element (namely the element x) and by (1) and (4) we have

$$(5) \qquad a + b = \alpha\, \varphi(a) \circ \beta\, \xi(b) \quad \text{for all} \quad a, b \in G\,.$$

Therefore $a = a + 0 = \alpha\, \varphi(a) \circ \beta\, \xi(0)$, $b = \alpha\, \varphi(0) \circ \beta\, \xi(b)$ and we see that there is an element $k \in G$ such that

$$(6) \qquad a + b = a \circ b \circ k \quad \text{for all} \quad a, b \in G\,.$$

However, the equality (6) implies that $G(+)$ is an Abelian group and consequently $a + b - j = a \circ b$. Hence $ab = \alpha(a) \circ \beta(b) = \alpha(a) - \alpha(0) + \beta(b) + \alpha(0) - j = \sigma(a) + \varrho(b)$, where $\sigma(a) = \alpha(a) - \alpha(0)$, $\varrho(b) = \beta(b) + \alpha(0) - j$. Now we can write

$$\begin{aligned} xa \,.\, bc = 0a \,.\, bc &= \sigma\, \varrho(a) + \varrho(\sigma(b) + \varrho(c)) = \\ = 0b \,.\, ac &= \sigma\, \varrho(b) + \varrho(\sigma(a) + \varrho(c)) \end{aligned}$$

for all $a, b, c \in G$.

From this we can deduce that there are mappings π, τ of G into G with the property $\varrho(a + b) = \pi(a) + \tau(b)$ for all $a, b \in G$ and hence we complete the proof of (ii) by applying Lemma 17 from [2].

(ii) implies (i). Obvious.

2.2. Proposition. *Let G be a division μ-groupoid with non-empty $\mathscr{B}_l(G)$. Then $\mathscr{B}_l(G) = \{a \mid a \in G,\ \forall b, c \in G\ \exists d \in G$ such that $ab \,.\, cd = ac \,.\, bd\}$.*

Proof. Similar to that of 1.2.

2.3 Theorem. *Let G be a division μ-groupoid. Then the following conditions are equivalent:*

(i) *$\mathscr{B}_l(G)$ is non-empty and the map $a \to aa$ is an endomorphism of G.*

(ii) *At least two of the sets $\mathscr{A}(G)$, $\mathscr{B}_l(G)$, $\mathscr{B}_r(G)$ are non-empty.*

(iii) *G is Abelian.*

Proof. (i) implies (iii). Consider $(G(+), \sigma, \psi, g)$, the right linear form of G by 2.1. Since $a \to aa$ is an endomorphism of G, we get $\sigma(a + b) = \sigma(a + \psi\sigma^{-1}(a) + g) + \psi\sigma\psi^{-1}(b - g) - \psi(a) = \alpha(a) + \beta(b)$ for all $a, b \in G$. Hence, by Lemma 17 ([2]), there are an endomorphism φ of $G(+)$ and $k \in G$ such that $\sigma(a) = \varphi(a) + k$

for every $a \in G$. Since $\sigma(0) = 0$, it must be $k = 0$ and consequently the four-tuple $(G(+), \sigma, \psi, g)$ is a linear form of G. Further, $\sigma\,\psi(a) + \sigma(g) = \sigma(\psi(a) + g) = \sigma(g) + \psi\,\sigma(a)$, so $\psi\sigma = \sigma\psi$ and by 1.3 (v), G is an Abelian groupoid.

(ii) implies (iii). By 1.2 and 2.2.

(iii) implies (i) and (ii). Obvious.

2.4. Corollary. *Let G be a division μ-groupoid with non-empty $\mathscr{B}_l(G)$. Then G is Abelian, provided at least one of the following conditions holds:*

(i) *G is commutative.*

(ii) *G is idempotent.*

(iii) *G is unipotent.*

3. F-QUASIGROUPS ISOTOPIC TO A GROUP

3.1 Proposition. *Let a quasigroup Q have a linear form $(Q(\circ), \varphi, \psi, g)$. Then Q is an F-quasigroup if and only if φ, ψ are central automorphisms of $Q(\circ)$ and $\varphi\psi = \psi\varphi$.*

Proof. (i) Let Q be an F-quasigroup. Put $\psi_1(x) = g \circ \psi(x) \circ g^{-1}$. Then we have $e(x) = \psi_1^{-1}(\varphi(x^{-1}) \circ x \circ g^{-1})$ for all $x \in Q$, and hence the law (δ) may be written as

$$\varphi(a) \circ \psi_1\,\varphi(b) \circ \psi_1^2(c) \circ \psi_1(g) \circ g = \varphi^2(a) \circ \varphi\,\psi_1(b) \circ \varphi(g) \circ$$
$$\circ\, \psi_1\varphi\psi_1^{-1}(\varphi(a^{-1}) \circ a \circ g^{-1}) \circ \psi_1^2(c) \circ \psi_1(g) \circ g$$

for all $a, b, c \in Q$.

From this,

$$(1) \qquad \varphi(a) \circ \psi_1\,\varphi(b) = \varphi^2(a) \circ \varphi\,\psi_1(b) \circ \varphi(g) \circ \psi_1\varphi\psi_1^{-1}(\varphi(a^{-1}) \circ a \circ g^{-1})$$

for all $a, b \in Q$.

Now in (1) substitute $a = b = j$ to obtain

$$(2) \qquad j = \varphi(g) \circ \psi_1\varphi\psi_1^{-1}(g^{-1})\,.$$

From (1) and (2) we see (setting $a = j$) that $\varphi\psi_1 = \psi_1\varphi$, and consequently

$$(3) \qquad a \circ \psi_1(b) = \varphi(a) \circ \psi_1(b) \circ g \circ \varphi(a^{-1}) \circ a \circ g^{-1}$$

for all $a, b \in Q$.

So $\varphi(a^{-1}) \circ a \circ \psi_1(b) \circ g = \psi_1(b) \circ g \circ \varphi(a^{-1}) \circ a$, i.e., $\varphi(a^{-1}) \circ a \in C(Q(\circ))$ and φ is a central automorphism. Finally, $\varphi\,\psi_1(a) = \varphi(g \circ \psi(a) \circ g^{-1}) = \varphi(g) \circ \varphi\,\psi(a) \circ \varphi(g^{-1}) = \psi_1\,\varphi(a) = g \circ \psi\,\varphi(a) \circ g^{-1}$ and hence $g^{-1} \circ \varphi(g) \circ \varphi\,\psi(a) = \psi\,\varphi(a) \circ$

$\circ\, g^{-1} \circ \varphi(g)$. However, $g^{-1} \circ \varphi(g) \in C(Q(\circ))$ and therefore $\varphi\psi = \psi\varphi$. Similarly, using ($\varepsilon$), we can prove that ψ is a central automorphism.

(ii) Let the linear form $(Q(\circ), \varphi, \psi, g)$ have the required properties. Then, for all $a, b, c \in Q$ we have

$$ab \,.\, e(a)\, c = \varphi^2(a) \circ \varphi(g) \circ \varphi\, \psi(b) \circ g \circ \varphi(g^{-1}) \circ \varphi^2(a^{-1}) \circ \varphi(a) \circ$$
$$\circ\, \psi(g) \circ \psi^2(c) = \varphi(a) \circ g \circ \psi\, \varphi(b) \circ \psi(g) \circ \psi^2(c) = a \,.\, bc\,.$$

Thus Q satisfies (δ). The law (ε) may be proved in a similar way.

3.2 Theorem. *Let Q be a quasigroup. Then the following statements are equivalent:*

(i) *Q is an F-quasigroup with non-empty $\mathscr{A}(Q)$.*

(ii) *Q is an F-quasigroup isotopic to a group.*

(iii) *Q has a linear form $(Q(\circ), \varphi, \psi, g)$ such that $\varphi\psi = \psi\varphi$, φ, ψ are central automorphisms of $Q(\circ)$ and $g \in C(Q(\circ))$.*
In this case, $\mathscr{A}(Q) = C(Q(\circ))$.

Proof. (i) implies (iii). By 1.6, Q has a linear form $(Q(\circ), \varphi, \psi, g)$ such that $\varphi\, \psi(a) \circ g = g \circ \psi\, \varphi(a)$ for every $a \in Q$. According to 3.1, φ and ψ are central and $\varphi\psi = \psi\varphi$. So $\varphi\, \psi(a) \circ g = g \circ \varphi\, \psi(a)$ for all $a \in Q$, and consequently $g \in C(Q(\circ))$.

(iii) implies (ii). By 3.1.

(ii) implies (i). Since Q is isotopic to a group, there are permutations α, β of the set Q and a group $Q(\circ)$ such that $ab = \alpha(a) \circ \beta(b)$ for all $a, b \in Q$. The law (δ) yields the equality

$$\alpha(a) \circ \beta(\alpha(b) \circ \beta(c)) = \alpha(\alpha(a) \circ \beta(b)) \circ \beta(\alpha\, e(a) \circ \beta(c))$$

for all $a, b, c \in Q$.

Therefore $\beta(b \circ c) = \gamma(b) \circ \delta(c)$, where γ and δ are suitable permutations. By Lemma 17 [2], there is $x \in Q$ such that the mapping ψ, defined by $\psi(a) = x^{-1} \circ \beta(a)$, is an automorphism of $Q(\circ)$. Similarly, using (ε), we can show that there is $y \in Q$ such that the mapping φ with $\varphi(a) = \alpha(a) \circ y^{-1}$ is an automorphism of $Q(\circ)$. So $(Q(\circ), \varphi, \psi, y \circ x)$ is a linear form of Q and hence $\varphi\psi = \psi\varphi$ and φ, ψ are central (by 3.1.). Now it is easy to verify that $\varphi^{-1}\psi^{-1}(g^{-1}) \in \mathscr{A}(Q)$, $g = y \circ x$.

3.3 Proposition. *Let Q be an F-quasigroup with non-empty $\mathscr{A}(Q)$. Then $e(a), f(a) \in \mathscr{A}(Q)$ for each $a \in Q$. In particular, any idempotent element is contained in $\mathscr{A}(Q)$.*

Proof. Let $(Q(\circ), \varphi, \psi, g)$ be the linear form of Q from 3.2. Then $\mathscr{A}(Q) = C(Q(\circ))$ and $g \in C(Q(\circ))$. Since φ is central, $\varphi(a^{-1}) \circ a \in C(Q(\circ))$ for every $a \in Q$. Further

$\psi^{-1}(g^{-1}) \in C(Q(\circ))$, and hence $e(a) = \psi^{-1}(\varphi(a^{-1}) \circ a) \circ \psi^{-1}(g^{-1}) \in C(Q(\circ)) = \mathscr{A}(Q)$. Similarly, $f(a) \in \mathscr{A}(Q)$.

3.4 Theorem. *Let Q be an F-quasigroup with non-empty $\mathscr{A}(Q)$. Then $\mathscr{A}(Q)$ is a normal subquasigroup of Q. Moreover, $\mathscr{A}(Q)$ is an Abelian quasigroup and the factor-quasigroup $Q/\mathscr{A}(Q)$ is a group.*

Proof. Consider $(Q(\circ), \varphi, \psi, g)$, the linear form of Q from 3.2. Then $\mathscr{A}(Q) = C(Q(\circ))$ and $j \in \mathscr{A}(Q)$, $jj = g \in \mathscr{A}(Q)$. Now 1.7 may be used. Finally, with respect to 3.3, $Q/\mathscr{A}(Q)$ has a unit, and since it is an F-quasigroup, it is a group.

3.5 Corollary. *Let Q be an F-quasigroup and let the set $\mathscr{A}(Q)$ have exactly one element. Then Q is a group.*

3.6 Theorem. *Let Q be an F-quasigroup and let $\mathscr{B}_l(Q)$ (or $\mathscr{B}_r(Q)$) be non-empty. Then Q is Abelian.*

Proof. By 2.1, 3.2 and 2.3.

4. WA-QUASIGROUPS

Let Q be a loop and $\alpha : Q \to Q$ a mapping. We shall say that the loop Q satisfies the N_α-law (N^α-law) if

$$(\varphi) \qquad (\alpha(a)\,.\,a)(bc) = (\alpha(a)\,.\,b)(ac)\ ((bc)(\alpha(a)\,.\,a) = (b\,.\,\alpha(a))(ca)) \quad \text{for all} \quad a, b, c \in Q\,.$$

Further we shall say that α is a nuclear mapping if $\varrho(x) \in N(Q)$ for all $x \in Q$, where $x\,\varrho(x) = \alpha(x)$.

4.1 Proposition. *Let Q be a loop satisfying the N_α-law for a mapping $\alpha : Q \to Q$. Then:*

(i) $\qquad \alpha(a)\,a\,.\,b = \alpha(a)\,b\,.\,a = \alpha(a)\,.\,ab$ *for all* $a, b \in Q$.

(ii) $\qquad (\alpha(a)\,a)(bc) = (\alpha(a)\,b)(ac) = \alpha(a)(a\,.\,bc) = (\alpha(a)\,.\,bc)\,a$ *for all* $a, b, c \in Q$.

(iii) $\qquad$ *Q is a CI-loop (i.e., $a\,.\,ba^{-1} = b$ with $aa^{-1} = j$).*

Proof. (i) Immediately by (φ) setting $b = j$ or $c = j$.

(ii) By (i) and (φ).

(iii) According to (ii) we can write

$$\alpha(a)\,b = (\alpha(a)\,b)\,(aa^{-1}) = \alpha(a)\,(a\,.\,ba^{-1})\,.$$

Therefore $b = a\,.\,ba^{-1}$.

4.2 Proposition. *Let Q be a commutative loop and $\alpha : Q \to Q$ a mapping. Then the following assertions are equivalent:*

(i) *Q satisfies the N_α-law.*

(ii) *Q is a Moufang loop and α is a nuclear mapping.*

Proof. (i) implies (ii). Using 4.1 (ii) and the commutativity of Q, we get $(ab)\,.\,(c\,.\,\alpha(a)) = (a\,.\,bc)\,\alpha(a)$, i.e., the loop Q satisfies the M_α-law introduced in [4]. Now we may apply Theorem 1 from [4].

(ii) implies (i). By Theorem 2 [4], the loop Q satisfies the M_α-law, i.e., $(ab)\,.\,(c\,.\,\alpha(a)) = (a\,.\,bc)\,\alpha(a)$ for all $a, b, c \in Q$. In particular we have $(ab)\,(a\,.\,\alpha(a)) = (a\,.\,ba)\,\alpha(a)$, which may be written as $\alpha(a)\,.\,ac = \alpha(a)\,a\,.\,c$ for all $a, c \in Q$. Thus $(\alpha(a)\,a)\,(bc) = \alpha(a)\,(a\,.\,bc) = (\alpha(a)\,b)\,(ac)$.

4.3 Proposition. *Let a WA-quasigroup Q be isotopic to a group. Then it is an Abelian quasigroup.*

Proof. We have $ab = \alpha(a) \circ \beta(b)$ for all $a, b \in Q$, where $Q(\circ)$ is a group and α, β are some permutations of the set Q. Therefore, using the law (β), we see that there are permutations γ, δ of Q such that $\beta(a \circ b) = \gamma(a) \circ \delta(b)$ for all $a, b \in Q$. By Lemma 17 [2], there are $k \in Q$ and $\psi \in \operatorname{Aut} Q(\circ)$ such that $\beta(a) = k \circ \psi(a)$ for all $a \in Q$. For the same reason (considering the law (γ)), there are $l \in Q$ and $\varphi \in \operatorname{Aut} Q(\circ)$ such that $\alpha(a) = \varphi(a) \circ l$. Hence the four-tuple $(Q(\circ), \varphi, \psi, l \circ k)$ is a linear form of Q. Now, if we set $g = l \circ k$, we may write the law (β) as $aa\,.\,bc = \varphi^2(a) \circ \varphi(g) \circ \varphi\,\psi(a) \circ g \circ \psi\,\varphi(b) \circ \psi(g) \circ \psi^2(c) = ab\,.\,ac = \varphi^2(a) \circ \varphi(g) \circ \varphi\,\psi(b) \circ g \circ \psi\,\varphi(a) \circ \psi(g) \circ \psi^2(c)$, and consequently

$$\varphi\,\psi(a) \circ g \circ \psi\,\varphi(b) = \varphi\,\psi(b) \circ g \circ \psi\,\varphi(a)$$

for all $a, b \in Q$.

From this we can easily deduce that $xa\,.\,by = xb\,.\,ay$ for all $a, b, x, y \in Q$. Thus Q is an Abelian quasigroup.

4.4 Lemma. *Let Q be an LWA-quasigroup. Then:*

(i) *$L_{xx}R_y = R_{xy}L_x$, $L_{xx}L_y = L_{xy}L_x$ and $L_{xx}R_x = R_{xx}L_x$ for all $x, y \in Q$.*

(ii) *$L_{xx}(ab) = L_x(a)\,.\,L_x(b)$ and $L_{xx}^{-1}(ab) = L_x^{-1}(a)\,.\,L_x^{-1}(b)$ for all $x, a, b \in Q$.*

Proof. Obvious from (β).

4.5 Proposition. *Let Q be an LWA-quasigroup and let there be an element $x \in Q$ such that $ax = xa$ for all $a \in Q$. Then Q is commutative.*

Proof. Let $z \in Q$ be arbitrary. There exists $y \in Q$ such that $yx = z$. By Lemma 4.4 we have $L_{yy}R_x = R_{yx}L_y = R_zL_y$. However, $R_x = L_x$ and $L_{yy}L_x = L_{yx}L_y = L_zL_y$. Thus $L_{yy}R_x = R_zL_y = L_{yy}L_x = L_zL_y$ and consequently, $L_z = R_z$.

4.6 Proposition. *Let Q be an LWA-quasigroup and $x \in Q$ an element. Put $xx = y$, $yy = z$ and $a * b = L_x^{-1}(a) \,.\, L_x^{-1}(b)$ for all $a, b \in Q$. Then:*

(i) *The quasigroup $Q(*)$ possesses a left unit, namely the element y.*

(ii) *For all $a, b, c, d \in Q$, $(a * b) * (c * d) = L_y^{-1}L_z^{-1}(ab \,.\, cd)$.*

(iii) *$Q(*)$ is an LWA-quasigroup.*

(iv) *$Q(*)$ is an RWA-quasigroup iff Q is.*

(v) *$Q(*)$ is a loop iff Q is commutative.*

Proof. (i) $y * a = L_x^{-1}(xx) \,.\, L_x^{-1}(a) = x \,.\, L_x^{-1}(a) = a$.

(ii) By 4.4 we have $a * b = L_x^{-1}(a) \,.\, L_x^{-1}(b) = L_y^{-1}(ab)$. Hence $(a * b) * (c * d) =$
$= L_y^{-1}(ab) * L_y^{-1}(cd) = L_y^{-1}(L_y^{-1}(ab) \,.\, L_y^{-1}(cd)) = L_y^{-1}L_{yy}^{-1}(ab \,.\, cd) =$
$= L_y^{-1}L_z^{-1}(ab \,.\, cd)$.

(iii) and (iv). Immediately by (ii).

(v) If Q is commutative then obviously $Q(*)$ is commutative, thus being a loop. On the other hand, if $Q(*)$ is a loop then $a * y = a$ for all $a \in Q$, i.e. $L_y^{-1}Ry(a) = a$. Hence $L_y = R_y$ and Q is commutative by 4.5.

4.7 Theorem. *Let Q be a commutative quasigroup. Then the following assertions are equivalent:*

(i) *Q is a WA-quasigroup.*

(ii) *There are a commutative Moufang loop $Q(*)$, $\alpha \in \operatorname{Aut} Q(*)$ and $g \in Q$ such that $ab = \alpha(a * b) * g$ for all $a, b \in Q$.*

Proof. (i) implies (ii). Let $x \in Q$ be arbitrary, $xx = y$ and $a * b = L_x^{-1}(a) \,.\, L_x^{-1}(b)$ for all $a, b \in Q$. Then, by 4.6 (i), (iii) and (v), $Q(*)$ is a commutative Moufang loop with the unit $j = y$ (all the properties of commutative Moufang loops used in this paper can be found in [5] or [6]). Therefore $L_y(a * b) = L_y(L_x^{-1}(a) \,.\, L_x^{-1}(b)) =$ $= ab = L_x(a) * L_x(b)$, and so $L_y(a) = L_x(a) * L_x(j) = L_x(a) * k$. Further, we have

$$\begin{aligned} L_x(a * b) * k^{-1} &= (L_y(a * b) * k^{-1}) * k^{-1} = L_y(a * b) * (k^{-1} * k^{-1}) = \\ &= (L_x(a) * L_x(b)) * (k^{-1} * k^{-1}) = (L_x(a) * k^{-1}) * (L_x(b) * k^{-1}) \end{aligned}$$

and we see that the map $a \to L_x(a) * k^{-1}$ is an automorphism of $Q(*)$. Now we can put $\alpha(a) = L_x(a) * k^{-1}$ and $g = k * k$.

(ii) implies (i). We may write

$$aa \,.\, bc = \alpha((\alpha(a * a) * g) * (\alpha(b * c) * g)) * g =$$
$$= \alpha((\alpha(a * a) * \alpha(b * c)) * (g * g)) * g = \alpha((\alpha(a * b) * \alpha(a * c)) * (g * g)) * g =$$
$$= \alpha((\alpha(a * b) * g) * (\alpha(a * c) * g)) * g = ab \,.\, ac \,.$$

4.8 Proposition. *Let Q be a WA-quasigroup and $x \in Q$ an element. Put $xx = y$, $yy = z$ and $a \circ b = R_y^{-1}(a) \,.\, L_y^{-1}(b)$. Then:*

(i) *$Q(\circ)$ is a loop with the unit $j = z$.*

(ii) *The mapping $\alpha = R_y L_y^{-1}$ is an automorphism of $Q(\circ)$.*

(iii) *The loop $Q(\circ)$ satisfies the N_α-law.*

(iv) *The loop $Q(\circ)$ satisfies the N^α-law.*

(v) *The loop $Q(\circ)$ is commutative iff $ya \,.\, by = yb \,.\, ay$ for all a, b.*

Proof. (i). Obvious.

(ii) We may write (using 4.4)

$$\alpha(a \circ b) = R_y L_y^{-1}(R_y^{-1}(a) \,.\, L_y^{-1}(b)) = R_y(L_x^{-1}R_y^{-1}(a) \,.\, L_x^{-1}L_y^{-1}(b)) =$$
$$= R_y(R_x^{-1}L_y^{-1}(a) \,.\, L_x^{-1}L_y^{-1}(b)) = R_x R_x^{-1}L_y^{-1}(a) \,.\, R_x L_x^{-1}L_y^{-1}(b) =$$
$$= L_y^{-1}(a) \,.\, R_x L_x^{-1}L_y^{-1}(b) = R_y^{-1}R_y L_y^{-1}(a) \,.\, L_y^{-1}R_y L_y^{-1}(b) = \alpha(a) \circ \alpha(b) \,.$$

(iii) We have

$$(\alpha(a) \circ a) \circ (b \circ c) =$$
$$= R_y^{-1}(R_y^{-1}\,\alpha(a) \,.\, L_y^{-1}(a)) \,.\, L_y^{-1}(R_y^{-1}(b) \,.\, L_y^{-1}(c)) =$$
$$= (R_x^{-1}L_y^{-1}(a) \,.\, R_x^{-1}\,L_y^{-1}(a)) \,.\, (L_x^{-1}\,R_y^{-1}(b) \,.\, L_x^{-1}\,L_y^{-1}(c)) =$$
$$= (R_x^{-1}\,L_y^{-1}(a) \,.\, L_x^{-1}\,R_y^{-1}(b)) \,.\, (R_x^{-1}\,L_y^{-1}(a) \,.\, L_x^{-1}\,L_y^{-1}(c)) =$$
$$= (R_x^{-1}R_y^{-1}\,\alpha(a) \,.\, R_x^{-1}\,L_y^{-1}(b)) \,.\, (L_x^{-1}\,R_y^{-1}(a) \,.\, L_x^{-1}\,L_y^{-1}(c)) =$$
$$= R_y^{-1}(R_y^{-1}\,\alpha(a) \,.\, L_y^{-1}(b)) \,.\, L_y^{-1}(R_y^{-1}(a) \,.\, L_y^{-1}(c)) = (\alpha(a) \circ b) \circ (a \circ c) \,.$$

(iv) Similarly as for (iii).

(v) Let the loop $Q(\circ)$ be commutative. Then $R_y^{-1}(a) \,.\, L_y^{-1}(b) = R_y^{-1}(b) \,.\, L_y^{-1}(a)$ for all $a, b \in Q$. Therefore $a \,.\, L_y^{-1}\,R_y(b) = b \,.\, L_y^{-1}\,R_y(a)$ and consequently,

$$L_z(a \,.\, L_y^{-1}\,R_y(b)) = L_y(a) \,.\, R_y(b) = ya \,.\, by =$$
$$= L_z(b \,.\, L_y^{-1}\,R_y(a)) = yb \,.\, ay \,.$$

If, on the contrary, $ya \,.\, by = yb \,.\, ay$ for all $a, b \in Q$, then we can reverse our argument.

If Q is a quasigroup then let $\mathscr{D}(Q) = \{a \mid ab \,.\, ca = ac \,.\, ba \text{ for all } b, c \in Q\}$.

4.9 Theorem. *Let Q be a quasigroup. Then the following statements are equivalent*:

(i) *Q is a WA-quasigroup and $aa \in \mathscr{D}(Q)$ for every $a \in Q$.*

(ii) *Q is a WA-quasigroup and there is $x \in Q$ such that $xx \in \mathscr{D}(Q)$.*

(iii) *There are a commutative Moufang loop $Q(\circ)$, $\varphi, \psi \in \operatorname{Aut} Q(\circ)$ and $g \in Q$ such that $\varphi\psi = \psi\varphi$, $\varphi\psi^{-1}$ is a nuclear automorphism of $Q(\circ)$ and $ab = (\varphi(a) \circ \psi(b)) \circ g$ for all $a, b \in Q$.*

Proof. (i) implies (ii) trivially.

(ii) implies (iii). Let $y = xx$, $yy = z$ and $a \circ b = R_y^{-1}(a) \, . \, L_y^{-1}(b)$. Then, by 4.8, $Q(\circ)$ is a loop with the unit $j = z$ satisfying the N_α and N^α-laws for $\alpha = R_y L_y^{-1}$. Moreover, $\alpha \in \operatorname{Aut} Q(\circ)$ and $Q(\circ)$ is commutative. Hence (by 4.2), $Q(\circ)$ is a commutative Moufang loop and α is a nucelar mapping of $Q(\circ)$. Now, in view of 4.4, we can write

$$R_y(a \circ b) = R_y(R_y^{-1}(a) \, . \, L_y^{-1}(b)) = \gamma(a) \circ \delta(b) \, ;$$

$$\gamma(a) = R_y R_x R_y^{-1}(a) \, , \quad \delta(b) = L_y R_x L_y^{-1}(b) \, .$$

So $R_y(a) = \gamma(a) \circ \delta(j) = \gamma(a) \circ m$ and $R_y(b) = \gamma(j) \circ \delta(b) = n \circ \delta(b)$. From this we see

$$R_y(a \circ b) = \gamma(a) \circ \delta(b) = (R_y(a) \circ m^{-1}) \circ (R_y(b) \circ n^{-1}) \, . \tag{1}$$

However,

$$\alpha(m) = R_y L_y^{-1} \, \delta(j) = R_y L_y^{-1} L_y R_x \, L_y^{-1}(j) = R_y R_x \, L_y^{-1}(j) =$$
$$= R_y R_x \, L_y^{-1}(yy) = R_y \, R_x(y) = R_y R_x \, R_y^{-1}(yy) = \gamma(j) = n$$

and consequently, $\alpha(m^{-1}) = n^{-1}$ since α is an automorphism of $Q(\circ)$. Applying the N_α-law to (1) we get

$$R_y(a \circ b) = (R_y(a) \circ R_y(b)) \circ k^{-1} \, , \quad k^{-1} = m^{-1} \circ n^{-1} \, . \tag{2}$$

The equality (2) implies, as one may check easily, $\varphi \in \operatorname{Aut} Q(\circ)$ where $\varphi(a) = R_y(a) \circ k^{-1}$ for each $a \in Q$. Furthermore, $R_y(j) = (R_y(j) \circ R_y(j)) \circ k^{-1}$, and hence $k = R_y(j)$ (here we use the fact that $Q(\circ)$ is a loop with the inverse property). Similarly we can show that $\psi \in \operatorname{Aut} Q(\circ)$; $\psi(a) = L_y(a) \circ L_y(j) = L_y(a) \circ l^{-1}$. Further, $\alpha(l) = R_y L_y^{-1} \, L_y(j) = R_y(j) = k$, hence $\alpha(l^{-1}) = k^{-1}$ and we have

$$ab = R_y(a) \circ L_y(b) = (\varphi(a) \circ k) \circ (\psi(b) \circ l) =$$
$$= (\varphi(a) \circ \alpha(l)) \circ (\psi(b) \circ l) = (\varphi(a) \circ \psi(b)) \circ (\alpha(l) \circ l) = (\varphi(a) \circ \psi(b)) \circ g \, .$$

Now it remains to prove that $\varphi\psi = \psi\varphi$ and $\varphi\psi^{-1}$ is nuclear. However,

$$\begin{aligned}\varphi\,\psi^{-1}(a) &= \varphi L_y^{-1}(a \circ l) = R_y L_y^{-1}(a \circ l) \circ \alpha(l^{-1}) = \\ &= \alpha(a \circ l) \circ \alpha(l^{-1}) = (\alpha(a) \circ \alpha(l)) \circ \alpha(l^{-1}) = \alpha(a)\end{aligned}$$

for all $a \in Q$.

Finally

$$\begin{aligned}&(\psi\,\varphi(a) \circ \varphi(g)) \circ g = \\ &\quad = (\varphi((\varphi(j) \circ \psi(j)) \circ g) \circ \psi((\varphi(a) \circ \psi\,\psi^{-1}(g^{-1})) \circ g)) \circ g = \\ &\quad = jj\,.\,a\psi^{-1}(g^{-1}) = ja\,.\,j\,\psi^{-1}(g^{-1}) = \\ &\quad = (\varphi((\varphi(j) \circ \psi(a)) \circ g) \circ \psi((\varphi(j) \circ \psi\,\psi^{-1}(g^{-1})) \circ g)) \circ g = (\varphi\,\psi(a) \circ \varphi(g)) \circ g\,.\end{aligned}$$

Thus $\psi\,\varphi(a) = \varphi\,\psi(a)$ for all $a \in Q$ which completes the proof.

(iii) implies (i). Since $\varphi\psi^{-1}$ is a nuclear mapping, the loop $Q(\circ)$ satisfies the $N_{\varphi\psi^{-1}}$-law. Further $\varphi\psi = \psi\varphi$, and so we can write

$$\begin{aligned}&aa\,.\,bc = \\ &\quad = ((\varphi^2(a) \circ \varphi\,\psi(a)) \circ \varphi(g)) \circ ((\psi\,\varphi(b) \circ \psi^2(c)) \circ \psi(g)) \circ g = \\ &\quad = ((\varphi^2(a) \circ \varphi\,\psi(a)) \circ \varphi\psi^{-1}\,\psi(g)) \circ ((\psi\,\varphi(b) \circ \psi^2(c)) \circ \psi(g)) \circ g = \\ &\quad = (((\varphi\psi^{-1}\varphi\,\psi(a) \circ \varphi\,\psi(a)) \circ (\psi\,\varphi(b) \circ \psi^2(c))) \circ (\varphi(g) \circ \psi(g))) \circ g = \\ &\quad = (((\varphi^2(a) \circ \varphi\,\psi(b)) \circ (\psi\,\varphi(a) \circ \psi^2(c))) \circ (\varphi(g) \circ \psi(g))) \circ g = ab\,.\,ac\,.\end{aligned}$$

Similarly we can show that Q is an RWA-quasigroup. Let now $a \in Q$ be arbitrary. Put $b * c = R_{aa}^{-1}(b)\,.\,L_{aa}^{-1}(c)$. Then $Q(*)$ is a loop satisfying the N_β-law for $\beta = R_{aa}L_{aa}^{-1}$ (see 4.8). On the other hand, the quasigroup Q is isotopic to a Moufang loop and consequently $Q(*)$ must be a Moufang loop. However, $Q(*)$ is a CI-loop and therefore $Q(*)$ is commutative. Hence, by 4.8 (v), $aa \in \mathscr{D}(Q)$.

4.10 Proposition. *Let a loop Q satisfy the N_α- and N^α-laws for some $\alpha \in \operatorname{Aut} Q$. For every $a, b \in Q$ put $a \circ b = \alpha(a)\,.\,b$. Then $Q(\circ)$ is a WA-quasigroup and a left loop.*

Proof. We have

$$(a \circ a) \circ (b \circ c) = (\alpha^2(a)\,.\,\alpha(a))\,(\alpha(b)\,.\,c) = (\alpha^2(a)\,.\,\alpha(b))\,(\alpha(a)\,.\,c) = (a \circ b) \circ (a \circ c)\,.$$

Similarly $(b \circ c) \circ (a \circ a) = (b \circ a) \circ (c \circ a)$. Finally, $j \circ a = \alpha(j)\,.\,a = a$ for every $a \in Q$.

Remark. The author does not know whether there exists a quasigroup Q with the following properties:

(i) Q is a WA-quasigroup,

(ii) there is $a \in Q$ such that $aa \notin \mathscr{D}(Q)$.

This problem is equivalent to the one whether any loop satisfying the N_α- and N^α-laws for some automorphism α need be Moufang.

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