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Časopis pro pěstování matematiky, roč. 99 (1974), Praha

THE EQUIVALENCE OF SOME INTEGRAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

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1. PRELIMINARIES

1.1. Notation. For $m, n = 1, 2, \ldots, \boldsymbol{R}^{m \times n}(\boldsymbol{K}^{m \times n})$ will stand for the space of all real (complex) $m \times n$-matrices with the Euclidean norm (denoted by $|\cdot|$). We shall denote the identity matrices by I and the zero matrices by 0 or, more detailed, I_n, 0_{mn}. The m-dimensional vectors will be identified with column matrices and we shall write shortly $\boldsymbol{R}^m(\boldsymbol{K}^m)$ in stead of $\boldsymbol{R}^{m \times 1}(\boldsymbol{K}^{m \times 1})$. $\chi(A)$ will be the range of a matrix $A \in \boldsymbol{K}^{m \times n}$, A^{T} its transpose. $\boldsymbol{R}^{n\mathrm{R}n}(\boldsymbol{K}^{n\mathrm{R}n})$ will be the set of all regular $n \times n$-matrices. Let $m_i > 0$, $n_j > 0$ be integers for $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$; $Y_{ij} \in \boldsymbol{K}^{m_i \times n_j}$. We shall identify the matrix

$$\begin{bmatrix} Y_{11} & Y_{12} & \ldots & Y_{1q} \\ Y_{21} & Y_{22} & \ldots & Y_{2q} \\ \ldots & \ldots & \ldots & \ldots \\ Y_{p1} & Y_{p2} & \ldots & Y_{pq} \end{bmatrix}$$

with the corresponding element of $\boldsymbol{K}^{M \times N}$, where $M = m_1 + \ldots + m_p$, $N = n_1 + \ldots + n_q$. The partial derivatives of a function f with the domain in $\boldsymbol{R}^p$ will be denoted by

$$D^i f(u_1, \ldots, u_p) = D^{i_1, \ldots, i_p} f(u_1, \ldots, u_p) = \frac{\partial^{i_1 + \ldots + i_p} f(u_1, \ldots, u_p)}{\partial u_1^{i_1} \ldots \partial u_p^{i_p}}$$

where $i = (i_1, \ldots, i_p)$ denotes some multiindex, $p = 1, 2, \ldots$ We define the 0-th derivative of a given function to be equal to the function itself. Let $\mathscr{U}$ be a subset of the domain of a function f. Then $f/\mathscr{U}$ will denote the corresponding partial function. $\overline{\mathscr{U}}$ will be the closure of a set $\mathscr{U}$, $\mathscr{U}^0$ the set of its inner points, $\mathscr{I} = \langle 0, 1 \rangle$. Let $\mathscr{G}_0, \mathscr{G}_1, \ldots, \mathscr{G}_r$ be disjoint domains in $\boldsymbol{R}^p$, $\mathscr{G} = \mathscr{G}_0 \cup \ldots \cup \mathscr{G}_r$, $f : \mathscr{G} \to \boldsymbol{K}^{m \times n}$, $f_i = f/\mathscr{G}_i$, $i = 1, \ldots, r$, $k \geqq 0$ an integer. We shall denote by $\boldsymbol{C}^{(k)}_{m \times n}(\widetilde{\mathscr{G}})$ the space of those f for which all derivatives of f_i of all orders $\leqq k$ exist and may be continuously extended on $\bar{\mathscr{G}}_i$, $i = 1, 2, \ldots, r$. We shall suppose that f_i are defined on $\bar{\mathscr{G}}_i$ by these extensions, $i = 1, 2, \ldots, r$. $C^{(k)}_{nRn}(\tilde{\mathscr{G}})$ will be the subset of those $f \in C^{(k)}_{n\times n}(\tilde{\mathscr{G}})$ for which $f_i(t)$ are regular matrices for all $t \in \bar{\mathscr{G}}_i$, $i = 1, \ldots, r$. We shall sometimes omit $k = 0$ and $n = 1$ in the notation introduced.

$L^p_{m\times n}(\Omega)$ will be the Lebesgue space of all $m \times n$-matrix „functions" f for which $|f|^p$ is integrable on Ω.

Now, we shall introduce some simple lemmas concerning the so called adjoint and complementary adjoint matrices – see mostly O. Vejvoda, M. Tvrdý [2]. l, m, n will denote positive integers, $l + m = 2n$.

1.2. Lemma. *Let*

$$M_0, M_1 \in \boldsymbol{K}^{m\times n}, \tag{2,1}$$

$$\chi([M_0, M_1]) = m. \tag{2,2}$$

Then there exist

$$P_0, P_1 \in \boldsymbol{K}^{n\times l} \tag{2,3}$$

so that it holds

$$\chi\left(\begin{bmatrix} P_0 \\ P_1 \end{bmatrix}\right) = l, \tag{2,4}$$

$$M_0 P_0 + M_1 P_1 = 0. \tag{2,5}$$

1.3. Remark. Let us denote

$$M = [M_0, M_1], \tag{3,1}$$

$$P = \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}. \tag{3,2}$$

We can express Lemma 2 in the following equivalent form: *let*

$$M \in \boldsymbol{K}^{m\times 2n}, \quad \chi(M) = m. \tag{3.3}$$

Then there exists

$$P \in \boldsymbol{K}^{2n\times l}. \tag{3.4}$$

so that $\chi(P) = l$,

$$MP = 0. \tag{3.5}$$

(The columns of the matrix P form a basis of all solutions of the equation $Mx = 0$. It holds $\tilde{P} = PQ$, where $Q \in \boldsymbol{K}^{l\mathrm{R}l}$, if both P and $\tilde{P}$ fulfil (3,4–5).)

1.4. Lemma. *Let* (2,1–5) *hold. Let us denote*

(4,1)
$$G = \begin{bmatrix} M_0 & M_1 \\ P_0^{\mathrm{T}} & P_1^{\mathrm{T}} \end{bmatrix}.$$

Then

(4,2)
$$G \in \boldsymbol{K}^{2n\mathrm{R}2n},$$

(4,3)
$$GG^{\mathrm{T}} = \begin{bmatrix} G_0 & 0 \\ 0 & G_1 \end{bmatrix},$$

where

(4,4)
$$G_0 \in \boldsymbol{K}^{m\mathrm{R}m}, \quad G_1 \in \boldsymbol{K}^{l\mathrm{R}l}.$$

1.5. Remark. G_0, G_1 are the Gramm matrices formed by the rows of M and columns of P, respectively (the notation (3.1–2) being used).

1.6. Lemma. *Let*

(6,1)
$$U_0, U_1 \in \boldsymbol{K}^{n\mathrm{R}n},$$

let (2,1–5) *hold. Let us put*

(6,2)
$$S_0 = U_1^{-1}P_1, \quad S_1 = U_0^{-1}P_0.$$

Then it holds

(6,3)
$$S_0, S_1 \in \boldsymbol{K}^{n \times l},$$

(6,4)
$$\chi\left(\begin{bmatrix} S_1 \\ S_0 \end{bmatrix}\right) = l,$$

(6,5)
$$M_0U_0S_1 + M_1U_1S_0 = 0.$$

Conversely, (2,1–2), (6,1–5) *imply* (2,3–5).

1.7. Lemma. *Let* (6,1), (2,1–2) *hold. Then there exist* S_0, S_1 *so that* (6,3–5) *hold. Conversely, the existence of* M_0, M_1 *satisfying* (2,1–2), (6,5) *follows from* (6,1), (6,3–4).

1.8. Lemma. *Let $m = l = n$. Then the equivalence of*

(8,1) $$\chi(M_0U_0 + M_1U_1) = n\,,$$

(8,2) $$\chi(S_0 - S_1) = n$$

follows from (2,1–2), (6,1), (6,3–5).

Proof. Let us define P_0, P_1 by (6,2). Then using Lemma 1.4 with the corresponding notation we obtain

$$\begin{bmatrix} M_0 & M_1 \\ P_0^T & P_1^T \end{bmatrix} \begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} I & S_1 \\ I & S_0 \end{bmatrix} = \begin{bmatrix} M_0U_0 + M_1U_1 & 0 \\ P_0^TU_0 + P_1^TU_1 & G_1 \end{bmatrix}.$$

The first two matrices on the left hand side and G_1 are regular. Hence we conclude: the regularity of $S_0 - S_1$, left hand side, right hand side and $M_0U_0 + M_1U_1$ are equivalent.

2. DIFFERENTIAL EQUATIONS

2.1. Theorem. *Let $P \in C_{n\times n}(\mathscr{I})$; $f \in C_n(\mathscr{I})$; M_0, $M_1 \in K^{m\times n}$, $d \in K^m$, $\chi([M_0, M_1]) = m$. Then we obtain all solutions $x \in C_n^{(1)}(\mathscr{I})$ of the boundary value problem* (D)

(1,1) $$\dot{x} - P(t)\,x = f(t)\,,$$

(1,2) $$M_0\,x(0) + M_1\,x(1) = d$$

from the formula

(1,3) $$x(t) = H(t)\left[x_0 + \int_0^t H^{-1}f\right]$$

if x_0 ranges over all solutions of

(1,4) $$Nx_0 = f_0$$

where H is the fundamental matrix of (1,1),

$$f_0 = d - M_1H_1\int_0^t H^{-1}f\,,$$

(1,5) $$N = M_0H_0 + M_1H_1\,,$$

(1,6) $$H_0 = H(0)\,,\quad H_1 = H(1)\,.$$

2.2. Corollary. *The problem* (D) *has a unique solution for arbitrary f, d if and only if $\chi(N) = n = m$. This solution is given by*

$$(2{,}1) \qquad x(t) = H(t)\,N^{-1}\left[d + M_0 H_0 \int_0^t H^{-1}f - M_1 H_1 \int_t^1 H^{-1}f\right].$$

Proof. Theorem follows from variation of constants formula. Putting $x_0 = N^{-1}f_0$ in (1,3) we get (2,1) after simple calculation.

3. FREDHOLM EQUATIONS AND BOUNDARY VALUE PROBLEMS

3.1. Notation. In the following, $n \geqq 1$ will be a fixed integer, $\mathscr{I} = \langle 0, 1\rangle \subset \mathbf{R}^1$, $\mathscr{G}_0 = \{[t, s] \in \mathbf{R}^2 : 1 > t > s > 0\}$, $\mathscr{G}_1 = \{[t, s] \in \mathbf{R}^2 : 1 > s > t > 0\}$, $\mathscr{G} = \mathscr{G}_0 \cup \cup \mathscr{G}_1 \subset \mathbf{R}^2$, $B \in C^{(1)}_{n\times n}(\bar{\mathscr{G}})$ kernel and $a \in C^{(1)}_n(\mathscr{I})$ the right hand side of the integral equation

$$(\mathrm{I}) \qquad x(t) = a(t) = \int_0^1 B(t, s)\, x(s)\,\mathrm{d}s\,, \quad t \in \mathscr{I}\,,$$

with the unknown function $x \in C_n(\mathscr{I})$. Using the notation introduced, we write

$$B(t, s) = \left\langle \begin{array}{ll} B_0(t, s)\,, & 1 > t > s > 0 \\ B_1(t, s)\,, & 1 > s > t > 0 \end{array}\right. .$$

So we shall also write the equation (I) in the form

$$(1{,}1) \qquad x(t) = a(t) + \int_0^t B_0(t, s)\, x(s)\,\mathrm{d}s + \int_t^1 B_1(t, s)\, x(s)\,\mathrm{d}s\,, \quad t \in \mathscr{I}\,,$$

where $B_i \in C^{(1)}_{n\times n}(\bar{\mathscr{G}}_i)$; $i = 0, 1$.

Finally, we de note

$$(1{,}2) \qquad \beta(t) = B_0(t, t) - B_1(t, t)\,, \quad t \in \mathscr{I}\,.$$

3.2. Problem. We shall study the convertibility of the equation (I) to the form of the boundary value problem (D) and vice versa, the problem having been mentioned to the author by J. Nagy. Equivalence of the following there assertions will be shown:

(i) the kernels B_0, B_1 are degenerate of a special type,

(ii) the function $B(\cdot, s)$ is a solution of (D) with $f = 0$, $d = 0$ on $\mathscr{I} \dotminus \{s\}$ for $s \in \mathscr{I}$,

(iii) all solutions of (I) satisfy (D) with P, M_0, M_1 independent of a.

These results include those in [1] concerning Volterra equations and initial value problems. First of all, we show that we may work with a special form of the kernels B_0, B_1 in (I) without loss of generality, if B_0, B_1 are degenerate.

3.3. Theorem. *Let*

$$B_0(t, s) = U_0(t)\, V_0(s)\,, \quad [t, s] \in \bar{\mathscr{G}}_0\,, \tag{3,1}$$

$$B_1(t, s) = U_1(t)\, V_1(s)\,, \quad [t, s] \in \bar{\mathscr{G}}_1\,,$$

where $U_0, U_1 \in C^{(k)}_{n\times m}(\mathscr{I})$; $V_0, V_1 \in C^{(k)}_{m\times n}(\mathscr{I})$.

Then there exist an integer $q > n$ *and degenerate kernels* $\tilde{B}_0, \tilde{B}_1 \in C^{(k)}_{q\times q}(\mathscr{I} \times \mathscr{I})$ *of the form*

$$\tilde{B}_0 = \begin{bmatrix} B_0 & 0 \\ W_0 & 0 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_1 & 0 \\ W_1 & 0 \end{bmatrix} \tag{3,2}$$

so that

$$\tilde{B}_0(t, s) = \tilde{U}(t)\, S_0\, \tilde{V}(s)\,, \quad \tilde{B}_1(t, s) = \tilde{U}(t)\, S_1\, \tilde{V}(s)\,; \quad t, s \in \mathscr{I}\,, \tag{3,3}$$

where $S_0, S_1, S_0 - S_1 \in K^{qRq}$, $\tilde{U} \in C^{(k)}_{qRq}(\mathscr{I})$, $\tilde{V} \in C^{(k)}_{q\times q}(\mathscr{I})$ *and it holds: Let* $\tilde{a}, \tilde{x} \in C^{(k)}_q(\mathscr{I})$; $a, x \in C^{(k)}_n(\mathscr{I})$, $y \in C^{(k)}_{q-n}(\mathscr{I})$,

$$\tilde{a} = \begin{bmatrix} a \\ 0 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3,4}$$

Then

(i) *if* $\tilde{x}$ *is a solution of*

$$\tilde{x}(t) = \tilde{a}(t) + \int_{\mathscr{I}} \tilde{B}(t, s)\, \tilde{x}(s)\, \mathrm{d}s\,, \tag{$\tilde{\mathrm{I}}$}$$

where

$$\tilde{B}(t, s) = \begin{cases} \tilde{B}_0(t, s)\,, & 1 > t > s > 0\,, \\ \tilde{B}_1(t, s)\,, & 1 > s > t > 0 \end{cases}$$

x *satisfies* (I),

(ii) *if* x *is a solution of* (I), *there exists (unique)* $y \in C^{(k)}_{q-n}(\mathscr{I})$ *so that* $\tilde{x}$ *defined by* (3,4) *satisfies* ($\tilde{\mathrm{I}}$). (*The assertion of the theorem or, more precisely, its easy modification, is fulfilled with* $q = n$ *for some kernels.*)

Proof. Let us put, for example, $q = n + 3m$,

$$\tilde{U} = \begin{bmatrix} I_n & U_0 & U_1 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} 0_{n,n} & \\ V_0 & \\ 0_{n,n} & 0_{q,3m} \\ V_1 & \end{bmatrix}, \quad S_1 = \begin{bmatrix} 2I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \\ 0 & -I_m & 0 & 0 \end{bmatrix},$$

$$S_0 = I_q\,; \quad W_0(t, s) = \begin{bmatrix} V_0(s) \\ 0_{m,n} \\ V_1(s) \end{bmatrix}, \quad W_1(t, s) = \begin{bmatrix} 0_{m,n} \\ V_1(s) \\ -V_0(s) \end{bmatrix}; \quad t, s \in \mathscr{I}\,;$$

and define $\tilde{B}_0$, $\tilde{B}_1$ using (3,2). Then both (3,3) and (3,2) hold. From (3,2) (i) follows immediately. We shall prove (ii). Let x be a solution of (I). Let us put

$$y(t) = \int_0^t W_0(t, s)\, x(s)\, \mathrm{d}s + \int_t^1 W_1(t, s)\, x(s)\, \mathrm{d}s\,, \quad t \in \mathscr{I}\,.$$

Then $\tilde{x}$ defined by (3,4) satisfies (I). Finally, the regularity of matrices $\tilde{U}(t)$, $t \in \mathscr{I}$, S_0, S_1 is evident.

3.4. Lemma. *Let* $U \in C^{(1)}_{nRn}(\mathscr{I})$; $V_0, V_1 \in C^{(1)}_{n\times n}(\mathscr{I})$;

(4,1) $\quad B_0(t, s) = U(t)\, V_0(s),\ [t, s] \in \mathscr{G}_0\,; \quad B_1(t, s) = U(t)\, V_1(s)\,, \quad [t, s] \in \mathscr{G}_1\,.$

Let l, m be nonnegative integers, $l + m = 2n$. Then the following assertions (i) *and* (ii) *are equivalent.*

(i) *There exist $M_0, M_1 \in K^{m\times n}$ so that it holds*

(4,2) $\quad M_0 B_1(0, s) + M_1 B_0(1, s) = 0\,, \quad s \in \mathscr{I}\,; \quad \chi([M_0, M_1]) = m\,.$

(ii) *There exist $S_0, S_1 \in K^{n\times l}$, $V \in C^{(1)}_{l\times n}(\mathscr{I})$ so that it holds*

(4,3) $\quad V_0(s) = S_0\, V(s)\,, \quad V_1(s) = S_1\, V(s)\,, \quad s \in \mathscr{I}\,; \quad \chi\left(\begin{bmatrix} S_1 \\ S_0 \end{bmatrix}\right) = l\,.$

Moreover, it holds:

(iii) *Let* (4,2) *be satisfied. Then we may choose the matrices S_0, S_1 in* (4,3) *arbitrarily so that they satisfy*

(4,4) $$M_0\, U(0)\, S_1 + M_1\, U(1)\, S_0 = 0$$

and have the required range. On the other hand, we may choose M_0, M_1 in (4,2) *arbitrarily so that they satisfy* (4,4) *and have the required range, supposing* (4,3). (*We can find S_0, S_1 (M_0, M_1) in the following assertions* (iv), (v) *similarly.*)

(iv) *Let there exist* $M_0, M_1 \in \boldsymbol{K}^{n\times n}$ *satisfying* (4,2) *with* $m = n$ *and*

$$\chi(M) = n \tag{4,5}$$

where

$$M = M_0\, U(0) + M_1\, U(1)\,. \tag{4,6}$$

Then there exist $S_0, S_1 \in \boldsymbol{K}^{n\times n}$ *satisfying* (4,3) *with* $l = n$ *and*

$$\chi(S) = n\,, \tag{4,7}$$

where

$$S = S_0 - S_1\,. \tag{4,8}$$

(v) *Similarly, the existence of* M_0, M_1 *satisfying* (4,2), (4,5) *follows from the existence of* S_0, S_1 *satisfying* (4,3), (4,8).

Proof. Let

$$U_0 = U(0)\,,\quad U_1 = U(1)\,,\quad \tilde{U} = \begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix}.$$

(i) $\Rightarrow$ (ii): Let (4,2) hold. Let us choose P_0, P_1 from Lemma 1.2 and let us define G, G_0, G_1, S_0, S_1 similarly as in Lemmas 1.4, 1.6. It follows

$$G\tilde{U}\begin{bmatrix} V_1 \\ V_0 \end{bmatrix} = \begin{bmatrix} 0 \\ G_1 V \end{bmatrix}$$

from (4,2) and Lemma 1.4, where

$$V = G_1^{-1}(P_0^{\mathrm{T}} U_0 V_1 + P_1^{\mathrm{T}} U_1 V_0)\,.$$

It holds

$$G^{-1} = G^{\mathrm{T}}(G^{\mathrm{T}})^{-1}\, G^{-1} = G^{\mathrm{T}}(GG^{\mathrm{T}})^{-1} = G^{\mathrm{T}}\begin{bmatrix} G_0^{-1} & 0 \\ 0 & G_1^{-1} \end{bmatrix}$$

so that

$$\begin{bmatrix} V_1 \\ V_0 \end{bmatrix} = \tilde{U}^{-1} G^{\mathrm{T}} \begin{bmatrix} G_0^{-1} & 0 \\ 0 & G_1^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ G_1 V \end{bmatrix} = \begin{bmatrix} U_0^{-1} & 0 \\ 0 & U_1^{-1} \end{bmatrix} \begin{bmatrix} M_0^{\mathrm{T}} & P_0 \\ M_1^{\mathrm{T}} & P_1 \end{bmatrix} \begin{bmatrix} 0 \\ V \end{bmatrix} =$$

$$= \begin{bmatrix} U_0^{-1} M_0^{\mathrm{T}} & S_1 \\ U_1^{-1} M_1^{\mathrm{T}} & S_0 \end{bmatrix} \begin{bmatrix} 0 \\ V \end{bmatrix}$$

and (4,3) holds.

(ii) $\Rightarrow$ (i): Let (4,3) hold. Let us choose M_0, M_1 from Lemma 1.7. Then (4,2) follows from the identity

$$M_0 B_1(0, s) + M_1 B_0(1, s) = (M_0 U_0 S_1 + M_1 U_1 S_0) V(s).$$

Now we easily obtain (iii) from the above and, using Lemma 1.8, (iv) and (v) as well.

3.5. Remark. We may write general degenerate kernels B_0, B_1 in the form (3,1). (See [1].) It follows from Theorems 3.3–3.4 that we may complement an arbitrary equation (I) with degenerate B_0, B_1 to the equivalent form of the same type where (4,1–5), (4,7) hold and S_0, S_1 are regular. Generally, (4,4) does not follow from (4,1–3) but only a weaker assertion

$$(M_0 U(0) S_1 + M_1 U(1) S_0) V(s) x = 0, \quad s \in \mathscr{I}, \quad x \in \boldsymbol{K}^n.$$

(4,4) follows from here if $\{V(s) x : s \in \mathscr{I}, \ x \in \boldsymbol{K}^n\} = \boldsymbol{K}^n$, which is satisfied if, for example, $V(s)$ is regular for some $s \in \mathscr{I}$.

3.6. Theorem. *The following assertions* (i), (ii), (iii) *are equivalent.*

(i) *There exist* $P \in C_{n \times n}(\mathscr{I})$, *integer* $m \geqq 0$ *and* $M_0, M_1 \in \boldsymbol{K}^{m \times n}$ *so that*

$$\chi([M_0, M_1]) = m; \tag{6,1}$$

$$D^{1,0} B(t, s) - P(t) B(t, s) = 0; \quad t, s \in \mathscr{I}; \quad t \neq s; \tag{6,2}$$

$$M_0 B_1(0, s) + M_1 B_0(1, s) = 0; \quad s \in \mathscr{I}. \tag{6,3}$$

(ii) *There exist integer* $l \geqq 0$, $U \in C^{(1)}_{nRn}(\mathscr{I})$, $V \in C^{(1)}_{l \times n}(\mathscr{I})$; $S_0, S_1 \in \boldsymbol{K}^{n \times l}$ *so that*

$$\chi\left(\begin{bmatrix} S_0 \\ S_1 \end{bmatrix}\right) = l, \tag{6,4}$$

$$B_0(t, s) = U(t) S_0 V(s), \quad [t, s] \in \mathscr{G}_0, \tag{6,5}$$

$$B_1(t, s) = U(t) S_1 V(s), \quad [t, s] \in \mathscr{G}_1. \tag{6,6}$$

(iii) *There exist* $P \in C_{n \times n}(\mathscr{I})$, *integer* $m \geqq 0$ *and* $M_0, M_1 \in \boldsymbol{K}^{m \times n}$ *so that* (6,1) *holds and all solutions* x *of* (I) *satisfy the boundary value problem* (*where* β *is given by* (1,2))

$$\dot{x}(t) - [P(t) + \beta(t)] x(t) = \dot{a}(t) - P(t) a(t), \quad t \in \mathscr{I}, \tag{6,7}$$

$$M_0 x(0) + M_1 x(1) = M_0 a(0) + M_1 a(1) \tag{6,8}$$

for arbitrary $a \in C^{(1)}_n(\mathscr{I})$.

Remark. Let (i) or (iii) hold with some m, M_0, M_1, P. Then both (i) and (iii) hold with the same m, M_0, M_1, P. Moreover, (ii) holds with l, U, S_0, S_1 satisfying $M_0U_0S_1 + M_1U_1S_0 = 0$, $m + l = 2n$.

Proof. (i) $\Rightarrow$ (ii): Let us suppose (6,2). Let U be the fundamental matrix of (6,2). Then (4,1) holds and (ii) follows from Lemma 3,4.

(ii) $\Rightarrow$ (i): (6,2) and $P = \dot{U}U^{-1}$ follow from (6,5–6). Then we apply Lemma 3.4.

(i) $\Rightarrow$ (iii): Let x be a solution of (I). Then

$$\dot{x}(t) - [P(t) + \beta(t)]\, x(t) = \dot{a}(t) - P(t)\, a(t) + \int_0^1 [D^{1,0} B(t, s) - P(t)\, B(t, s)]\, x(s)\, \mathrm{d}s\,, \tag{6,9}$$

$$M_0\, x(0) + M_1\, x(1) = M_0\, a(0) + M_1\, a(1) + \int_0^1 [M_0\, B(0, s) + M_1\, B(1, s)]\, x(s)\, \mathrm{d}s\,. \tag{6,10}$$

This together with (6,2–3) implies (6,7–8) with the same P.

(iii) $\Rightarrow$ (i): Let us choose $x \in C_n^{(1)}(\mathscr{I})$ and find a so that (I) holds. (6,9–10) follows from (I). The integrals in (6,9–10) equal zero if (6,7–8) also hold. Since x is arbitrary, we obtain (6,2–3).

3.7. Theorem. *Let* $U \in C_{nRn}^{(1)}(\mathscr{I})$, $P = \dot{U}U^{-1}$; $M_0, M_1, S_0, S_1 \in K^{n\times n}$, $U_0 = U(0)$, $U_1 = U(1)$, $Z = M_0U_0 + M_1U_1$

$$\chi([M_0, M_1]) = \chi\left(\begin{bmatrix} S_0 \\ S_1 \end{bmatrix}\right) = n\,, \tag{7,1}$$

$$M_0U_0S_1 + M_1U_1S_0 = 0\,. \tag{7,2}$$

Then the following four assertions are equivalent:

(i) *It holds*

$$D^{1,0} B(t, s) - P(t)\, B(t, s) = 0\,; \quad t, s \in \mathscr{I}\,, \quad t \neq s\,, \tag{7,3}$$

$$M_0\, B_1(0, s) + M_1\, B_0(1, s) = 0\,, \quad s \in \mathscr{I}\,, \tag{7,4}$$

$$\chi(Z) = n\,. \tag{7,5}$$

(ii) *There exists* $V \in C_{n\times n}(\mathscr{I})$ *so that*

$$B_0(t, s) = U(t)\, S_0\, V(s)\,, \quad [t, s] \in \mathscr{G}_0\,, \tag{7,6}$$

$$B_1(t, s) = U(t)\, S_1\, V(s)\,, \quad [t, s] \in \mathscr{G}_1\,, \tag{7,7}$$

and

$$(7{,}8) \qquad \chi(S_0 - S_1) = n\,.$$

(iii) *The equation* (I) *is equivalent to the boundary value problem (with the unknown function* x)

$$(7{,}9) \qquad \dot{x}(t) - [P(t) + \beta(t)]\,x(t) = \dot{a}(t) - P(t)\,a(t)\,, \quad t \in \mathscr{I}\,,$$

$$(7{,}10) \qquad M_0\,x(0) + M_1\,x(1) = M_0\,a(0) + M_1\,a(1)$$

for all $a \in \boldsymbol{C}_n^{(1)}(\mathscr{I})$.

(iv) *For all* $f \in \boldsymbol{C}_n(\mathscr{I})$, $d \in \boldsymbol{K}^n$, *there exists an* $a \in \boldsymbol{C}_n^{(1)}(\mathscr{I})$ *so that the boundary value problem*

$$(7{,}13) \qquad \dot{x}(t) - [P(t) + \beta(t)]\,x(t) = f(t)\,, \quad t \in \mathscr{I}\,,$$

$$(7{,}14) \qquad M_0\,x(0) + M_1\,x(1) = d$$

is equivalent to the equation (I) *and*

$$(7{,}15) \qquad \dot{a}(t) - P(t)\,a(t) = f(t)\,, \quad t \in \mathscr{I}\,,$$

$$(7{,}16) \qquad M_0\,a(0) + M_1\,a(1) = d$$

hold. (This a *is unique and is given by*

$$(7{,}17) \qquad a(t) = U(t)\,Z^{-1}\left[d + M_0 U_0 \int_0^t U^{-1} f - M_1 U_1 \int_t^1 U^{-1} f\right].$$

Proof. We conclude the equivalence (i) $\Leftrightarrow$ (ii) from Lemma 3.4 (iv), (v) and (iii) following the arguments proving Theorem 3.6.

(i) $\Rightarrow$ (iii): Let us suppose (i). Let x satisfy (I). Then (6,9–10) hold and (i) yields (7,9–10). On the other hand, let (7,9–10) hold and let us put

$$\lambda(t) = x(t) - a(t) - \int_0^t B(t, s)\,x(s)\,\mathrm{d}s\,.$$

We get $\dot{\lambda}(t) = P(t)\,\lambda(t)$, $t \in \mathscr{I}$ using (7,9) and $M_0\,\lambda(0) + M_1\,\lambda(1) = 0$ using (7,4), (7,10). So we obtain $\lambda(t) = U(t)\,\lambda(0)$, $M_0\,\lambda(0) + M_1\,\lambda(1) = Z\,\lambda(0) = 0$. It follows $\lambda(0) = 0$ from (7,5) so that $\lambda(t) = 0$, $t \in \mathscr{I}$ and (I) holds.

(iii) $\Rightarrow$ (i): Let (iii) hold. We prove (7,3–4) following the arguments proving (iii) in Theorem 3.6. Now, let us choose x und let us find a so that (I) holds. (iii) follows from (7,9–10). If Z were not regular, we obtain by Theorem 2.1 that the boundary value problem (7,9–10) (with the unknown a) has a solution $\tilde{a} \neq a$. We get (I) with a substituted by $\tilde{a}$ from here and (iii), which is a contradiction and hence (7,5) holds.

(iii) $\Rightarrow$ (iv): Let us suppose (iii). Then we get (7,5). Let us define a using (7,17). We obtain (7,15–16) from Theorem 2.2. Let x satisfy (7,13–14). Then (7,9–10) hold and, using (iii), (I) holds as well. Conversely, we get (7,9–10) from (I) using (ii) and (7,13–14) follows from here and (7,15–16).

(iv) $\Rightarrow$ (iii): The unique solution a of (7,15–16) is given by (7,17). Let us define f, d using (7,15–16). Then (I) is equivalent to (7,13–14), that is with (7,9–10) so that (iii) holds.

3.8. Remark. We may see the interesting symmetry of a and x in the boundary value problem (7,9–10). We have proved: (7,9–10) follow from (I) if (7,6–7) hold. Using the mentioned symmetry we may expect similarly that (7,9–10) will follow from some integral equation of the type (I) with the unknown a. We shall prove that this equation is identical with the formula expressing the solution x of (I) by means of the resolvent kernel R of the kernel B, provided that R exists.

This theorem follows from the known theory of integral equations.

3.9. Theorem. *Let the equation* (I) *with the kernel* $B \in L^2_{n\times n}(\mathscr{I} \times \mathscr{I})$ *have for all* $a \in L^2_n(\mathscr{I})$ *a unique solution* $x \in L^2_n(\mathscr{I})$. *Then there exists the so called resolvent kernel* $R \in L^2_{n\times n}(\mathscr{I} \times \mathscr{I})$ *such that we may write the solution* x *in the form*

$$x(t) = a(t) + \int_0^1 R(t, s)\, a(s)\, \mathrm{d}s\,, \quad a \in L^2_n(\mathscr{I})\,. \tag{9,1}$$

It holds

$$R(t, s) - B(t, s) = \int_0^1 B(t, u)\, R(u, s)\, \mathrm{d}u\,, \tag{9,2}$$

$$R(t, s) - B(t, s) = \int_0^1 R(t, u)\, B(u, s)\, \mathrm{d}u\,, \tag{9,3}$$

for the kernel R *on* $\mathscr{I} \times \mathscr{I}$. *Moreover, if* $B \in C^{(k)}_{n\times n}(\tilde{\mathscr{G}})$ *then also* $R \in C^{(k)}_{n\times n}(\tilde{\mathscr{G}})$ *and*

$$R_0(t, t) - R_1(t, t) = B_0(t, t) - B_1(t, t)\,, \quad t \in \mathscr{I}\,. \tag{9,4}$$

3.10. Theorem. *Let* l, m, n *be nonnegative integers,* $l + m = 2n$; $M_0, M_1 \in K^{m\times n}$; $S_0, S_1 \in K^{n\times l}$; $U, H \in C^{(1)}_{nRn}(\mathscr{I})$, $V \in C^{(1)}_{l\times n}(\mathscr{I})$, $U_0 = U(0)$, $U_1 = U(1)$, $H_0 = H(0)$, $H_1 = H(1)$, $S = S_0 - S_1$, $P = \dot{U}U^{-1}$, $Q = \dot{H}H^{-1}$. *Let*

$$\chi([M_0, M_1]) = m\,, \quad \chi\left(\begin{bmatrix} S_1 \\ S_0 \end{bmatrix}\right) = l\,, \tag{10,1}$$

$$M_0 U_0 S_1 + M_1 U_1 S_0 = 0\,, \tag{10,2}$$

$$\dot{H}H^{-1} = \dot{U}U^{-1} + USV\,. \tag{10,3}$$

(P) *Let there exist a solution* $x \in C_n^{(1)}(\mathscr{I})$ *of the integral equation*

$$(10{,}4) \qquad x(t) = a(t) + U(t)\left[S_0 \int_0^t Vx + S_1 \int_t^1 Vx\right], \quad t \in \mathscr{I}$$

for all $a \in C_n^{(1)}(\mathscr{I})$.

Then there exist $T_0, T_1 \in K^{n \times l}$; $W \in C_{l \times n}^{(1)}(\mathscr{I})$ *such that it holds:* (i)

$$(10{,}5) \qquad \chi\left(\begin{bmatrix} T_1 \\ T_0 \end{bmatrix}\right) = l\,,$$

$$(10{,}6) \qquad M_0 H_0 T_1 + M_1 H_1 T_0 = 0\,,$$

$$(10{,}7) \qquad \dot{U}U^{-1} = \dot{H}H^{-1} + UTW\,,$$

where $T = T_0 - T_1$.

(ii) (10,4) *and the equation*

$$(10{,}8) \qquad a(t) = x(t) + H(t)\left[T_0 \int_0^t Wa + T_1 \int_t^1 Wa\right], \quad t \in \mathscr{I}$$

are equivalent (*for arbitrary* a, x).

(iii) *It follows*

$$(10{,}9) \qquad \dot{x}(t) - Q(t)\,x(t) = \dot{a}(t) - P(t)\,a(t)\,, \quad t \in \mathscr{I}\,,$$

$$(10{,}10) \qquad M_0\,x(0) + M_1\,x(1) = M_0\,a(0) + M_1\,a(1)$$

from (10,4) *or* (10,8).

(iv) *If* $l = m = n$,

$$(10{,}11) \qquad \chi(S) = n$$

and (10,9–10) *hold then* (10,4), (10,8) *also hold and the boundary value problem* (10,9–10) *has a unique solution* (*for the unknown* a) *for all* x.

(v) *If* $l = m = n$,

$$(10{,}12) \qquad \chi(T) = n$$

and (10,9–10) *hold then* (10,4), (10,8) *also hold and the boundary value problem* (10,9–10) (*with the unknown* x) *has a unique solution for all* a.

Remark. The solution of (10,4) is given by (10,8) and vice versa. If (10,11) holds, the solution of the boundary value problem (10,9–10) (with the corresponding unknown) is given by (10,4). If (10,12) holds, the solution is given by (10,8). The equations (10,4), (10,8) are completely symmetric as well as their relations to (10,9 to 10). The corresponding couples are

$$x \leftrightarrow a\,,\quad U \leftrightarrow H\,,\quad V \leftrightarrow W,\quad S_0 \leftrightarrow T_0\,,\quad S_1 \leftrightarrow T_1\,,\quad S \leftrightarrow T,\quad P \leftrightarrow Q\,.$$

Proof. We may write (10,4) in the operator form $(\mathbf{I} - \mathbf{B})\,x = a$ where $\mathbf{I}$ is the identity mapping and $\mathbf{B}$ is a totally continuous mapping of $\boldsymbol{L}_n^2(\mathscr{I})$ into itself. For $\boldsymbol{C}_n^{(1)}(\mathscr{I})$ is dense in $\boldsymbol{L}_n^2(\mathscr{I})$ there exists a solution of (10,4) for all $a \in \boldsymbol{L}_n^2(\mathscr{I})$. The number 1 is a regular point both of the operator B and its adjoint. Consequently, the solution is unique and we may write

$$x(t) = a(t) + \int_0^1 R(t, s)\, a(s)\, \mathrm{d}s\,, \tag{10,13}$$

where $R \in \boldsymbol{L}^2_{n\times n}(\mathscr{I} \times \mathscr{I})$. We obtain $R \in \boldsymbol{C}^{(1)}_{n\times n}(\widetilde{\mathscr{G}})$ from Theorem 9. The equation (10,4) has the form (I) where the kernel B satisfies (6,4–6). Using Theorem 3.6 we obtain (6,7–8) (that is (10,9–10)) from (10,4) since, using (10,3), we may write $P = \dot{U}U^{-1}$, $P + \beta = \dot{U}U^{-1} + USV = \dot{H}H^{-1} = Q$ in (6,7). (10,13) and (10,7) are equivalent. Therefore (10,9–10) follow from (10,13). We shall apply Theorem 3.6 to the equation (10,13) (with the unknown a). Since (by Theorem 3.9 and (10,3)) it holds

$$Q(t) + R(t+, t) - R(t-, t) = Q(t) - \beta(t) = Q(t) - U(t)\, S\, V(t) = P(t) \tag{10,14}$$

we get the assertion (iii) of Theorem 3.6. (The equation which we obtain, using (6,7), from (10,13) is identical with the equation (10,9), which is satisfied by the solution of (10,13).) So both the assertions (i), (ii) of Theorem 3.6 and the remark following this theorem hold. From here it follows that we may put the function U in Theorem 3.6 equal to H and choose T_0, T_1, W so that (10,5–6) hold and

$$\begin{aligned} R_0(t, s) &= H(t)\, T_0\, W(s)\,, \quad [t, s] \in \mathscr{G}_0\,,\\ R_1(t, s) &= H(t)\, T_1\, W(s)\,, \quad [t, s] \in \mathscr{G}_1\,. \end{aligned} \tag{10,15}$$

Substituting this into (10,14) and using the definitions of P, Q we obtain (10,7). Substituting again from (10,15) to (10,13) we get (ii). (iii) has been already proved and (iv), (v) follow immediately from Theorem 3.7.

3.11. Remark. The resolvent equation of (10,4) is

$$\begin{aligned} R_0(t, s) &= U(t)\, S_0\, V(s) + \\ &+ U(t)\left[S_0 \int_0^s V R_1(\cdot, s) + S_0 \int_s^t V R_0(\cdot, s) + S_1 \int_t^1 V R_0(\cdot, s)\right], \quad t > s\,,\\ R_1(t, s) &= U(t)\, S_1\, V(s) + \\ &+ U(t)\left[S_0 \int_0^t V R_1(\cdot, s) + S_1 \int_t^s V R_1(\cdot, s) + S_1 \int_s^1 V R_0(\cdot, s)\right], \quad t < s\,. \end{aligned} \tag{11,1}$$

The resolvent kernel is given by (if it exists)

$$(11,2)\qquad R_0(t,s) = H(t)\,Y^{-1}S_0\left[I - \left(\int_0^s VH\right)H(s)^{-1}\,U(s)\,S\right]V(s)\,,\quad t > s\,,$$

$$R_1(t,s) = H(t)\,Y^{-1}S_1\left[I + \left(\int_s^1 VH\right)H(s)^{-1}\,U(s)\,S\right]V(s)\,,\quad t < s\,,$$

where

$$Y = U_0^{-1}H_0 - S_1\int_0^1 VH = U_1^{-1}H_1 - S_0\int_0^1 VH\,.$$

From (10,3) we obtain $(U^{-1}H)^{\cdot} = SVH$. Using this and substituting (11,2) into (11,1) we prove the relations (11,2). It follows

$$R_0(t,t) - R_1(t,t) = B_0(t,t) - B_1(t,t)\,,$$
$$M_0\,R_1(0,s) + M_1\,R_0(1,s) = 0\,,$$

from (11,1). We obtain (10,15) from here, using Lemma 3.4 and (11,2). It also holds

$$M_0H_0 + M_1H_1 = (M_0U_0 + M_1U_1)\,Y\,.$$

Provided that the left hand side is a regular matrix, the unique solution of (10,9–10) is given by (10,8).

3.12. Example. We shall study the equation (10,4) if $S_0 = S_1$. Then we may put $S_0 = S_1 = I$ without loss of generality. So we get the Fredholm integral equation

$$(12,1)\qquad x(t) = a(t) + U(t)\int_0^1 V(s)\,x(s)\,\mathrm{d}s\,,\quad t\in\mathcal{I}\,,$$

with degenerate kernel. The corresponding boundary value problem (10,9–10) is

$$(12,2)\qquad \dot{x}(t) - P(t)\,x(t) = \dot{a}(t) - P(t)\,a(t)\,,\quad t\in\mathcal{I}\,,$$

$$(12,3)\qquad M_0\,x(0) + M_1\,x(1) = M_0\,a(0) + M_1\,a(1)\,,$$

where $P = \dot{U}U^{-1}$ and where we assume

$$(12,4)\qquad M_0U_0 + M_1U_1 = 0\,.$$

(I) has a unique solution

$$x(t) = a(t) + U(t)\left[I - \int_0^1 V(s)\,U(s)\,\mathrm{d}s\right]^{-1}\int_0^1 V(s)\,a(s)\,\mathrm{d}s\,,\quad t\in\mathcal{I}\,,$$

if

$$\int_0^1 V(s)\,U(s)\,\mathrm{d}s \neq \mathcal{I}\,.$$

On the other hand, it follows from (12,4) that the boundary value problem (12,2 – 3) has not a unique solution.

3.13. Example. Let (10,4) be of the form

$$(13,1) \qquad x(t) = a(t) + \frac{M}{1 - e^{-M}} \int_0^t x + \frac{Me^{-M}}{1 - e^{-M}} \int_t^1 x, \quad t \in \mathscr{I},$$

where $1 - \exp(-M) \neq 0$. Using the notation of Theorem 3.10 we obtain

$$S_0 = \frac{M}{1 - e^{-M}}, \quad S_1 = \frac{Me^{-M}}{1 - e^{-M}}, \quad S = M, \quad U = V = 1,$$

$$H(t) = e^{Mt}, \quad t \in \mathscr{I}, \quad P = 0, \quad Q = M.$$

We put $M_0 = \exp(-M)$, $M_1 = -1$ so that (10,1 – 3) hold with $l = m = 1$. The boundary value problem (10,9 – 10) has the form

$$(13,2) \qquad \dot{x}(t) - M\,x(t) = \dot{a}(t), \quad t \in \mathscr{I},$$

$$(13,3) \qquad e^{-M}\,x(0) - x(1) = e^{-M}\,a(0) - a(1).$$

So the solution of (13,1) is

$$(13,4) \qquad x(t) = e^{Mt}\left[x_0 + \int_0^t e^{-Ms}\,\dot{a}(s)\,\mathrm{d}s\right], \quad t \in \mathscr{I}.$$

Substituting into (13,1) we obtain that (13,1) is not uniquely solvable for all a. We shall prove that there does not exist an equation of the form (10.8) such that all its solutions a for arbitrary x satisfy (13,2 – 3). Indeed, if the opposite were true, the relations

$$(13,5) \qquad P = Q + HTW,$$

$$(13,6) \qquad (M_0H_0T_1 + M_1H_1T_0)\,W = 0$$

would hold. Substituting into (13,6) we should obtain

$$(e^{-M}T_1 - e^{-M}T_0)\,W = 0$$

so that

$$TW = (T_0 - T_1)\,W = 0.$$

Substituting into (13,5) we should obtain $M = 0$, which is a contradiction.

3.14. Remark. Let (6,5 – 6) hold for the kernel of the equation (I), that is, let (I) be an equation of the form (10,4). Let the propositions of Theorem 3.10 hold with the exception of (P). Then (10,4) follows from (10,9 – 10) (Theorem 3.6). The reverse assertion is not generally true. (Example 3.12) (10,4) follows from (10,9 – 10) (Theorem 3.7) if the boundary value problem (10,9 – 10) is uniquely solvable for the unknown a. The solution of (10,9) is given by the completely analogous formula (10,8) if also (P) holds and then (10,9 – 10) also follows from (10,8) analogously as from (10,4). If (P) does not hold such an equation need not exist. (Example 3.13.)

References

[1] *Cerha, J.:* A note on Volterra integral equations with degenerate kernel, CMUC *13*, 4 (1972), 659—672.

[2] *Vejvoda, O., Tvrdý, M.:* Existence of solutions to a linear integroboundary-differential equation with additional conditions. Anuali di matematica pura ed applicata IV—*LXXXIX* (1971), 169—216.

[3] *Schmeidler, W.:* Integralgleichungen mit Anwendungen in Physik und Technik, Leipzig 1955.

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