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INTEGRALS INVOLVING PRODUCTS OF *E*-FUNCTIONS AND APPELL'S FUNCTIONS

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1. Introduction. In this paper we have evaluated a number of infinite integrals involving products of Mac-Robert's E-functions and Appells' functions F_1, F_2, F_3, F_4 with the help of the following

Main theorem.

$$(1)\qquad \int_0^\infty x^{k-1}\,E(p;\phi_r\colon q;\theta_u : ax)\,F\left(\begin{matrix}\mu\\ \tau\\ \varepsilon\\ \sigma\end{matrix}\left|\begin{matrix}\alpha_1,\ldots,\alpha_\mu\\ \beta_1,\beta'_1;\ldots;\beta_\tau,\beta'_\tau\\ \gamma_1,\ldots,\gamma_\varepsilon\\ \delta_1,\delta'_1;\ldots;\delta_\sigma,\delta'_\sigma\end{matrix}\right|\,-bx,\,-c\right)\mathrm{d}x =$$

$$= -\pi\frac{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j)\,\Gamma(\delta'_j)\}}{a^k\sin(k\pi)\prod_{j=1}^{\mu}\Gamma(\alpha_j)\prod_{j=1}^{\tau}\{\Gamma(\beta_j)\,\Gamma(\beta'_j)\}}\sum_{m=0}^{\infty}\frac{(-c)^m\prod_{j=1}^{\tau}\Gamma(\beta'_j+m)}{m!\prod_{j=1}^{\sigma}\Gamma(\delta'_j+m)}\times$$

$$\times\left[E\left(\begin{matrix}\alpha_1+m,\ldots,\alpha_\mu+m,\beta_1,\ldots,\beta_\tau,\phi_1+k,\ldots,\phi_p+k : e^{\pm i\pi}\dfrac{a}{b}\\ 1+k,\gamma_1+m,\ldots,\gamma_\varepsilon+m,\delta_1,\ldots,\delta_\sigma,\theta_1+k,\ldots,\theta_q+k\end{matrix}\right)-\right.$$

$$\left.-\left(\frac{a}{b}\right)^k E\left(\begin{matrix}\alpha_1-k+m,\ldots,\alpha_\mu-k+m,\beta_1-k,\ldots,\beta_\tau-k,\phi_1,\ldots,\phi_p : e^{\pm i\pi}\dfrac{a}{b}\\ 1-k,\gamma_1+k-m,\ldots,\gamma_\varepsilon+k-m,\delta_1-k,\ldots,\delta_\sigma-k,\theta_1,\ldots,\theta_q\end{matrix}\right)\right]$$

where $R(k+\phi_r)>0$ $(r=1,2,\ldots,p)$, $\mu+\tau\leqq\varepsilon+\sigma+1$, $R(\alpha_r-k)>0$ $(r= =1,2,\ldots,\mu)$, $R(\beta_r-k)>0$ $(r=1,2,\ldots,\tau)$, *a, b, c are real and positive.*

The function F appearing in (1) is Kampé de Fériet's function of higher order in two variables whose properties are given in [1] pp. 401 and 489. This function is defined as:

$$(2)\quad F\left(\begin{array}{c|c|c}\mu & \alpha_1, \ldots, \alpha_\mu & \\ \tau & \beta_1, \beta'_1; \ldots; \beta_\tau, \beta'_\tau & x, y \\ \varepsilon & \gamma_1, \ldots, \gamma_\varepsilon & \\ \sigma & \delta_1, \delta'_1; \ldots; \delta_\sigma, \delta'_\sigma & \end{array}\right) =$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j; m + n) \prod_{j=1}^{\tau} \{(\beta_j, m)(\beta'_j, n)\} x^m y^n}{m!\, n! \prod_{j=1}^{\varepsilon} (\gamma_j; m + n) \prod_{j=1}^{\sigma} (\delta_j; m)(\delta'_j; n)},$$

where $\mu + \tau \leqq \varepsilon + \sigma + 1$ and neither of the quantities γ, δ, δ' is a negative integer.

This function is reduced to the four functions of P. Appell F_1, F_2, F_3, F_4 by specialising the numbers μ, τ, ε, and σ. Thus we have (see [2], p. 151):

$$(3)\quad F\left(\begin{array}{c|c|c}1 & \alpha & \\ 1 & \beta, \beta' & x, y \\ 1 & \gamma & \\ 0 & \ldots & \end{array}\right) = F_1[\alpha; \beta, \beta'; \gamma; x, y],$$

$$(4)\quad F\left(\begin{array}{c|c|c}1 & \alpha_1, & \\ 1 & \beta_1, \beta'_1 & x, y \\ 0 & \ldots & \\ 1 & \delta_1, \delta'_1 & \end{array}\right) = F_2[\alpha_1; \beta_1, \beta'_1; \delta_1, \delta'_1; x, y],$$

$$(5)\quad F\left(\begin{array}{c|c|c}0 & \ldots & \\ 2 & \beta_1, \beta'_1; \beta_2, \beta'_2 & x, y \\ 1 & \gamma & \\ 0 & \ldots & \end{array}\right) = F_3(; \beta_1, \beta'_1; \beta_2, \beta'_2; \gamma; x, y),$$

and

$$(6)\quad F\left(\begin{array}{c|c|c}2 & \alpha_1, \alpha_2 & \\ 0 & \ldots & x, y \\ 0 & \ldots & \\ 1 & \delta_1, \delta'_1 & \end{array}\right) = F_4[\alpha_1, \alpha_2; \delta_1, \delta'_1; x, y].$$

Also we have ([2], p. 151):

$$(7)\quad F\left(\begin{array}{c|c|c}\mu & \alpha_1, \ldots, \alpha_\mu & \\ 0 & \ldots & x, y \\ \varepsilon & \gamma_1, \ldots, \gamma_\varepsilon & \\ 0 & \ldots & \end{array}\right) = F\left(\begin{matrix}\alpha_1, \ldots, \alpha_\mu; x + y \\ \gamma_1, \ldots, \gamma_\varepsilon\end{matrix}\right)$$

and

$$(8)\qquad F\left(\begin{array}{c|c|c} 0 & \dots & \\ \tau & \beta_1, \beta_1'; \dots; \beta_\tau\ \beta_\tau' & \\ 0 & \dots & x, y \\ \sigma & \delta_1, \delta_1'; \dots; \delta_\sigma, \delta_\sigma' & \end{array}\right) = F\left(\begin{matrix}\beta_1, \dots, \beta_\tau; x \\ \delta_1, \dots, \delta_\sigma\end{matrix}\right) F\left(\begin{matrix}\beta_1', \dots, \beta_\tau'; y \\ \delta_1', \dots, \delta_\sigma'\end{matrix}\right),$$

$$(9)\qquad F\left(\begin{array}{c|c|c} \omega & \alpha_1, \dots, \alpha_\omega & \\ 1 & \beta, \beta' & \\ \omega & \gamma_1, \dots, \gamma_\omega & x, y \\ 0 & \dots & \end{array}\right) = F\left(\begin{matrix}\alpha_1, \dots, \alpha_\omega, \beta + \beta'; x \\ \gamma_1, \dots, \gamma_\omega\end{matrix}\right).$$

Again Kampé de Fériet's function is expressed as a double complex integral

$$(10)\qquad F\left(\begin{array}{c|c|c} \mu & \alpha_1, \dots, \alpha_\mu & \\ \tau & \beta_1, \beta_1'; \dots; \beta_\tau, \beta_\tau' & \\ \varepsilon & \gamma_1, \dots, \gamma_\varepsilon & x, y \\ \sigma & \delta_1, \delta_1'; \dots; \delta_\sigma, \delta_\sigma' & \end{array}\right) = \frac{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j)\,\Gamma(\delta_j')\}}{\prod_{j=1}^{\mu}\Gamma(\alpha_j)\prod_{j=1}^{\tau}\{\Gamma(\beta_j)\,\Gamma(\beta_j')\}} \times$$

$$\times \frac{1}{(2\pi i)^2}\iint\Gamma(s)\,\Gamma(t)\,\frac{\prod_{j=1}^{\mu}\Gamma(\alpha_j - s - t)\prod_{j=1}^{\tau}\{\Gamma(\beta_j - s)\,\Gamma(\beta_j' - t)\}}{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j - s - t)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j - s)\,\Gamma(\delta_j' - t)\}}(-x)^{-s}(-y)^{-t}\,\mathrm{d}s\,\mathrm{d}t$$

where the contours are of Barne's type and are curved (if necessary) to separate the increasing sequences of poles from the decreasing sequences of poles.

The definitions and properties of Mac-Robert's E-functions are given in [3], pp. 348–358 and which will be discussed further in section 2.

The proof of (1) will be given in section 3. Section 4 contains the derivation of infinite integrals of products of Bessel functions and Appell's function as particular cases of the main theorem (1).

The following formulae will be required in the proof:

The Mellin transform pair ([4]; p. 7)

$$(11)\qquad g(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x\,;$$

$$(12)\qquad f(x) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} x^{-s} g(s)\,\mathrm{d}s\,;$$

([3], p. 262)

$$(13)\qquad F\left(\begin{matrix}-n, n+1, -x \\ 1-h\end{matrix}\right) = \Gamma(1-h)\left(\frac{x}{1+x}\right)^{h/2} P_h^n(2x+1)\,;$$

where P_h^n is the associated Legendre function of the first kind.

2. Properties of the E-function. If $p \leqq q$, then the E-function is defined as

$$(14)\qquad E(p; \alpha_r : q; \varepsilon_s : z) = \frac{\Gamma(\alpha_1), \ldots, \Gamma(\alpha_p)}{\Gamma(\varepsilon_1), \ldots, \Gamma(\varepsilon_q)} F\begin{pmatrix} \alpha_1, \ldots, \alpha_p; -1/z \\ \varepsilon_1, \ldots, \varepsilon_q \end{pmatrix}.$$

When $p \geqq q +, |\arg z| < \pi$, then then E-function is defined as:

$$(15)\qquad E(p; \alpha_r : q; \varepsilon_s : z) = \sum_{r=1}^{p} \prod_{s=1}^{p}{}' \Gamma(\alpha_s - \alpha_r)\{\prod_{t=1}^{q} \Gamma(\varepsilon_t - \alpha_r)\}^{-1} \Gamma(\alpha_r) z^{\alpha_r} \times$$

$$\times {}_{q+1}F_{p-1}\begin{pmatrix} \alpha_r, \alpha_r - \varepsilon_1 + 1, \ldots, \alpha_r - \varepsilon_q + 1; (-1)^{p-q}z \\ \alpha_r - \alpha_1 + 1, \quad \ldots, \alpha_r - \alpha_p + 1 \end{pmatrix}$$

where the asterik means that the factor $\alpha_r - \alpha_r + 1$ is ommitted.

From (14) and (15) it is clear that the E-function is immediately related to the generalized hypergeometric function and reduces to simple expressions in the ordinary or Gauss hypergeometric function when $p = 2$, $q = 1$. For $p = q = 1$ it is also evident that the E-function reduces to the confluent hypergeometric function or Kummer's function. The case $p = 2$, $q = 0$ yields the relations (see [3], p. 351):

$$(16)\qquad \cos n\pi\, E(\tfrac{1}{2} + n, \tfrac{1}{2} - n : 2z) = \sqrt{(2\pi z)}\, e^z K_n(z),$$

and

$$(17)\qquad E(\tfrac{1}{2} - k + m, 1 - k - m :: z) =$$
$$= \Gamma(\tfrac{1}{2} - k + m)\, \Gamma(\tfrac{1}{2} - k - m)\, z^{-k} e^{1/2z}\, W_{k,m}(z)$$

where $K_n(z)$ and $W_{k,m}(z)$ are the modified Bessel function of the second kind and Whittkar function respectively. When $p = q = 0$, the E-function is just $e^{-1/z}$. When $p = 0$, $q = 1$, then we obtain the Bessel function of the first kind. Thus we have

$$(18)\qquad E(: \tau + 1 : z) = z^{\tau/2} J_\tau(2z^{-1/2}).$$

The case $p = 1$, $q = 0$ gives

$$(19)\qquad E(\alpha :: z) = \Gamma(\alpha)\,(1 + 1/z)^{-\alpha}.$$

More parameters in the E-function lead to the equivalence of the E-function with products of Hankel functions, with Lommel functions, Bessel functions and products of Whittaker functions. The following are some examples:

$$(20)\qquad x^\tau H_\tau^{(1)}(x)\, H_\tau^{(2)}(x) = 2\cos(\tau\pi)\, \pi^{-5/2} x^{\mu-1} \cdot E(\tfrac{1}{2} + \tau, \tfrac{1}{2} - \tau, \tfrac{1}{2} :: x),$$

$$(21)\qquad S_{\mu,\tau}(x) = \{\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\tau)\, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2} - \tau)\}^{-1} x^{\mu-1} \times$$
$$\times E(1, \tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\tau, \tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2} - \tau :: \tfrac{1}{4}x^2),$$

(22) $W_{k,m}(2ix)\, W_{k,m}(-2ix) = \pi^{-1/2}(\tfrac{1}{2}x)^{2k}\{\Gamma(\tfrac{1}{2}-k+m)\,\Gamma(\tfrac{1}{2}-k-m)\}^{-1} \times$
$\times\; E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m, \tfrac{1}{2}-k, 1-k : 1-2k : \tfrac{1}{4}x^2)$;

(23) $J_\tau(x)\, J_{-\tau}(x) = \pi^{-1/2}\, E(\tfrac{1}{2}; 1-\tau, 1+\tau : 1/x^2)$;

(24) $J_\tau^2(x) = \pi^{-1/2}\, E(\tfrac{1}{2}+\tau : 1+\tau, 1+2\tau : 1/x^2)$;

Also the following formula which can be deduced by calculating the residues of the complex integral (see [3], p. 374) will be utilized

$$E(p; \alpha_r : q; \varepsilon_s : z) = \frac{1}{2\pi i}\int \Gamma(\phi)\, \frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\phi)}{\prod_{s=1}^{q}\Gamma(\varepsilon_s-\phi)}\, z^{\phi}\, d\phi\,; \tag{25}$$

where the integral is taken along the θ-axis with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ lie to the right of the contour. Zero and neagative values of the α's and ε's are ommitted. When $p < q + 1$ the contour is bent to the left at both ends.

Convergence is secured if $|\arg z| < \tfrac{1}{2}(p-q+1)$ if $p > q+1$ and $|z| > 1$ if $p = q+1$. When Mellin transform pair (11), (12) is applied to (25), we obtain the following formula which is needed also in the proof:

$$\int_0^\infty x^{k-1}\, E(p; \alpha_r : q; \varepsilon_s : x)\, dx = \Gamma(-k)\, \frac{\prod_{r=1}^{p}\Gamma(\alpha_r+k)}{\prod_{s=1}^{q}\Gamma(\varepsilon_s+k)}\,; \tag{26}$$

where $R(\alpha_r + k) > 0$, $r = 1, 2, \ldots, p$ and $R(-k) > 0$.

3. Proof of the main theorem. To prove (1), replace the F function on the left by a double complex integral by means of (10; then, on changing the order of integration, the left hand side of (1) becomes

$$\frac{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j)\,\Gamma(\delta'_j)\}}{\prod_{j=1}^{\mu}\Gamma(\alpha_\mu)\prod_{j=1}^{\tau}\{\Gamma(\beta_\tau)\,\Gamma(\beta'_\tau)\}\,.\,a^k}\;\frac{1}{(2\pi i)^2}\iint \Gamma(s)\,\Gamma(t)\,\times$$

$$\times\;\frac{\prod_{j=1}^{\mu} I(\alpha_j-s-t)\prod_{j=1}^{\tau}\{\Gamma(\beta_j-s)\,\Gamma(\beta'_j-t)\}}{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j-s-t)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j-s)\,\Gamma(\delta'_j-t\}}\left(\frac{b}{a}\right)^{-s}(c)^{-t}\,ds\,dt\;\times$$

$$\times\int_0^\infty x^{k-s-1}\, E(p; \phi_r : q; \theta_u : x)\, dx\,.$$

Here evaluate the last integral by means of (26) and the last expression becomes

$$\frac{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j)\,\Gamma(\delta_j')\}}{\prod_{j=1}^{\mu}\Gamma(\alpha_\mu)\prod_{j=1}^{\tau}\{\Gamma(\beta_\tau)\,\Gamma(\beta_\tau')\}\,a^k}\,\frac{1}{2\pi i}\int\Gamma(s)\,\Gamma(s-k)\,\times$$

$$\times\frac{\prod_{j=1}^{\tau}\Gamma(\beta_j-s)\prod_{j=1}^{p}\Gamma(\phi_j+k-s)}{\prod_{j=1}^{\sigma}\Gamma(\delta_j-s)\prod_{j=1}^{q}(\theta_j+k-s)}\left(\frac{a}{b}\right)^s\times$$

$$\times\left(\frac{1}{2\pi i}\int\Gamma(t)\frac{\prod_{j=1}^{\mu}\Gamma(\alpha_j-s-t)\prod_{j=1}^{\tau}\Gamma(\beta_j'-t)\,(c)^{-t}}{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j-s-t)\prod_{j=1}^{\sigma}\Gamma(\delta_j'-t)}\,dt\right)ds\,.$$

Now evaluate the last integral by means of (25), apply (14) since $\mu+\tau\leqq\varepsilon+$ $+\sigma+1$, and the last expression becomes

$$\frac{\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j)\prod_{j=1}^{\sigma}\{\Gamma(\delta_j)\,\Gamma(\delta_j')\}}{\prod_{j=1}^{\mu}\Gamma(\alpha_\mu)\prod_{j=1}^{\tau}\{\Gamma(\beta_\tau)\,\Gamma(\beta_\tau')\}}\,\frac{1}{2\pi i}\int\Gamma(s)\,\Gamma(s-k)\,\frac{\prod_{j=1}^{\tau}\Gamma(\beta_j-s)\prod_{j=1}^{p}\Gamma(\phi_j+k-s)}{\prod_{j=1}^{\sigma}\Gamma(\delta_j-s)\prod_{j=1}^{q}\Gamma(\theta_j+k-s)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{\mu}\Gamma(\alpha_j+m-s\prod_{j=1}^{\tau}\Gamma(\beta_j'+m)}{m!\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j+m-s)\prod_{j=1}^{\sigma}\Gamma(\delta_j'+m)}\,(-c)^m\,(a/b)^s\,ds\,.$$

Here change the order of integration and summation and evaluate the last integral by calculating the residues at the two poles:

$$s=-n\quad\text{and}\quad s=k-n\quad(0,1,2,\ldots)$$

applying (25) and the known relations:

$$\Gamma(k)\,\Gamma(1-k)=\pi\operatorname{cosec}(k\pi)\,,\quad\text{and}\quad(k;-n)=\frac{(-1)^n}{(1-k;n)}\,,$$

and so obtain the right hand side of (1). Thus (1) is proved.

4. Particular cases. We are now in a position to obtain a large number of infinite integrals specialising the parameters $\mu,\tau,\varepsilon,p,q,\sigma$.

Integrals involving F_1. Thus combination of (1) and (3) gives:

(27)
$$\int_0^\infty x^{k-1}\, E(p; \phi_r : q; \theta_u : ax)\, F_1[\alpha; \beta, \beta'; \gamma; -bx, -c]\, dx =$$
$$= -\pi \frac{\Gamma(\gamma) \operatorname{cosec}(k\pi)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')\, a^k} \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!} (-c)^m \times$$
$$\times \left[E\begin{pmatrix} \alpha + m, \beta, \phi_1 + k, \ldots, \phi_p + k : e^{\pm i\pi} \dfrac{a}{b} \\ 1 + k, \gamma + m, \theta_1 + k, \ldots, \theta_q + k \end{pmatrix} - \right.$$
$$\left. - \left(\frac{a}{b}\right)^k E\begin{pmatrix} \alpha - k + m, \beta - k, \phi_1, \ldots, \phi_p : e^{\pm i\pi} \dfrac{a}{b} \\ 1 - k, \gamma - k + m, \phi_1, \ldots, \theta_q \end{pmatrix} \right]$$

where $R(k + \phi_r) > 0$, $r = 1, 2, \ldots, p$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$ and a, b, c are real and positive.

In (27) take $p = 2$, $q = 0$, apply (16) and get

(28)
$$\int_0^\infty e^x x^{k-1/2} K_n(ax)\, F_1(\alpha; \beta, \beta'; \gamma; -bx, -c)\, dx =$$
$$= \frac{-\sqrt{(\pi/2)}\,\Gamma(\gamma)}{\sin(k\pi)\cos(n\pi)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!} (-c)^m \times$$
$$\times \left[E\left(\alpha + m, \beta, \tfrac{1}{2} + n + k, \tfrac{1}{2} - n + k : 1 + k, \gamma + m : e^{\pm i\pi} \frac{2a}{b}\right) - \right.$$
$$\left. - \left(\frac{2a}{b}\right)^k E\left(\alpha - k + m, \beta - k, \tfrac{1}{2} + n, \tfrac{1}{2} - n : 1 - k, \gamma - k + m : e^{\pm i\pi} \frac{a}{b}\right)\right]$$

where $R(k + \frac{1}{2} \pm n) < 0$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$ and a, b, c are real and positive.

In (27) take $p = 2$, $q = 0$, apply (17) with $\varphi_1 = \frac{1}{2} - k' + n$, $\varphi_2 = \frac{1}{2} - k' - n$ and get

(29)
$$\int_0^\infty e^{x/2} x^{k-k'-1} W_{k',n}(ax)\, F_1(\alpha; \beta, \beta'; \gamma; -bx, -c)\, dx =$$
$$= \frac{-\Gamma(\gamma)\,\pi}{\sin(k\pi)\,\Gamma(\frac{1}{2} - k' + n)\,\Gamma(\frac{1}{2} - k' - n)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')\, a^k} \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!} (-c)^m \times$$
$$\times \left[E\left(\alpha + m, \tfrac{1}{2} - k' + k + n, \tfrac{1}{2} - k' + k - n, \beta : 1 + k, \gamma + m : e^{\pm i\pi}\frac{a}{b}\right) - \right.$$
$$\left. - \left(\frac{a}{b}\right)^k E\left(\alpha - k + m, \beta - k, \tfrac{1}{2} - k' + m, \tfrac{1}{2} - k' - n : 1 + k, \gamma - k + m : e^{\pm i\pi}\frac{a}{b}\right)\right]$$

where $R(\frac{1}{2} - k' \pm n + k) > 0$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$ and a, b, c are real and positive.

In (27) write $4/x^2$, $1/a^2$, $b^2/4$ for x, a, b, respectively, take $p = 0$, $q = 1$, apply (18) and get:

(30) $$\int_0^\infty x^{k-1} J_\tau(ax)\, F_1\left(\alpha; \beta, \beta'; \gamma; -\frac{b^2}{x^2}, -c\right) dx = \frac{\Gamma(\gamma)\, a^{-k-\tau}}{2^k\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')} \times$$

$$\times \frac{\pi}{\sin(k+\tau)/2\,\pi} \left[\sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)\,(-c)^m\, \Gamma(\alpha+m)\, \Gamma(\beta)}{m!\, \Gamma(\gamma+m)\, \Gamma(1-\frac{1}{2}\tau-\frac{1}{2}k)\, \Gamma(1+\frac{1}{2}\tau-\frac{1}{2}k)} \times \right.$$

$$\times\ {}_2F_3\left(\begin{matrix} \alpha+m, \beta; \dfrac{b^2}{4a^2} \\ \gamma+m, 1-\frac{1}{2}\tau-\frac{1}{2}k, 1+\frac{1}{2}\tau-\frac{1}{2}k \end{matrix}\right) -$$

$$- \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)\,(-c)^m\, \Gamma(\alpha+\frac{1}{2}\tau+\frac{1}{2}k+m)\, \Gamma(\beta+\frac{1}{2}\tau+\frac{1}{2}k)}{m!\, \Gamma(\gamma+\frac{1}{2}\tau+\frac{1}{2}k+m)\, \Gamma(1+\frac{1}{2}\tau+\frac{1}{2}k)\, \Gamma(1+\tau)} \left(\frac{b}{2a}\right)^{k+\tau} \times$$

$$\left. \times\ {}_2F_3\left(\begin{matrix} \alpha+\frac{1}{2}\tau+\frac{1}{2}k+m, \beta+\frac{1}{2}\tau+\frac{1}{2}k; \dfrac{b^2}{4a^2} \\ \gamma+\frac{1}{2}\tau+\frac{1}{2}k+m, 1+\frac{1}{2}\tau+\frac{1}{2}k, 1+\tau \end{matrix}\right)\right]$$

where $R(\alpha + \frac{1}{2}k + \frac{1}{2}\tau) > 0$, $R(\beta + \frac{1}{2}k + \frac{1}{2}\tau) > 0$, $R(k) > \frac{1}{2}$ and a, b, c are real and positive.

In (27) take $p = 4$, $q = 1$; write $x^2/4$, $4a^2$, $4/b^2$ for x, a, b respectively apply (22) and get

(31) $$\int_0^\infty x^{q-1} W_{k,n}(4iax)\, W_{k,n}(-4iax)\, F_1(\alpha; \beta, \beta'; \gamma; -x^2/b^2, -c)\, dx =$$

$$= -\frac{-\pi\,\Gamma(\gamma)\, \{\Gamma(\frac{1}{2}-k-n)\, \Gamma(\frac{1}{2}-k+n)\}^{-1}}{2^{2k+1} a^{2k+q} \sin(k+\frac{1}{2}q)\,\pi\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)}{m!} (-c)^m \times$$

$$\times E(\alpha+m, \beta, \tfrac{1}{2}+\tfrac{1}{2}k+\tfrac{1}{2}q+n, \tfrac{1}{2}+\tfrac{1}{2}k+\tfrac{1}{2}q-n, \tfrac{1}{2}+\tfrac{1}{2}q, 1+\tfrac{1}{2}q : 1+\tfrac{1}{2}q +$$

$$+ k, 1-k+\tfrac{1}{2}q : e^{\pm i\pi} a^2 b^2 + \frac{\pi\,\Gamma(\gamma)\, \{\Gamma(\frac{1}{2}-k-n)\, \Gamma(\frac{1}{2}-k+n\}^{-1} (ab)^{q+2k}}{2^{2k+1} a^{2k+q} \sin(k+\frac{1}{2}q)\,\pi\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')} \times$$

$$\times \sum_{0}^{\infty} \frac{\Gamma(\beta'+m)}{m!} (-c)^m E(\alpha-k-\tfrac{1}{2}q+m, \beta-k-\tfrac{1}{2}q, \tfrac{1}{2}-k-n, \tfrac{1}{2}-k+n,$$

$$\tfrac{1}{2}-k, 1-k; 1-k, 1-2k : e^{\pm i\pi} a^2 b^2);$$

where $R(k + q + 1 \pm 2n) > 0$, $R(2\alpha - 2k - q) > 0$, $R(2\beta - 2k - q) > 0$ and a, b, c are real and positive.

Also (27) in combination with (23) gives:

$$(32) \qquad \int_0^\infty x^{-2k-1} J_n(ax) J_{-n}(ax) F_1(\alpha; \beta, \beta'; \gamma; -\frac{b^2}{x^2}, -c)\, dx =$$

$$= -\frac{\sqrt{\pi} a^{2k} \Gamma(\gamma) \Gamma(\beta) \Gamma(\frac{1}{2} + k)}{\sin k\pi\, \Gamma(\alpha) \Gamma(\beta) \Gamma(\beta') \Gamma(1 + k) \Gamma(1 - n + k) \Gamma(1 + n + k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m) \Gamma(\alpha + m)}{m!\, \Gamma(\gamma + m)} (-c)^m\, {}_3F_4\left(\begin{matrix} \alpha + m, \beta, \frac{1}{2} + k; b^2 a^2 \\ 1 + k, \gamma + m, 1 - n + k, 1 + n + k \end{matrix}\right) +$$

$$+ \frac{\pi \Gamma(\gamma) \Gamma(\beta - k)}{b^{2k} \sin (k\pi) \Gamma(\alpha) \Gamma(\beta) \Gamma(\beta') \Gamma(1 - k) \Gamma(1 + n) \Gamma(1 - n)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m) \Gamma(a - k + m)}{m!\, \Gamma(\gamma - k + m)} (-c)^m\, {}_3F_4\left(\begin{matrix} \alpha - k + m, \beta - k, \frac{1}{2}; b^2 a^2 \\ 1 - k, \gamma - k + m, 1 + n, 1 - n \end{matrix}\right);$$

where $R(\alpha - k) > 0$, $R(\beta - k) > 0$, $R(k) > -\frac{1}{2}$ and a, b, c are real and positive.

Again (27) in combination with (24) gives:

$$(33) \qquad \int_0^\infty x^{-2k-1} J_n^2(ax) F_1\left(\alpha; \beta, \beta'; \gamma; -\frac{b^2}{x^2}, -c\right) dx =$$

$$= -\frac{a^{2k} \sqrt{\pi}\, \Gamma(\beta) \Gamma(\frac{1}{2} + n + k)}{\sin (k\pi) \Gamma(\alpha) \Gamma(\beta') \Gamma(1 + k) \Gamma(1 + n + k) \Gamma(1 + 2n + k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m) \Gamma(\alpha + m) (-c)^m}{m!\, \Gamma(\gamma + m)}\, {}_3F_4\left(\begin{matrix} \alpha + m\, \beta, \frac{1}{2} + n + k; a^2 b^2 \\ 1 + k, \gamma + m, 1 + n + k, 1 + 2n + k \end{matrix}\right) +$$

$$+ \frac{\pi \Gamma(\gamma) \Gamma(\beta - k)}{b^{2k} \sin (k\pi) \Gamma(\alpha) \Gamma(\beta) \Gamma(\beta') \Gamma(1 - k) \Gamma(1 + n) \Gamma(1 + 2n)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m) \Gamma(\alpha - k + m)}{m!\, \Gamma(\gamma - k + m)} (-c)^m\, {}_3F_4\left(\begin{matrix} \alpha - k + m, \beta - k, \frac{1}{2} + n; a^2 b^2 \\ 1 - k, \gamma - k + m, 1 + n, 1 + 2n \end{matrix}\right);$$

where $R(\alpha - k) > 0$, $R(\beta - k) > 0$, $R(\alpha + n - k) > 0$, $R(\beta + n - k) > 0$, $R(k) >$ $> -\frac{1}{2}$ and a, b, c, are real and positive.

Integrals involving F_2. (1) in combination with (4) gives

$$(34) \qquad \int_0^\infty x^{k-1} E(p; \phi_r : q; \theta_u : ax) F_2(\alpha; \beta, \beta'; \delta, \delta'; -bx, -c)\, dx \times$$

$$= -\frac{\Gamma(\delta) \Gamma(\delta') \pi}{a^k \sin (k\pi) \Gamma(\alpha) \Gamma(\beta) \Gamma(\beta')} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!\,\Gamma(\delta' + m)} (-c)^m \times \left[E\begin{pmatrix} \alpha + m, \beta, \phi_1 + k, \ldots, \phi_p + k : e^{\pm i\pi} \dfrac{a}{b} \\ 1 + k, \delta, \theta_1 + k, \ldots, \theta_q + k \end{pmatrix} - \right.$$

$$\left. - \left(\frac{a}{b}\right)^k E\begin{pmatrix} \alpha - k + m, \beta - k, \phi_1, \ldots, \phi_p : e^{\pm i\pi} \dfrac{a}{b} \\ 1 - k, \delta - k, \theta_1, \ldots, \theta_q \end{pmatrix} \right]$$

where $R(k + \phi_r) > 0$ $(r = 1, 2, \ldots, p)$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$, a, b, c are real and positive.

In (34) write $2a$ for a, take $p = 2$, $q = 0$, apply (16) and get:

(35) $$\int_0^\infty x^{k-1/2} e^{ax} K_n(ax)\, F_2(\alpha; \beta, \beta'; \delta, \delta'; -bx, -c)\, dx =$$

$$= \frac{-\sqrt{\pi} \cos n\pi\, \Gamma(\delta)\, \Gamma(\delta')}{2^{k+1/2} a^{k+1/2} \sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!\,\Gamma(\delta' + m)} (-c)^m \times$$

$$\times \left[E\left(\alpha + m, \beta, \tfrac{1}{2} + n + k, \tfrac{1}{2} - n + k; 1 + k, \delta : e^{\pm i\pi} \frac{2a}{b}\right) - \right.$$

$$\left. - \left(\frac{2a}{b}\right)^k E\left(\alpha - k + m, \beta - k, \tfrac{1}{2} + n, \tfrac{1}{2} - n : 1 - k, \delta - k : e^{\pm i\pi} \frac{2a}{b}\right) \right]$$

where $R(k \pm n + \frac{1}{2}) > 0$, $R(\beta - k) > 0$, $R(\alpha - k) > 0$, and a, b, c, are real and positive.

In (34) take $p = 2$, $q = 0$ with $\alpha_1 = \frac{1}{2} - \tau + n$, $\alpha_2 = \frac{1}{2} - \tau - n$, apply (17) and get:

(36) $$\int_0^\infty x^{k-\tau-1} e^{(1/2)ax} W_{\tau,n}(ax)\, F_2(\alpha; \beta, \beta'; \delta, \delta'; -bx, -c)\, dx =$$

$$= \frac{-\pi\, \Gamma(\delta)\, \Gamma(\delta')}{a^{k+\tau} \sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')\, \Gamma(\frac{1}{2} - \tau + n)\, \Gamma(\frac{1}{2} - \tau - n)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!\,\Gamma(\delta' + m)} (-c)^m$$

$$\left[E\left(\alpha + m, \beta, \tfrac{1}{2} - \tau + n + k, \tfrac{1}{2} - \tau - n + k : 1 + k, \delta : e^{\pm i\pi} \frac{a}{b}\right) - \right.$$

$$\left. - \left(\frac{a}{b}\right)^k E\left(\alpha - k + m, \beta - k, \tfrac{1}{2} - \tau + n, \tfrac{1}{2} - \tau - n : 1 - k, \delta - k : e^{\pm i\pi} \frac{a}{b}\right) \right]$$

where $R(k + \frac{1}{2} - \tau \pm n) > 0$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$, and a, b, c are real and positive.

In (34), take $p = 0$, $q = 1$, apply (18) and get:

$$(37) \qquad \int_0^\infty x^{k+(\tau/2)-1} J_\tau\left(\frac{2}{\sqrt{(ax)}}\right) F_2(\alpha; \beta, \beta'; \delta, \delta', -bx, -c)\, dx =$$

$$= \frac{-\pi\, \Gamma(\delta)\, \Gamma(\delta')}{a^{k+1/2\tau} \sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!\, \Gamma(\delta' + m)} (-c)^m \times$$

$$\times \left[\frac{\Gamma(\alpha + m)\, \Gamma(\beta)}{\Gamma(1 + k)\, \Gamma(\delta)\, \Gamma(1 + \tau + k)}\, {}_2F_3\left(\begin{matrix} \alpha + m, \beta; \frac{b}{a} \\ 1 - k, \delta, 1 + \tau + k \end{matrix}\right) - \right.$$

$$\left. - \left(\frac{a}{b}\right)^k \frac{\Gamma(\alpha - k + m)}{\Gamma(\delta - k) \Gamma(1 - k)\, \Gamma(\delta - k)\, \Gamma(1 + \tau)}\, {}_2F_3\left(\begin{matrix} \alpha - k + m, \beta - k; \frac{b}{a} \\ 1 - k, \delta - k, 1 + \tau \end{matrix}\right)\right.$$

where $R(k + \frac{1}{2}\tau - \frac{1}{4}) > 0$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$, and a, b, c are real and positive.

In (34) take $p = 3$, $q = 0$, apply (21) and get:

$$(38) \qquad \int_0^\infty x^{k-\mu} S_{\mu,\tau}(2\sqrt{(ax)})\, F_2(\alpha; \beta, \beta'; \delta, \delta', -bx, -c)\, dx =$$

$$= \frac{-\pi\, \Gamma(\delta)\, \Gamma(\delta')}{a^{k+1-\mu} \sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta') \Gamma(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\tau)\, \Gamma(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\tau)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!\, \Gamma(\delta' + m)} - (-c)^m \times$$

$$\times \left[E\left(\alpha + m, \beta, \tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\tau + k, \tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\tau + k : \delta : e^{\pm i\pi} \frac{a}{b}\right) - \right.$$

$$\left. - \left(\frac{a}{b}\right)^k E\left(\alpha - k + m, \beta - k, \tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\tau, \tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\tau, 1 : 1 - k, \delta - k : e^{\pm i\pi} \frac{a}{b}\right)\right];$$

where $R(k) > -1$, $R(k + {}^1/_2 - ({}^1/_2)\mu \pm {}^1/_2\tau) > 0$, $R(\alpha - k) > 0$, $R(\beta - k) > 0$ and a, b, c are real and positive.

Also in (34) take $p = 4$, $q = 1$, apply (22) and get:

$$(39)\quad \int_0^\infty x^{k-\tau-1} W_{\tau,n}(4i\sqrt{x})\, W_{\tau,n}(-4i\sqrt{x})\, F_2(\alpha; \beta, \beta'; \delta, \delta', -bx, -c)\, dx =$$

$$= \frac{-\pi a^{\tau-k}\, \Gamma(\delta)\, \Gamma(\delta')}{\sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')\, \Gamma(\tfrac{1}{2}-\tau+n)\, \Gamma(\tfrac{1}{2}-\tau-n)} \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)}{m!\, \Gamma(\delta'+m)} - (c)^m \times$$

$$\times \left[E\left(\begin{matrix} \alpha+m, \beta, \tfrac{1}{2}-\tau+n+k, \tfrac{1}{2}-\tau-n+k, \tfrac{1}{2}-\tau+k, 1-\tau+k : e^{\pm i\pi}\dfrac{a}{b} \\ 1+k, \delta, 1-2\tau+k \end{matrix} \right) - \right.$$

$$\left. - \left(\frac{a}{b}\right)^k . E\left(\begin{matrix} \alpha-k+m, \beta-k, \tfrac{1}{2}-\tau+n, \tfrac{1}{2}-\tau-n, \tfrac{1}{2}-\tau, 1-\tau : e^{\pm i\pi}\dfrac{a}{b} \\ 1-k, \delta-k, 1-2\tau \end{matrix} \right) \right] i$$

where $R(k+\tfrac{1}{2}-\tfrac{1}{2}\tau \pm n) > 0$, $R(\alpha-k) > 0$, $R(\beta-k) > 0$ and a, b, c are real and positive.

(34) in combination with (23) gives:

$$(40)\quad \int_0^\infty x^{-2k-1} J_\tau(ax)\, J_{-\tau}(ax)\, F_2\left(\alpha; \beta, \beta'; \delta, \delta'; \frac{-b^2}{x^2}, -c\right) dx =$$

$$= \frac{-\sqrt{\pi}\, a^{2k}\, \Gamma(\delta')\, \Gamma(\tfrac{1}{2}+k)}{2\sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta')\, \Gamma(1+\tau-k)\, \Gamma(1-\tau-k)\, \Gamma(1+k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)\, \Gamma(\alpha+m)}{m!\, \Gamma(\delta'+m)} (-c)^m\, {}_3F_4\left(\begin{matrix} \alpha+m, \beta, \tfrac{1}{2}+k; a^2b^2 \\ 1+k, \delta, 1+\tau-k, 1-\tau-k \end{matrix} \right) +$$

$$+ \frac{\pi\, a^{2k}\, \Gamma(\delta)\, \Gamma(\delta')\, \{\Gamma(1+\tau)\, \Gamma(1-\tau)\}^{-1}}{2\sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\beta')\, (a^2b^2)^k\, \Gamma(1-k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)\, \Gamma(\alpha-k+m)}{m!\, \Gamma(\delta'+m)} (-c)^m\, {}_3F_4\left(\begin{matrix} \alpha-k+m, \beta-k, \tfrac{1}{2}; a^2b^2 \\ 1-k, \delta-k, 1+\tau, 1-\tau \end{matrix} \right)$$

where $R(\alpha-k) > 0$, $R(\beta-k) > 0$, $R(k) > -\tfrac{1}{2}$ and a, b, c are real and positive.

Again (34) in combination with (24) gives:

$$(41)\quad \int_0^\infty x^{-2k-1} J_\tau^2(ax)\, F_2\left(\alpha; \beta, \beta'; \delta, \delta'; -\frac{b^2}{x^2}, -c\right) dx =$$

$$= \frac{-a^{2k}\sqrt{\pi}\, \Gamma(\delta')\, \Gamma(\tfrac{1}{2}+\tau+k)}{2\sin(k\pi)\, \Gamma(\alpha)\, \Gamma(\beta')\, \Gamma(1+\tau+k)\, \Gamma(1+2\tau+k)\, \Gamma(1+k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)\, \Gamma(\alpha+m)}{m!\, \Gamma(\delta'+m)} (-c)^m\, {}_3F_4\left(\begin{matrix} \alpha+m, \beta, \tfrac{1}{2}+\tau+k; a^2b^2 \\ 1+k, \delta, 1+\tau+k, 1+2\tau+k \end{matrix} \right) +$$

$$+\frac{\sqrt{\pi}\,\Gamma(\delta)\,\Gamma(\delta')\,\Gamma(\beta-k)}{2b^{2k}\,\Gamma(\alpha)\,\Gamma(\beta')\,\Gamma(1-k)\,\Gamma(\delta-k)\,\Gamma(1+2\tau)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\beta'+m)\,\Gamma(\alpha-k+m)}{m!\,\Gamma(\delta'+m)}(-c)^m\,{}_3F_4\left(\begin{matrix}\alpha-k+m,\beta-k;\frac{1}{2}+\tau;a^2b^2\\1-k,\delta-k,1+\tau,1+2\tau\end{matrix}\right);$$

where $R(\alpha+\tau-k)>0$, $R(\beta+\tau-k)>0$, $R(k)>-\frac{1}{2}$ and a, b, c are real and positive.

Integrals involving F_3. (1) in combination with (5) gives:

(42) $$\int_0^\infty x^{k-1}\,E(p;\phi_r:q,\theta_u:ax)\,F_3(;\beta_1,\beta_1';\beta_2,\beta_2';\gamma;-bx,-c)\,\mathrm{d}x=$$

$$=\frac{-\pi\,\Gamma(\gamma)}{a^k\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')}\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,(-c)^{-m}}$$

$$\left[E\left(\beta_1,\beta_2,\phi_1+k,\ldots,\phi_p+k:1+k,\gamma+m,\theta_1+k,\ldots,\theta_q+k:e^{\pm i\pi}\frac{a}{b}\right)-\right.$$

$$\left.-\left(\frac{a}{b}\right)^kE\left(\beta_1-k,\beta_2-k,\phi_1,\ldots,\phi_p:1-k,\gamma+m-k,\theta_1,\ldots,\theta_q:e^{\pm i\pi}\frac{a}{b}\right)\right]$$

where $R(k+\phi_r)>0$ $(r=1,2,\ldots,p)$, $R(\beta_1-k)>0$, $R(\beta_2-k)>0$; a, b, c real and positive.

In (42) take $p=2$, $q=0$, apply (16) and get:

(43) $$\int_0^\infty x^{k-1/2}e^{ax}\,K_n(ax)\,F_3(;\beta_1,\beta_1';\beta_2,\beta_2';\gamma;-bx,-c)\,\mathrm{d}x=$$

$$=\frac{-\pi\,\Gamma(\gamma)\cos n\pi}{\sqrt{(2\pi)}\,a^{k+1/2}\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')}\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,(-c)^{-m}}\times$$

$$\times\left[E\left(\beta_1,\beta_2,\tfrac{1}{2}+n+k,\tfrac{1}{2}-n+k:1+k,\gamma+m:e^{\pm i\pi}\frac{2a}{b}\right)-\right.$$

$$\left.-\left(\frac{2}{ab}\right)^kE\left(\beta_1-k,\beta_2-k,\tfrac{1}{2}+n,\tfrac{1}{2}-n:1-k,\gamma+m-k:e^{\pm i\pi}\frac{2a}{b}\right)\right];$$

where $R(\frac{1}{2}+k\pm n)>0$, $R(\beta_1-k)>0$, $R(\beta_2-k)>0$ a, b, c real and positive.

In (42) take $p = 2$, $q = 0$ with $\phi_1 = \frac{1}{2} - \tau + n$, $\phi_2 = \frac{1}{2} - \tau - n$, apply (17) and get:

$$\text{(44)} \qquad \int_0^\infty x^{k-\tau-1} e^{1/2ax} W_{\tau,n}(ax)\, F_3(;\beta_1, \beta_1'; \beta_2, \beta_2'; \gamma; -bx, -c)\, dx =$$

$$= \frac{-\pi\,\Gamma(\gamma)}{a^{k-\tau}\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')\,\Gamma(\frac{1}{2}-\tau+n)\,\Gamma(\frac{1}{2}-\tau-n)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,(-c)^{-m}} \times$$

$$\times \left[E\left(\beta_1, \beta_2, \tfrac{1}{2}-\tau+n+k, \tfrac{1}{2}-\tau-n+k; 1+k, \gamma+m; e^{\pm i\pi}\frac{a}{b}\right) -\right.$$

$$\left. - \left(\frac{a}{b}\right)^k E\left(\beta_1 - k, \beta_2 - k, \tfrac{1}{2}-\tau+n, \tfrac{1}{2}-\tau-n : 1-k, \gamma-k+m : e^{\pm i\pi}\frac{a}{b}\right)\right]$$

where $R(\frac{1}{2}+k-\tau\pm n) > 0$, $R(\beta_1 - k) > 0$, $R(\beta_2 - k) > 0$, a, b, c real and positive.

In (42) take $p = 0$, $q = 1$; apply (18) and get:

$$\text{(45)} \qquad \int_0^\infty x^{k+(\tau/2)-1} J_\tau\left(\frac{2}{\sqrt{ax}}\right) F_3(;\beta_1, \beta_1'; \beta_2, \beta_2' : \gamma; -bx, -c)\, dx =$$

$$= \frac{-\pi\,\Gamma(\gamma)}{a^{k-\tau/2}\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,(-c)^{-m}} \times$$

$$\times \left[\frac{\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(1+k)\,\Gamma(1+\tau+m)}\, {}_2F_2\left(\begin{matrix}\beta_1, \beta_2, \dfrac{a}{b}\\ 1+k, 1+\tau+m\end{matrix}\right) -\right.$$

$$\left. - \left(\frac{a}{b}\right)^k \frac{\Gamma(\beta_1-k)\,\Gamma(\beta_2-k)}{\Gamma(1+\tau-k+m)\,\Gamma(1-k)}\, {}_2F_2\left(\begin{matrix}\beta_1-k, \beta_2-k : \dfrac{b}{a}\\ 1-k, 1+\tau-k+m\end{matrix}\right)\right]$$

where $R(k+\frac{1}{2}\tau) > -\frac{1}{4}$, $R(\beta_1 - k) > 0$, $R(\beta_2 - k) > 0$, a, b, c real and positive.

In (42) take $p = 3$, $q = 0$, apply (20) and get:

$$\text{(46)} \qquad \int_0^\infty x^{2k-\mu} S_{\mu,\tau}(ax)\, F_3(;\beta_1, \beta_1'; \beta_2, \beta_2'; \gamma; -b^2x^2, -c)\, dx =$$

$$= -2^{2k-1}\pi\, a^{\mu-2k-1}\,\Gamma(\gamma)\,\{\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\tau) \times$$

$$\times \Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\tau\}^{-1} \sum_{m=0}^{\infty} \frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,(-c)^{-m}} \times$$

$$\times E\begin{pmatrix}\beta_1, \beta_2, \tfrac{1}{2} - \tfrac{1}{2}\tau - \tfrac{1}{2}\mu + k, \tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\tau + k : e^{\pm i\pi}\dfrac{a^2}{4b^2} \\ \gamma_1 + m\end{pmatrix} +$$

$$+ 2^{2k-1}\pi\, a^{\mu-2k-1}\,\Gamma(\gamma)\,\{\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')\,\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\tau)\times$$

$$\times\,\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\tau)\}^{-1}\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1' + m)\,\Gamma(\beta_2' + m)}{m!\,(-c)^{-m}}\times$$

$$\times E\left(\beta_1 - k, \beta_2 - k, 1, \tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\tau, \tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\tau : 1 - k, \gamma + m - k : e^{\pm i\pi}\frac{a^2}{4b^2}\right)$$

where $R(k + \tfrac{1}{2} - \tfrac{1}{2}\mu \pm \tfrac{1}{2}\tau) > 0$, $R(\beta_1 - k) > 0$, $R(\beta_2 - k) > 0$, a, b, c real and positive.

In (42) take $p = 4$, $q = 1$; apply (22) and get:

$$(47)\quad \int_0^\infty x^{2k-2\tau-1}\,W_{\tau,n}(2iax)\,W_{\tau,n}(-2iax)\,F_3(;\beta_1,\beta_1';\beta_2,\beta_2';\gamma;\,-b^2x^2,\,-c)\,dx =$$

$$= -\frac{2^{2k-2\tau-1}\sqrt{\pi}\,\Gamma(\gamma)}{a^{2k-2\tau}\sin(\pi k)\sqrt{\pi}\,\Gamma(\tfrac{1}{2} - \tau + n)\,\Gamma(\tfrac{1}{2} - \tau - n)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1' + m)\,\Gamma(\beta_2' + m)}{m!\,(-c)^{-m}}\times$$

$$\times\left[E\begin{pmatrix}\beta_1, \beta_2, \tfrac{1}{2} - \tau + n + k, \tfrac{1}{2} - \tau - n + k, \tfrac{1}{2} - \tau + k, 1 - \tau + k : e^{\pm i\pi}\dfrac{a^2}{4b^2} \\ 1 + k, \gamma + m, 1 - 2\tau + k\end{pmatrix} -\right.$$

$$\left. - \left(\frac{a^2}{4b^2}\right)^k E\begin{pmatrix}\beta_1 - k, \beta_2 - k, \tfrac{1}{2} - \tau + n, \tfrac{1}{2} - \tau - n, \tfrac{1}{2} - \tau, 1 - \tau : e^{\pm i\pi}\dfrac{a^2}{4b^2} \\ 1 - k, \gamma - k + m, 1 - 2\tau\end{pmatrix}\right]$$

where $R(\tfrac{1}{2} + k - \tau \pm n) > 0$, $R(\beta_1 - k) > 0$, $R(\beta_2 - k) > 0$, a, b, c real and positive.

In (42) take $p = 1$, $q = 2$; apply (23) and get:

$$(48)\quad \int_0^\infty x^{-2k-1}\,J_\tau(ax)\,J_{-\tau}(ax)\,F_3\left(;\beta_1,\beta_1';\beta_2,\beta_2';\gamma;\,-\frac{b^2}{x^2},\,-c\right)dx =$$

$$= \frac{-\sqrt{\pi}\,a^{2k}\,\Gamma(\gamma)\,\Gamma(\tfrac{1}{2} + k)}{2\sin(k\pi)\,\Gamma(\beta_1')\,\Gamma(\beta_2')\,\Gamma(1 + k)\,\Gamma(1 + \tau + k)\,\Gamma(1 - \tau + k)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1' + m)\,\Gamma(\beta_2' + m)}{m!\,\Gamma(\gamma + m)\,(-c)^{-m}}\,{}_3F_4\begin{pmatrix}\beta_1, \beta_2\, \tfrac{1}{2} + k;\, a^2b^2 \\ 1 + k, 1 - \tau + k, 1 + \tau + k.\, \gamma + m\end{pmatrix} +$$

$$+\frac{\pi\,\Gamma(\gamma)\,\Gamma(\beta_1-k)\,\Gamma(\beta_2-k)}{2b^{2k}\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_1')\,\Gamma(\beta_2')\,\Gamma(1-k)\,\Gamma(1+\tau)\,\Gamma(1-\tau)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,\Gamma(\gamma-k+m)}(-c)^m\,{}_3F_4\left(\begin{matrix}\beta_1-k,\beta_2-k,\tfrac{1}{2};\,a^2b^2\\ 1-k,\gamma-k+m,1+\tau,1-\tau\end{matrix}\right)$$

where $R(\beta_1-k)>0$, $R(\beta_2-k)>0$, $R(k)>-\frac{1}{2}$, a, b, c real and positive.

Similarly (42) in combination with (24) gives:

(49) $$\int_0^{\infty}x^{-2k-1}\,J_\tau^2(ax)\,F_3\left(;\beta_1,\beta_1';\beta_2,\beta_2';\gamma;-\frac{b^2}{x^2},-c\right)\mathrm{d}x=$$

$$=-\frac{\sqrt{\pi}\,a^{2k}\,\Gamma(\gamma)\,\Gamma(\tfrac{1}{2}+\tau+k)}{2\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_1')\,\Gamma(\beta_2')\,\Gamma(1+k)\,\Gamma(1+\tau+k)\,\Gamma(1+2\tau+k)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)\,\Gamma(\beta_1)\,\Gamma(\beta_2)}{m!\,\Gamma(\gamma+m)\,(-c)^{-m}}\times$$

$$\times\,{}_3F_4\left(\begin{matrix}\beta_1,\beta_2,\tfrac{1}{2}+\tau+k;\,a^2b^2\\ 1+k,1+\tau+k,1+2\tau+k,\gamma-k+m\end{matrix}\right)+$$

$$+\frac{\sqrt{\pi}\,b^{-2k}\,\Gamma(\gamma)\,\Gamma(\tfrac{1}{2}+\tau)\,\Gamma(\beta_1-k)\,\Gamma(\beta_2-k)}{2\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_1')\,\Gamma(\beta_2')\,\Gamma(1-k)\,\Gamma(1+\tau+k)\,\Gamma(1+2\tau+k)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,\Gamma(\gamma-k+m)}(-c)^m\,{}_3F_4\left(\begin{matrix}\beta_1-k,\beta_2-k,\tfrac{1}{2}+\tau;\,a^2b^2\\ 1-k,\gamma-k+m,1+\tau,1+2\tau\end{matrix}\right)$$

where $R(2\tau-2k+2\beta_1)>0$, $R(\tau-k+\beta_2)>0$, $R(k)>-\frac{1}{2}$, a, b, c real and positive.

Integrals involving F_4. (1) in combination with (6) gives

(50) $$\int_0^{\infty}x^{k-1}\,E(p;\phi_r:q;\theta_u:ax)\,F_4(\alpha_1,\alpha_2;\delta_1,\delta_2;-bx,-c)\,\mathrm{d}x=$$

$$=\frac{-\pi\,\Gamma(\delta_1)\,\Gamma(\delta_2)}{a^k\sin(k\pi)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\sum_{m=0}^{\infty}\frac{(-c)^m}{m!\,\Gamma(\delta_2+m)}\times$$

$$\times\left[E\left(\begin{matrix}\alpha_1+m,\alpha_2+m,\phi_1+k,\ldots,\phi_p+k:e^{\pm i\pi}\dfrac{a}{b}\\ 1+k,\delta_1,\theta_1+k,\ldots,\theta_q+k\end{matrix}\right)-\right.$$

$$\left.-\left(\frac{a}{b}\right)^kE\left(\begin{matrix}\alpha_1-k+m,\alpha_2-k+m,\phi_1,\ldots,\phi_q:e^{\pm i\pi}\dfrac{a}{b}\\ 1-k,\delta_1-k,\theta_1,\ldots,\theta_q\end{matrix}\right)\right]$$

where $R(k + \phi_r) > 0$ $(r = 1, 2, \ldots, p)$, $R(\alpha_1 - k) > 0$, $R(\alpha_2 - k) > 0$ and a, b, c are real and positive.

(50) in combination with (16) gives:

$$(51)\qquad \int_0^\infty x^{k-1/2} e^{ax} K_n(ax)\, F_4(\alpha_1, \alpha_2; \delta_1, \delta_2; -bx, -c)\, dx =$$

$$= \frac{-\cos n\pi\, \Gamma(\delta_1)\, \Gamma(\delta_2)}{\sqrt{2\pi a^{k-\frac{1}{2}}}\, \Gamma(\alpha_1)\, \Gamma(\alpha_2)} \frac{\pi}{\sin(k\pi)} \times$$

$$\times \sum_{m=0}^{\infty} \left[E\left(\alpha_1 + m, \alpha_2 + m, \tfrac{1}{2} + n + k, \tfrac{1}{2} - n + k; 1 + k, \delta_1 : e^{\pm i\pi} \frac{2a}{b}\right) - \right.$$

$$\left. - \left(\frac{2a}{b}\right)^k E\left(\alpha_1 - k + m, \alpha_2 - k + m, \tfrac{1}{2} + n, \tfrac{1}{2} - n : 1 - k, \delta_1 - k : e^{\pm i\pi} \frac{2a}{b}\right)\right] \times$$

$$\times \frac{(-c)^m}{m!\, \Gamma(\delta_2 + m)}$$

where $R(\frac{1}{2} + k \pm n) > 0$, $R(\alpha_1 - k) > 0$, $R(\alpha_2 - k) > 0$, a, b, c, real and positive.

Also (50) in combination with (17) gives:

$$(52)\qquad \int_0^\infty x^{k-\tau-1} e^{1/2ax}\, W_{\tau,n}(ax)\, F_4(\alpha_1, \alpha_2; \delta_1, \delta_2: -bx, -c)\, dx =$$

$$= \frac{-\pi\, \Gamma(\delta_1)\, \Gamma(\delta_2)}{a^{k-\tau} \sin(k\pi)\, \Gamma(\alpha_1)\, \Gamma(\alpha_2)\, \Gamma(\frac{1}{2} - \tau + n)\, \Gamma(\frac{1}{2} - \tau - n)} \sum_{m=0}^{\infty} \frac{(-c)^m}{m!\, \Gamma(\delta_2 + m)} \times$$

$$\times \left[E\left(\begin{matrix}\alpha_1 + m, \alpha_2 + m, \frac{1}{2} - \tau + n + k, \frac{1}{2} - \tau - n + k : e^{\pm i\pi} \frac{a}{b} \\ 1 + k, \delta_1\end{matrix}\right) - \right.$$

$$\left. - \left(\frac{a}{b}\right)^k E\left(\begin{matrix}\alpha_1 - k + m, \alpha_2 - k + m, \frac{1}{2} - \tau + n, 1 - \tau - n : e^{\pm i\pi} \frac{a}{b} \\ 1 - k, \delta_1 - k\end{matrix}\right)\right]$$

where $R(\frac{1}{2} - \tau + k \pm n) > 0$, $R(\alpha_1 - k) > 0$, $R(\beta_1 - k) > 0$, a, b, c real and positive.

(50) in combination with (18) gives:

$$(53)\qquad \int_0^\infty x^{k+(\tau/2)-1} J_\tau\left(\frac{2}{\sqrt{(ax)}}\right) F_4(\alpha_1, \alpha_2; \delta_1, \delta_2; -bx, -c)\, dx =$$

$$= \frac{-\pi\, \Gamma(\delta_1)\, \Gamma(\delta_2)}{a^{k+\tau/2} \sin(k\pi)\, \Gamma(\alpha_1)\, \Gamma(\alpha_2)} \sum_{m=0}^{\infty} \left[\frac{\Gamma(\alpha_1 + m)\, \Gamma(\alpha_2 + m)\, (-c)^m}{m!\, \Gamma(\delta_2 + m)\, \Gamma(\delta_1)\, \Gamma(1 + k)\, \Gamma(1 + \tau + k)} \times \right.$$

$$\times {}_2F_3\left(\begin{matrix}\alpha_1+m,\alpha_2+m;\dfrac{a}{b}\\ 1+k,1+\tau+k,\delta\end{matrix}\right)-\left(\frac{a}{b}\right)^k\frac{\Gamma(\alpha_1+m-k)\,\Gamma(\alpha_2-k+m)\,(-c)^m}{m!\,\Gamma(\delta_2+m)\,\Gamma(1-k)\,\Gamma(\delta_1-k)\,\Gamma(1+\tau)}\times$$

$$\times\, {}_2F_3\left(\begin{matrix}\alpha_1-k+m,\alpha_2-k+m;\dfrac{b}{a}\\ 1-k,\delta_1-k,1+\tau\end{matrix}\right)\Bigg]$$

where $R(k+\frac{1}{2}\tau)>-\frac{1}{4}$, $R(\alpha_1-k)>0$, $R(\alpha_2-k)>0$ and a, b, c are real and positive.

Again (50) in combination with (21) gives:

$$(54)\quad \int_0^\infty x^{2k-2\tau-1}\,W_{\tau,n}(2iax)\,W_{\tau,n}(-2iax)\,F_4\left(;\alpha_1,\alpha_2;\delta_1,\delta_2;-\frac{b^2x^2}{4},-c\right)\mathrm{d}x=$$

$$=\frac{-2^{2k-2\tau-1}\,\Gamma(\delta_1)\,\Gamma(\delta_2)\sqrt{\pi}}{a^{2k-2\tau}\sin(k\pi)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\frac{1}{2}-\tau+n)\,\Gamma(\frac{1}{2}-\tau-n)}\times\sum_{m=0}^{\infty}\frac{(-c)^m}{m!\,\Gamma(\delta_2+m)}\times$$

$$\times\left[E\left(\begin{matrix}\alpha_1+m,\alpha_2+m,\frac{1}{2}-\tau+n+k,\frac{1}{2}-\tau-n+k,\frac{1}{2}-\tau+k,\\ 1-\tau+k:e^{\pm i\pi}\dfrac{a^2}{b^2}\\ 1+k,\delta_1,1-2\tau+k\end{matrix}\right)-\right.$$

$$\left.-\left(\frac{a}{b}\right)^{2k}E\left(\begin{matrix}\alpha_1+m-k,\alpha_2+m-k,\frac{1}{2}-\tau+n,\frac{1}{2}-\tau-n,\frac{1}{2}-\tau,1-\tau:e^{\pm i\pi}\dfrac{a^2}{b^2}\\ 1-k,\delta_1-k,1-2\tau\end{matrix}\right)\right]$$

where $R(\frac{1}{2}-\tau+k\pm n)>0$, $R(\alpha_1-k)>0$, $R(\beta_1-k)>0$ and a, b, c are real and positive.

In (50) take $p=1$, $q=2$ apply (23) and get:

$$(55)\qquad \int_0^\infty x^{-2k-1}\,J_\tau(ax)\,J_{-\tau}(ax)\,F_4\left(\alpha_1,\alpha_1;\delta_1,\delta_2;-\frac{b^2}{x^2},-c\right)\mathrm{d}x=$$

$$=\frac{-\sqrt{\pi}\,a^{2k}\,\Gamma(\frac{1}{2}+k)\,\Gamma(\delta_2)}{2\sin(k\pi)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(1+k)\,\Gamma(1-\tau+k)\,\Gamma(1+\tau+k)}\times$$

$$\sum_{m=0}^{\infty}\frac{\Gamma(\alpha_1+m)\,\Gamma(\alpha_2+m)}{m!\,\Gamma(\delta_2+m)}(-c)^m\,{}_3F_4\left(\begin{matrix}\alpha_1+m,\alpha_2+m,\frac{1}{2}+k;a^2b^2\\ 1+k,\delta_1\,1-\tau+k,1+\tau+k\end{matrix}\right)\times$$

$$+\frac{\pi\,\Gamma(\delta_1)\,\Gamma(\delta_2)}{2b^{2k}\sin(k\pi)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(1-k)\,\Gamma(1+\tau)\,\Gamma(1-\tau)\,\Gamma(\delta_1-k)}\times$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\alpha_1-k+m)\,\Gamma(\alpha_2-k+m)}{m!\,\Gamma(\delta_2+m)}(-c)^m\,{}_3F_4\left(\begin{matrix}\alpha_1-k+m,\alpha_2-k+m,\frac{1}{2};a^2b^2\\ 1-k,\delta_1-k,1+\tau,1-\tau\end{matrix}\right)$$

where $R(\alpha_1-k)>0$, $R(\alpha_2-k)>0$, $R(k)>-\frac{1}{2}$ and a, b, c are real and positive.

In (50) take $p = 1$, $q = 2$, apply (24) and get:

(56)
$$\int_0^\infty x^{-2k-1} J_\tau^2(ax) F_4\left(\alpha_1, \alpha_2; \delta_1, \delta_2; -\frac{b^2}{x^2}, -c\right) dx =$$

$$= \frac{-\sqrt{\pi}\, a^{2k} \Gamma(\tfrac{1}{2} + \tau + k) \Gamma(\delta_2)}{2 \sin(k\pi) \Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(1 + k) \Gamma(1 + \tau + k) \Gamma(1 + 2\tau + k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1 + m) \Gamma(\alpha_2 + m)}{m!\, \Gamma(\delta_2 + m)} (-c)^m {}_3F_4\begin{pmatrix}\alpha_1 + m, \alpha_2 + m, \tfrac{1}{2} + \tau + k; a^2 b^2 \\ 1 + k, \delta_1, 1 + \tau + k, 1 + 2\tau + k\end{pmatrix} +$$

$$+ \frac{\sqrt{\pi}\, \Gamma(\delta_1) \Gamma(\delta_2) \Gamma(\tfrac{1}{2} + \tau)}{2b^{2k} \sin(k\pi) \Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(1 - k) \Gamma(1 + \tau) \Gamma(1 + 2\tau) \Gamma(\delta_1 - k)} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1 - k + m) \Gamma(\alpha_2 - k + m)}{m!\, \Gamma(\delta_2 + m)} (-c)^m \times$$

$$\times {}_3F_4\begin{pmatrix}\alpha_1 - k + m, \alpha_2 - k + m, \tfrac{1}{2} + \tau; a^2 b^2 \\ 1 - k, \delta_1 - k, 1 + \tau, 1 + 2\tau\end{pmatrix}$$

where $R(\alpha_1 + \tau - k) > 0$, $R(k) > -\tfrac{1}{2}$ and a, b, c are real and positive.

Integral sinvolving $P_l^n(2ax + 1)$. In (1) take $p = 2$, $q = 1$, with $\phi_1 = -n$, $\phi_2 = n + 1$, $\theta_1 = 1 - l$, write $1/x$, $1/\alpha$ for x, α respectively, apply (3) and (13), so getting:

(57)
$$\int_0^\infty x^{l/2-k-1}(1 + ax)^{-l/2} P_1^n(2ax + 1) F_1\left(\alpha; \beta, \beta'; \gamma, -\frac{b}{x}, -c\right) dx =$$

$$= -\frac{\Gamma(\gamma)\, a^{k-(l/2)} \sin(n\pi)}{\sin(k\pi) \Gamma(\alpha) \Gamma(\beta) \Gamma(\beta')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta' + m)}{m!\, (-c)^{-m}} \times$$

$$\times \left[E\begin{pmatrix}\alpha_1 + m, \beta, -n + k, 1 + n + k; \frac{1}{ab} \\ 1 + k, 1 - l + k, \gamma + m\end{pmatrix} - \left(-\frac{1}{ab}\right)^k \times\right.$$

$$\left.\times E\begin{pmatrix}\alpha_1 - k + m, \beta - k, -n, 1 + n; \frac{1}{ab} \\ 1 - k, \gamma - k + m, 1 - l\end{pmatrix}\right]$$

where $R(k) > 0$, $R(\alpha_1 - k) > 0$, $R(\beta - k) > 0$, $R(k + n) < 0$, $R(k - 1 - n) < 0$ and a, b, c are real and positive.

Again in combination of (1) with (4) and (13) we get:

$$(58)\quad \int_0^\infty x^{l/2-k-1}(1+ax)^{-l/2} P_l^n(2ax+1)\, F_2\left(\alpha; \beta, \beta'; \delta, \delta'; -\frac{b}{x}, -c\right) \mathrm{d}x =$$

$$= -\frac{\Gamma(\delta)\,\Gamma(\delta')\sin n\pi}{\mathrm{a}^{l/2-k}\sin(k\pi)\,\Gamma(\alpha)} \times \sum_{m=0}^{\infty} \frac{\Gamma(\beta'+m)}{m!\,\Gamma(\delta'+m)} \times$$

$$\times (-c)^m \left[E\left(\begin{matrix}\alpha+m, \beta, -n+k, 1+n+k : \dfrac{1}{ab} \\ 1+k, \delta, 1-l+k\end{matrix}\right) - \left(-\frac{1}{ab}\right)^k \times \right.$$

$$\left. \times\, E\left(\begin{matrix}\alpha-k+m, \beta-k, -n, 1+n : \dfrac{1}{ab} \\ 1-k, \delta-k, 1-l\end{matrix}\right)\right]$$

where $R(\alpha - k) > 0$, $R(\beta - k) > 0$, $R(k - n) > 0$, $R(1 + n + k) > 0$ and a, b, c are real and positive.

(1) in combination with (15) and (13) gives

$$(59)\quad \int_0^\infty x^{l/2-k-1}(1+ax)^{-l/2} P_l^n(2ax+1)\, F_3\left(; \beta_1, \beta_1'; \beta_2, \beta_2'; \gamma; -\frac{b}{x}, -c\right) \mathrm{d}x =$$

$$= -\frac{\Gamma(\gamma)\sin(n\pi)}{a^{l/2-k}\sin(k\pi)\,\Gamma(\beta_1)\,\Gamma(\beta_1')\,\Gamma(\beta_2)\,\Gamma(\beta_2')} \sum_{m=0}^{\infty} \frac{\Gamma(\beta_1'+m)\,\Gamma(\beta_2'+m)}{m!\,(-c)^{-m}} \times$$

$$\times \left[E\left(\begin{matrix}\beta_1, \beta_2, -n+k, 1+n+k; \dfrac{1}{ab} \\ 1+k, \gamma+m, 1-l+k\end{matrix}\right) - \left(-\frac{1}{ab}\right)^k \times \right.$$

$$\left. \times\, E\left(\begin{matrix}\beta_1-k, \beta_2-k, -n, 1+n; \dfrac{1}{ab} \\ 1-k, \gamma-k+m, 1-l\end{matrix}\right)\right]$$

where $R(\beta_1 - k) > 0$, $R(\beta_2 - k) > 0$, $R(k - n) > 0$, $R(1 + n + k) > 0$ and a, b, c are real and positive.

Again (1) in combination with (6) and (13) gives:

$$(60)\quad \int_0^\infty x^{l/2-k-1}(1+ax)^{-l/2} P_1^n(1+2ax)\, F_4\left(\alpha_1, \alpha_2; \delta_1, \delta_2; -\frac{b}{x}, -c\right) \mathrm{d}x =$$

$$= -\frac{a^{k-l/2}\sin(n\pi)\,\Gamma(\delta_1)\,\Gamma(\delta_2)}{\sin(k\pi)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)} \sum_{m=0}^{\infty} \frac{(-c)^m}{m!\,\Gamma(\delta_2+m)} \times$$

$$\times\left[E\begin{pmatrix}\alpha_1+m,\alpha_2+m,-n+k,1+n+k:\dfrac{1}{ab}\\1+k,\delta_1,1-l+k\end{pmatrix}-\left(-\frac{1}{ab}\right)^k\times\right.$$
$$\left.E\begin{pmatrix}\alpha_1-k+m,\alpha_2-k+m,-n,1+n:\dfrac{1}{ab}\\1-k,\delta_1-k,1-l\end{pmatrix}\right]$$

where $R(\alpha_1 - k) > 0$, $R(\alpha_2 - k) > 0$, $R(k - n) > 0$, $R(1 + n + k) > 0$ and a, b, c are real and positive.

Finally; (1) in combination with (8) gives

$$\int_0^\infty x^{k-1}E(p;\phi_r:q;\theta_u:ax)\,F\begin{pmatrix}\beta_1,\ldots,\beta_\tau;-bx\\\delta_1,\ldots,\delta_\tau\end{pmatrix}dx= \tag{61}$$

$$=\frac{\prod_{j=1}^{\sigma}\Gamma(\delta_j)}{\prod_{j=1}^{\tau}\Gamma(\beta_j)}\frac{\pi}{a^k\sin(k\pi)}\left[-E\begin{pmatrix}\beta_1,\ldots,\beta_\tau,\phi_1+k,\ldots,\phi_p+k;e^{\pm i\pi}\dfrac{a}{b}\\1+k,\delta_1,\ldots,\delta_\sigma,\theta_1+k,\ldots,\theta_q+k\end{pmatrix}+\right.$$
$$\left.+\left(\frac{a}{b}\right)^kE\begin{pmatrix}\beta_1-k,\ldots,\beta_\tau-k,\phi_1,\ldots,\phi_p:e^{\pm i\pi}\dfrac{a}{b}\\1-k,\delta_1-k,\ldots,\delta_\sigma-k\,\theta_1,\ldots,\theta_q\end{pmatrix}\right]$$

where $R(k + \phi_r) > 0$ $(r = 1, 2, \ldots, p)$, $R(\beta_j - k) > 0$ $(j = 1, 2, \ldots, \tau)$ and a, b, c are real and positive.

This result was obtained by F. M. Ragab in [5]. And (1) in combination with (7) gives:

$$\int_0^\infty x^{k-1}E(p;\phi_r:q;\theta_u:ax)\,{}_\mu F_\varepsilon\begin{pmatrix}\alpha_1,\ldots,\alpha_\mu;-bx,-c\\\gamma_1,\ldots,\gamma_\varepsilon\end{pmatrix}dx= \tag{62}$$

$$=-\frac{\pi\prod_{j=1}^{\varepsilon}\Gamma(\gamma_j)}{a^k\sin(k\pi)\prod_{j=1}^{\mu}\Gamma(\alpha_j)}\sum_{m=0}^{\infty}\frac{(-1)^m}{m}\times$$
$$\times\left[E\begin{pmatrix}\alpha_1+m\ldots,\alpha_\mu+m,\phi_1+k,\ldots,\phi_p+k;e^{\pm i\pi}\dfrac{a}{b}\\1+k,\gamma_1+m,\ldots,\gamma_\varepsilon+m,\theta_1+k,\ldots,\theta_q+k\end{pmatrix}-\right.$$
$$\left.-\left(\frac{a}{b}\right)^kE\begin{pmatrix}\alpha_1-k+m,\ldots,\alpha_\mu-k+m,\phi_1,\ldots,\phi_p:e^{\pm i\pi}\dfrac{a}{b}\\1-k,\gamma_1-k-m,\ldots,\gamma_\varepsilon-k+m,\theta_1,\ldots,\theta_q\end{pmatrix}\right]$$

where $\mu \leqq \varepsilon$, $R(\phi_r + k) > 0$, $R(\alpha_j - k) > 0$, $(j = 1, 2, \ldots, \mu)$ and a, b, c are real and positive.

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