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FUNCTIONS CONTINUOUS IN THE FINE TOPOLOGY FOR THE HEAT EQUATION

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Introduction. One of the most important concepts of potential theory is the notion of superharmonic (or hyperharmonic) function. Superharmonic functions having continuous second partial derivatives date back to the nineteenth century, but lower semicontinuous superharmonic functions were first introduced by F. RIESZ in 1926. Nowadays, in some axiomatic systems of potential theory, hyperharmonic function becomes a primitive notion (compare [5]).

In view of the fact that hyperharmonic functions form the core of potential theory, for certain problems an intrinsically defined topology appears to be more suitable than the Euclidean topology or, in the general context of harmonic spaces, the original locally compact topology. The new topology, called the fine topology, can be defined as the smallest topology making continuous all hyperharmonic functions. The notion of the fine topology corresponding to the Laplace case and first important results on the subject are connected with the names of French mathematicians M. BRELOT and H. CARTAN (about 1945). The former also studied systematically the fine topology and the closely related concept of thinness in a more or less general setting in a long series of papers and lectures (see [4] where the corresponding bibliography may be found).

Although some special problems for the heat equation were treated by means of methods of potential theory, in the first half of the century there existed no unified potential theory including the case of both elliptic and parabolic equations. In 1954, J. L. DOOB introduced an axiomatic system of potential theory and remarked that the theory may be applied also to the heat equation drawing by this parabolical equations to potential theory. This axiomatic approach was extended, deepened and developed chiefly by H. BAUER and C. CONSTANTINESCU and A. CORNEA (see [2] and 5]) and all important results of potential theory were rediscovered in this more general frame. In particular, the notion of the fine topology τ for the heat equation in R^{n+1}, the Euclidean $(n + 1)$-space, has a good sense. Some properties of τ are known, for example, τ is completely regular and (R^{n+1}, τ) is a Baire space (see [4] or [7]). Of course, the topology τ essentially differs from the Euclidean topology. Let us notice that there is no infinite compact set in the fine topology and τ is neither normal nor paracompact. Moreover, it does not possess the Lindelöf's property and is totally disconnected (see [3]). One may ask how complicated, from the point of view of the topology of R^{n+1}, can be the nature of τ-continuous functions. It is the aim of this note to show that they cannot behave badly. More precisely, any τ-continuous function is of Baire class 1 in the Euclidean topology. From this fact it will be deduced that each τ-continuous function vanishing at the infinity can be expressed as a pointwise limit of heat potentials with special properties.

In the classical potential theory, that is in the Laplace case, it is known that a function continuous in the corresponding fine topology is approximately continuous and, consequently, of Baire class 1 (compare [7], p. 165).

Definitions and notation. In what follows, $n \geqq 1$ will be a fixed integer. For any open set U of R^{n+1} we shall denote by $\mathscr{H}(U)$ the collection of all functions h having continuous second partials on U and satisfying there

$$\sum_{j=1}^{n} \frac{\partial^2 h}{\partial x_j^2} = \frac{\partial h}{\partial x_{n+1}}.$$

Then $\mathscr{H}$ is a harmonic sheaf on R^{n+1} possessing the Doob convergence property and R^{n+1} endowed with $\mathscr{H}$ is a $\mathfrak{P}$-Bauer space (Theorem 3.3.1 in [5]). The functions belonging to $\mathscr{H}(U)$ and the hyperharmonic functions (with respect to $(R^{n+1}, \mathscr{H})$) will be called parabolic and hyperparabolic functions, respectively.

Recall the following definition (see [5], § 5.1). The fine topology on R^{n+1} is the coarsest topology on R^{n+1} in which any hyperparabolic functions on any open set of R^{n+1} is continuous. Concepts relative to the fine topology will be prefixed by "fine" or "finely".

The symbol λ will stand for the $(n + 1)$-dimensional Lebesgue measure in R^{n+1}. If $M \subset R^{n+1}$ and f is a function defined on M, we shall write

$$\int_M^* f \,\mathrm{d}\lambda$$

for the upper Lebesgue integral of f over M.

For $x = [x_1, \ldots, x_{n+1}] \in R^{n+1}$ and $\alpha > 0$ we shall denote

$$H_x = \{y \in R^{n+1};\ y_{n+1} \leqq x_{n+1}\},$$

$$K_x^\alpha = \{y \in R^{n+1};\ |y_i - x_i| < \tfrac{1}{2}\alpha,\ i = 1, \ldots, n,\ x_{n+1} - \alpha^2 < y_{n+1} < x_{n+1}\},$$

$$\Omega_x^\alpha = \{y \in R^{n+1};\ \sum_{i=1}^{n+1} (x_i - y_i)^2 \leqq \alpha^2\}.$$

We shall sometimes write $K(x, \alpha)$ and $\Omega(x, \alpha)$ instead of K_x^α and Ω_x^α, respectively.

For our purposes the following result from [6] (p. 224) will be useful. There is a non-negative function Q_0 continuous in H_0 such that the following assertion is true: If u is a hyperparabolic function defined in a neighbourhood U of the origin 0 and $\alpha > 0$ such that the closure of K_0^α is contained in U, then

$$u(0) \geqq \alpha^{-n-2} \int_{K(0,\alpha)} u \,.\, Q_0 \, \mathrm{d}\lambda \,. \tag{1}$$

Moreover, the sign of equality holds in (1), provided u is parabolic on U.

Given $x \in R^{n+1}$, put

$$Q_x(\xi) = Q_0(\xi - x)$$

for $\xi \in H_x$ while $Q_x(\xi) = 0$ elsewhere and set

$$v_x^*(M) = \int_M^* Q_x \, \mathrm{d}\lambda \,, \quad M \subset R^{n+1} \,.$$

In the case that M is measurable (λ) we shall write $v_x(M)$ in place of $v_x^*(M)$.

Taking $u = 1$ in (1) we get easily

$$v_x(K_x^\alpha) = \alpha^{n+2} \,, \quad x \in R^{n+1} \,. \tag{2}$$

Finally, we shall define the function G on R^{n+1} by

$$G(z) = z_{n+1}^{-n/2} \,.\, \exp\big(-\sum_{i=1}^{n} z_i^2/4z_{n+1}\big) \quad \text{for} \quad z_{n+1} > 0 \,,$$
$$G(z) = 0 \qquad\qquad \text{for} \quad z_{n+1} \leqq 0 \,.$$

For any finite signed Borel measure μ with compact support in R^{n+1} the function

$$x \mapsto \int G(x - y) \, \mathrm{d}\mu(y)$$

on R^{n+1} will be denoted by $G\mu$ and called the heat potential of μ.

Proposition. *Let $x \in R^{n+1}$ and V be a fine neighbourhood of x. Then*

$$\lim_{\alpha \to 0+} \frac{v_x^*(K_x^\alpha \setminus V)}{v_x(K_x^\alpha)} = 0 \,. \tag{3}$$

Proof. We shall assume that V is not an Euclidean neighbourhood of x, for otherwise (3) holds trivially. V being a fine neighbourhood of x there is a hyperparabolic function u defined on an open neighbourhood M of x such that

$$\infty > \beta = \liminf_{V \not\ni y \to x} u(y) - u(x) > 0$$

([5], Proposition 5.5.1). Fix an arbitrary $\varepsilon > 0$ and define the function v on M by

$$v = 1 + (\varepsilon\beta)^{-1} (u - u(x)).$$

It is clear that v is hyperparabolic on M, $v(x) = 1$ and

$$\liminf_{V \not\ni y \to x} v(y) > \varepsilon^{-1}.$$

Let $\alpha_0 > 0$ be chosen in such a way that

$$\inf \{v(y);\quad y \in K_x^{\alpha_0} \setminus V\} > \varepsilon^{-1}. \tag{4}$$

By (1), (2) and (4) we get for $0 < \alpha \leqq \alpha_0$

$$1 = v(x) = \alpha^{-n-2} \int_{K(x,\alpha)} v \,.\, Q_x \, d\lambda \geqq \alpha^{-n-2} \varepsilon^{-1} \int^*_{K(x,\alpha)\setminus V} Q_x \, d\lambda = \varepsilon^{-1} \,.\, \frac{v_x^*(K_x^\alpha \setminus V)}{v_x(K_x^\alpha)}$$

and the equality (3) follows.

Corollary 1. *Let U be an open subset of R^{n+1}. If x is a boundary point of U and*

$$\limsup_{\alpha \to 0+} \alpha^{-n-2} \int^*_{K(x,\alpha)\setminus U} Q_x \, d\lambda > 0,$$

then x is a regular point of U.

Proof. If the assertion were false, then $U \cup \{x\}$ would be a fine neighbourhood of x ([5], Theorem 6.3.3, Proposition 6.3.3), which would contradict the proposition.

Lemma. *Let $x \in R^{n+1}$, $M \subset R^{n+1}$, $\alpha > 0$, $a > 0$ and*

$$v_x^*(K_x^\alpha \setminus M) < a \,.\, v_x(K_x^\alpha).$$

Then there is a $\delta > 0$ such that the inequality

$$v_y^*(K_y^\alpha \setminus M) < a \,.\, v_y(K_y^\alpha) \tag{5}$$

holds for any $y \in \Omega_x^\delta$.

Proof. Observing that

$$v_x^*(K_x^\alpha \setminus M) < a \,.\, v_x(K_x^\alpha) = a \,.\, v_y(K_y^\alpha), \quad y \in R^{n+1},$$

we arrive at the following inequality:

$$v_y^*(K_y^\alpha \setminus M) < a \,.\, v_y(K_y^\alpha) + v_y(K_y^\alpha \setminus K_x^\alpha) + \\ + v_y^*(K_x^\alpha \setminus M) - v_x^*(K_x^\alpha \setminus M).$$

In view of the obvious fact that

$$\lim_{y \to x} [v_y(K_y^\alpha \setminus K_x^\alpha)] = 0\,,$$

it is sufficient to prove that

$$\lim_{y \to x} |v_y^*(K_x^\alpha \setminus M) - v_x^*(K_x^\alpha \setminus M)| = 0\,.$$

In order to do this, denote by C the λ-measurable hull of the set $K_x^\alpha \setminus M$. Then

$$v_y^*(K_x^\alpha \setminus M) = \int_C Q_0(\xi - y)\,\mathrm{d}\lambda(\xi)\,, \quad v_x^*(K_x^\alpha \setminus M) = \int_C Q_0(\xi - x)\,\mathrm{d}\lambda(\xi)$$

and the quantity in question admits the following estimate:

$$|v_y^*(K_x^\alpha \setminus M) - v_x^*(K_x^\alpha \setminus M)| \leqq \int_C |Q_0(\xi - y) - Q_0(\xi - x)|\,\mathrm{d}\lambda(\xi)\,.$$

The integral can be shown to approach zero as $y \to x$ (see e.g. [8], Theorem 13.24).

This concludes the proof of the lemma.

Theorem. *Any finely continuous function in R^{n+1} is of Baire class 1 in the Euclidean topology.*

Proof. The proof of the theorem is based on the characterization of function of Baire class 1 given in [9].

Let us assume that the function f is finely continuous in R^{n+1}. Suppose that there is a closed set $\emptyset \neq F \subset R^{n+1}$ and numbers $a < b$ such that the sets $A = \{x \in F;\ f(x) \leqq a\}$, $B = \{x \in F;\ f(x) \geqq b\}$ are dense in F. Fix $\varepsilon > 0$ in such a way that $a + \varepsilon < b - \varepsilon$ and for any positive integer k put $A_{2k-1} = A$, $A_{2k} = B$, $J_{2k-1} = (-\infty, a + \varepsilon)$, $J_{2k} = (b - \varepsilon, \infty)$.

We shall start with an arbitrary $x_1 \in A$. Recalling that $f^{-1}(J_1)$ is a fine neighbourhood of x_1 and applying the proposition, we obtain

$$v_{x_1}^*(K(x_1, \alpha_1) \setminus f^{-1}(J_1)) < \tfrac{1}{4} v_{x_1}(K(x_1, \alpha_1))$$

for a suitable $\alpha_1 \in (0, 1)$. The lemma (cf. (5)) guarantees the existence of $\delta_1 \in (0, 1)$ such that

$$v_y^*(K(y, \alpha_1) \setminus f^{-1}(J_1)) < \tfrac{1}{4} v_y(K(y, \alpha_1))$$

provided $y \in \Omega(x_1, \delta_1)$.

Further we shall proceed inductively. Put $x_0 = x_1$, $\alpha_0 = \delta_0 = 2$ and let $k \geqq 1$

be an integer. Let us suppose that the points x_i and the numbers α_i, δ_i $(i = 1, \ldots, k)$ have already been defined such that

$$x_i \in A_i \cap \Omega(x_{i-1}, \tfrac{1}{2}\delta_{i-1}),$$
$$0 < \delta_i < \tfrac{1}{2}\delta_{i-1}, \quad 0 < \alpha_i < \tfrac{1}{2}\alpha_{i-1},$$
$$v_y^*(K(y, \alpha_i) \setminus f^{-1}(J_i)) < \tfrac{1}{4} v_y(K(y, \alpha_i))$$

whenever $y \in \Omega(x_i, \delta_i)$. Since the set A_{k+1} is dense in F, there is $x_{k+1} \in A_{k+1} \cap \cap \Omega(x_k, \frac{1}{2}\delta_k)$. The set $f^{-1}(J_{k+1})$ is a fine neighbourhood of x_{k+1} and the proposition and the lemma yield the existence of $\alpha_{k+1} \in (0, \frac{1}{2}\alpha_k)$, $\delta_{k+1} \in (0, \frac{1}{2}\delta_k)$ such that

$$v_y^*(K(y, \alpha_{k+1}) \setminus f^{-1}(J_{k+1})) < \tfrac{1}{4} v_y(K(y, \alpha_{k+1}))$$

for an arbitrary $y \in \Omega(x_{k+1}, \delta_{k+1})$. In this manner we have defined the sequence of points $\{x_i\} \subset F$ and two sequences $\{\alpha_i\}$, $\{\delta_i\}$ of positive numbers. It is obvious that $\bigcap_{j=1}^{\infty} \Omega(x_j, \delta_j)$ contains exactly one point, say x. Of course, $x \in F$. Since x belongs to all $\Omega(x_j, \delta_j)$ we conclude that for any $k = 1, 2, \ldots$

$$v_x^*(K(x, \alpha_{2k+1}) \setminus f^{-1}((-\infty, a + \varepsilon))) < \tfrac{1}{4} v_x(K(x, \alpha_{2k+1})), \tag{6}$$

$$v_x^*(K(x, \alpha_{2k}) \setminus f^{-1}((b - \varepsilon, \infty))) < \tfrac{1}{4} v_x(K(x, \alpha_{2k})). \tag{7}$$

The last two inequalities are in contradiction to the assumption that f is finely continuous at x. Indeed, supposing $f(x) < b - \varepsilon$ and using (7) we derive for any k

$$\frac{3}{4} \leqq \frac{v_x^*(K(x, \alpha_{2k}) \setminus f^{-1}((-\infty, b - \varepsilon\rangle))}{v_x(K(x, \alpha_{2k}))},$$

which is impossible in view of the proposition, becauses $f^{-1}((-\infty, b - \varepsilon\rangle)$ is a fine neighbourhood of x and $\alpha_{2k} \to 0$. The same type of arguments together with (6) may be used for the case that $f(x) > a + \varepsilon$.

We have established the following assertion: For each closed set $F \subset R^{n+1}$ and any real numbers $a < b$ at most one of the sets $\{x \in F; f(x) \leqq a\}$, $\{x \in F; f(x) \geqq b\}$ is dense in F. It follows ([9], Theorem 1) that f is of Baire class 1 and the proof of the theorem is complete.

Remark. The theorem may be used to show directly that the fine topology is not normal. It is sufficient to repeat the reasonings from [7], p. 165.

Corollary 2. *Let f be a finely continuous function in R^{n+1} vanishing at the infinity (i.e. at the ideal point of the Alexandrov compactification of R^{n+1}). Then* *there exist finite Borel signed measures μ_m with compact support such that $G\mu_m$ is a continuous function having compact support and*

$$f(x) = \lim_{m \to \infty} G\mu_m(x) , \quad x \in R^{n+1} .$$

Proof. If g is a continuous function vanishing at the infinity, continuous potentials p_k, q_k (in the sense of the harmonic space $(R^{n+1}, \mathscr{H})$) may be found such that p_k, q_k are parabolic outside a compact set X_k, $p_k - q_k$ has a compact support and g is the uniform limit in R^{n+1} of the functions $p_k - q_k$ (see [2], Satz 2.7.4). The Riesz representation theorem for potentials ([1], Bemerkung 14) can be used to assert that $p_k - q_k = G\nu_k$ for a suitable signed measure with compact support in R^{n+1}. The rest of the proof is easy.

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