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THINNESS AND THE HEAT EQUATION

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Introduction. At the beginning of the century diverse methods for solving the classical Dirichlet problem for the Laplace equation and the heat equation were known. In all of these methods, however, some restrictions on the region in question were imposed. For a long time it was believed that limitations of generality, in the case of Laplace equation, are only caused by special approaches used in that period. It was pointed out by ZAREMBA (1910) and LEBESGUE (1912) that there are regions, for which the classical Dirichlet problem need not have a solution for all continuous boundary conditions, or, in our terminology, these regions are not regular. On the other hand, for the heat equation the existence of non-regular domains has belonged to the obvious facts. For example, considering a rectangle S in the plane (with sides parallel to the axes), the physical reasonings tell us not to prescribe the temperature on the top line of S. In the mathematical terms, the points of the upper part of the boundary are irregular for the Dirichlet problem.

Similarly to the Laplace case, given a bounded open set $U \subset R^{n+1}$ and a continuous function f on the topological boundary ∂U, the Perron's construction may be used to obtain the generalized solution H_f^U of the Dirichlet problem for the heat equation (see [12], [1] or [2]). This solution satisfies the equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial u}{\partial t} \tag{1}$$

and it remains only to investigate, whether

$$\lim_{y \to x} H_f^U(y) = f(x) \tag{2}$$

holds for a given point $x \in \partial U$. The point $x \in \partial U$ is said to be regular, if (2) holds for any continuous function f defined on ∂U. The problem is to find connections between the geometry of a region and the notion of regularity. For regions with a smooth boundary, $x \in \partial U$ is regular provided the outer normal at x is not in $+t$ direction, while the matter is rather delicate if the outward pointing normal at x is vertical upward. Although important and fine results on this go back to PETROVSKY [11] (1935; see also [7]), complete results were obtained in 1971 by EFFROS and KAZDAN [4] (see also [3]). They gave a necessary and sufficient condition for regularity of a point lying on the sufficiently smooth (C^3 is enough) boundary in nice differential geometric terms. The key to the proof of their main result is an analogue of the Laplace equation "cone condition" on the boundary points. The cone is replaced by a parabolic "tusk", the shape of which is suggested by the form of invariant transformations for the heat equation. In all above mentioned results all regular points are also stable for the Dirichlet problem. (Recall that an $x \in \partial U$ is stable if it is regular for a larger open set, that is, if there is an open set V with $\overline{U} \subset V \cup \{x\}$ such that x is a regular boundary point for V.) A criterion of regularity which applies also to non-stable points is due to HANSEN [5] (1971).

The Wiener's type solution for the parabolic equation is investigated by LANDIS [8], [9]. A necessary and sufficient condition of regularity analogous to the classical Wiener's test is proved in [8] (1969) by means of a suitable notion of capacity and some criteria of the geometric nature are introduced in [9].

This paper sets out to prove a geometric regularity criterion for the heat equation. This result enables one to establish in some cases the regularity of a boundary point, not necessarily stable, and represents a generalization of the Hansen's result and Effros-Kazdan's "tusk condition". In fact, our criterion of regularity is obtained as a consequence of an assertion concerning the parabolic thinness. A similar theorem (of density type in character) is known for the Laplace equation (see e.g. [6], Corollary 10.5).

Further information on the subject may be found in [3], [4].

Notation. It is a known fact that the harmonic measure (corresponding to the heat equation) on an $(n + 1)$-dimensional interval K has a density with respect to the area measure on the boundary of K. This density can be expressed in terms of the function Q_I to be defined below (see also § 3.3 in [2]).

We denote by Z the set of integers and $N = Z \cap (0, \infty)$. For $m \in N$, the symbol R^m will stand for the m-dimensional Euclidean space. We shall write R instead of R^1.

Suppose that $k \in N$ and set $I = \{1, 2, \ldots, k\}$. Let Γ_I be the function on $R^I \times R$ which is equal to

$$(x, t) \mapsto (4\pi t)^{-k/2} \,.\, \exp\Big(-\sum_{i=1}^{k} x_i^2/4t\Big)$$

on $\{(x, t) \in R^I \times R;\, t > 0\}$ and zero elsewhere. For any $J \subset I$ and any $y \in R^I$ we denote by $|J|$ the number of elements of J and by y^J the point of R^I such that $y_i^J = y_i$ for $i \in I \setminus J$ and $y_i^J = -y_i$ for $i \in J$. We denote further for any $J \subset I$ and any $l \in Z^I$ by $\Gamma_l^{J,I}$ the function on $R^I \times R \times R^I \times R$ defined by

$$(x, t, y, s) \mapsto \Gamma_I(x - y^J + 2l, t - s)$$

and set

$$Q_I = \sum_{l\in Z^I} \sum_{J\subset I} (-1)^{|J|} \Gamma_l^{J,I} .$$

Note that the series defining Q_I is convergent and that Q_I is of class C^∞ outside the set $\{(x, t, y, s); t = s\}$ (see [2], p. 85).

The point of R^I each of which coordinates equals $\frac{1}{2}$ will be denoted by a^I.

The following lemma will be useful below.

Lemma. *If* $q \in R - \{0\}$, $n \in N$, $I = \{1, \ldots, n\}$, *then*

(3) $$Q_I(a^I, q^2, z, 0) > 0$$

for any $z \in (0, 1)^I$.

Proof. The proof is by induction on $|I|$. Suppose first that $|I| = 1$. By definition,

$$Q_{\{1\}}(a^{\{1\}}, q^2, z, 0) = \sum_{l\in Z} (\Gamma_{\{1\}}(\tfrac{1}{2} - z + 2l, q^2) - \Gamma_{\{1\}}(\tfrac{1}{2} + z + 2l, q^2)) =$$

$$= \frac{1}{2\sqrt{(\pi q^2)}} \sum_{l\in Z} [\exp(-(\tfrac{1}{4} - \tfrac{1}{2}z + l)^2/q^2) - \exp(-(\tfrac{1}{4} + \tfrac{1}{2}z + l)^2/q^2)] .$$

Putting

$$r = \exp(-\pi^2 q^2) , \quad r_0 = \prod_{s=1}^{\infty} (1 - r^{2s})$$

and using formulas for ϑ_3-function (see [10], pp. 140, 141), the last expression containing the sum can be transformed into

$$\tfrac{1}{2} r_0 [\prod_{s=1}^{\infty} (1 + 2r^{2s-1} \sin \pi z + r^{4s-2}) - \prod_{s=1}^{\infty} (1 - 2r^{2s-1} \sin \pi z + r^{4s-2})] .$$

If $z \in (0, 1)$, then the difference of these products is obviously strictly positive. Consequently,

(4) $$Q_{\{1\}}(a^{\{1\}}, q^2, z, 0) > 0 .$$

Let us assume that $j > 1$, $K = \{1, \ldots, j - 1\}$ and

(5) $$Q_K(a^K, q^2, \xi, 0) > 0$$

provided $\xi \in (0, 1)^K$. Put $L = \{1, \ldots, j\}$ and for each $x = [x_1, \ldots, x_j] \in R^L$ denote $\hat{x} = [x_1, \ldots, x_{j-1}] \in R^K$. Note that $x \in (0, 1)^L$ implies $\hat{x} \in (0, 1)^K$ and $\widehat{a^L} = a^K$. By definition of Q_L we have for any $y \in (0, 1)^L$

$$Q_L(a^L, q^2, y, 0) = \sum_{k\in Z^L} \sum_{J\subset L} (-1)^{|J|} \Gamma_k^{J,L}(a^L, q^2, y, 0) =$$

$$= \sum_{l\in Z^K} \sum_{J\subset K} (-1)^{|J|} \Gamma_l^{J,K}(a^K, q^2, \hat{y}, 0) . [\sum_{m\in Z} (\Gamma_{\{1\}}(\tfrac{1}{2} - y_j + 2m , q^2) -$$

$$- \Gamma_{\{1\}}(\tfrac{1}{2} + y_j + 2m, q^2))] = Q_K(a^K, q^2, \hat{y}, 0) . Q_{\{1\}}(a^{\{1\}}, q^2, y_j, 0) .$$

The last product is strictly positive by (5) and (4).

The proof is complete.

Definitions and notation. In what follows, n will be a fixed element of N and $I = \{1, \ldots, n\}$. For $q > 0$ we shall denote by Q_q the function defined on $(-1/2q, 1/2q)^n$ by

$$y \mapsto q^n \cdot Q_I(a^I, q^2, qy + \tfrac{1}{2}, 0)$$

and we put

$$\omega_q = \inf \{Q_q(y)\,; \quad y \in \langle -1/4q, 1/4q \rangle^n\} . \tag{6}$$

It follows immediately from the lemma that

$$\omega_q > 0 .$$

For $x = [y, s] \in R^n \times R$ and $\beta \in R$ set

$$H_\beta(x) = \{[z, t] \in R^n \times R\,; \quad t = s - \beta\} .$$

The n-dimensional outer Hausdorff measure in R^{n+1} will be denoted by λ^* and the symbol λ will stand for the corresponding measure. Hence for $M \subset R^n$ the outer n-dimensional Lebesgue measure λ_n^* of M coincides with $\lambda^*(M \times \{0\})$.

Suppose that $M \subset R^{n+1}$ and $x \in R^{n+1}$. We say that M lies parabolically below x provided there is $b > 0$ such that

$$y_{n+1} - x_{n+1} < -b \sum_{i=1}^{n} (y_i - x_i)^2$$

for any $y \in M$.

Before stating our theorem a few words on the Dirichlet problem for the heat equation will be useful. It is well known that the Dirichlet problem for the heat equation (1) in R^{n+1} can be investigated in the frame of the axiomatic theory of harmonic spaces (see [1], Standard-Beispiel (2) or [2], § 3.3). Recall that the set $M \subset R^{n+1}$ is said parabolically thin at $x \in R^{n+1}$ if either $x \notin \overline{M}$ (= the closure of M) or $x \in \overline{M}$ and there is a hyperparabolic function u defined on a neighborhood of x such that

$$u(x) < \liminf_{\substack{y \to x \\ y \in M \setminus \{x\}}} u(y) .$$

(See [1], III. § 1, § 3, Satz 3.3.3 and Satz 5.3.1 and Beispiel 2 in II, § 8, or [2], § 6.3, Corollary 6.3.2, Proposition 6.3.3 and Proposition 5.1.1.) (Hyperparabolic means here, of course, hyperharmonic in the corresponding harmonic space.) There is a close connection between the notion of thinness and regularity, namely a boundary point x of an open bounded subset U of R^{n+1} is regular for the heat equation if and only if $R^{n+1} \setminus U$ is not parabolically thin at x ([1], Satz 4.3.1 or [2], Theorem 6.3.3).

Theorem. *Suppose that $x \in R^{n+1}$ and $M \subset R^{n+1}$. If M lies parabolically below x and M is parabolically thin at x, then*

$$\lim_{\alpha \to 0+} \alpha^{-n} \lambda^*(M \cap H_{\alpha^2}(x)) = 0 . \tag{7}$$

Proof. We may assume that $x = 0$ and $0 \in \overline{M}$, the case $0 \notin \overline{M}$ being trivial. For the sake of brevity we shall write H_β instead of $H_\beta(0)$. According to the hypothesis of the theorem there is $q > 0$ such that

$$y \in M \Rightarrow y_{n+1} < -16q^2 \sum_{j=1}^{n} y_i^2 . \tag{8}$$

For $\alpha > 0$ put $F_\alpha = (-\alpha/2q, \alpha/2q)^n$,

$$U_\alpha = \{[y, s] \in R^n \times R ; \quad y \in F_\alpha , \quad s \in (-\alpha^2, \alpha^2)\}$$

and denote by μ_α the harmonic measure on U_α at 0 ([2], p. 19). Choose $\varepsilon > 0$ and a hyperparabolic function u defined on a neighborhood of 0 such that

$$+\infty > c = \liminf_{\substack{y \to 0 \\ y \in M \setminus \{0\}}} u(y) - u(0) > 0 .$$

Putting (for definition of ω_q see (6))

$$v = 1 + (c \cdot \varepsilon \cdot \omega_q)^{-1} [u - u(0)] ,$$

we obtain the hyperparabolic function such that $v(0) = 1$ and

$$\liminf_{\substack{y \to 0 \\ y \in M \setminus \{0\}}} v(y) > (\varepsilon \cdot \omega_q)^{-1} .$$

Fix now $\alpha_0 > 0$ in such a way that

$$v(z) > (\varepsilon \cdot \omega_q)^{-1} \tag{9}$$

for any $z \in M \cap \overline{U}_{\alpha_0}$, $z \neq 0$.

Let us consider $\alpha \in (0, \alpha_0)$. Since v is positive and hyperparabolic on a neighborhood of U_α we arrive at

$$1 = v(0) \geqq \int v \, d\mu_\alpha \geqq \int_{U_\alpha \cap H_{\alpha^2}} v \, d\mu_\alpha = \int_{F_\alpha} v([x, -\alpha^2]) \cdot \alpha^{-n} Q_q(x/\alpha) \, dx . \tag{10}$$

The last equality (with the n-dimensional Lebesgue integral on the right-hand side) follows from the definition of Q_q and from results of § 3.3. in [2] (in particular lemma 3). Denote by M_α the set of all $y = [y_1, \ldots, y_n] \in R^n$ such that $[y, -\alpha^2] \in M$. If $y \in M_\alpha$, then (8) yields

$$\sum_{i=1}^{n} \left(\frac{y_i}{\alpha}\right)^2 < \frac{1}{16q^2} ,$$

so that $y \in F_\alpha$ and $y/\alpha \in \langle -1/4q, 1/4q \rangle^n$. Consequently, $Q_q(y/\alpha) \geqq \omega_q$ (cf. (6)). This together with (10) and (9) implies

$$1 \geqq \alpha^{-n} \int_{M_\alpha}^{*} v([x, -\alpha^2)]) \cdot Q_q(x/\alpha) \, dx \geqq \varepsilon^{-1} \alpha^{-n} \lambda^*(M \cap H_{\alpha^2})$$

(the asterisk is used to denote the upper Lebesgue integral). We see that

$$\alpha^{-n} \lambda^*(M \cap H_{\alpha^2}) < \varepsilon$$

whenever $\alpha \in (0, \alpha_0)$ and (7) is established.

The proof of the theorem is complete.

Corollary 1. *Let $U \subset R^{n+1}$ be an open bounded set, $U' = R^{n+1} \setminus U$ and x be a boundary point of U. If there is a neighborhood V of x such that $U' \cap V$ lies parabolically below x and*

$$\limsup_{\alpha \to 0+} \alpha^{-n} \lambda(U' \cap V \cap H_{\alpha^2}(x)) > 0$$

holds, then x is regular for U.

Corollary 2. *Let $B_0 \subset R^n$, $\emptyset \neq T \subset (0, \infty)$, $\inf T = 0$ and*

$$B = \{[\alpha x, -\alpha^2] \in R^n \times R \, ; \quad x \in B_0 \, , \quad \alpha \in T\} \, .$$

If B is parabolically thin at 0, then $\lambda_n(B_0) = 0$.

Proof. Suppose that B is parabolically thin at 0. For $k \in N$ we shall denote by B_0^k the intersection of B_0 and the n-dimensional ball with the center 0 and radius k and

$$B^k = \{[\alpha x, -\alpha^2] \in R^n \times R \, ; \quad x \in B_0^k \, , \quad \alpha \in T\} \, .$$

Then B^k is parabolically thin at 0 and B^k lies parabolically below 0. By the theorem,

$$\lim_{\alpha \to 0+} \alpha^{-n} \lambda^*(B^k \cap H_{\alpha^2}(0)) = 0 \, . \tag{11}$$

Since $\lambda^*(B^k \cap H_{\alpha^2}(0)) = \alpha^n \cdot \lambda_n(B_0^k)$ provided $\alpha \in T$, (11) implies $\lambda_n^*(B_0^k) = 0$. Consequently, $\lambda_n(B_0) = 0$.

Remarks. In particular, taking in Corollary 2 B_0 a ball and $T = (0, \infty)$, we obtain the "tusk condition" of Effros-Kazdan ([4], lemma 3).

For the case that the set T in corollary 2 is countable and shrinkable to 0 in the sense that $\alpha T \subset T$ for arbitrarily small $\alpha > 0$, the assertion of the last corollary is included in Satz 4.3 of [5].

Note that not every countable $T \subset (0, \infty)$ with $\inf T = 0$ is shrinkable to 0.

Choosing e.g. $T = \{2^{-n^2}; n \in N \cap \{0\}\}$ we see at once that $\alpha T \subset T$ if and only if $\alpha = 1$. Indeed, $\alpha < 1$ would imply $\alpha = 2^{-m^2}$ with a suitable $m \in N$ and $\alpha T \subset T$ would yield $2^{-2m^2} \in T$. Consequently, $2m^2$ is a square, which is impossible.

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