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A BAIRE FUNCTION NOT COUNTABLY DECOMPOSABLE INTO CONTINUOUS FUNCTIONS

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In connection with a problem of KARTÁK [1], VRKOČ recently constructed [2] a measurable real function f on $I = [0, 1]$ such that I cannot be partitioned into countably many sets A_n with each restriction $f \mid A_n$ continuous. He asked whether for every Baire function there does exist such a partition of I into Borel sets. Here it will be shown that, on the contrary, there exists a function of Baire class 1 for which there exists no such partition whatever, even into non-Borel sets.

Theorem 1. *If $f: A \to I$ is continuous, where A is a subset of I, then given $\varepsilon > 0$ there exists a closed set $F \subseteq I \times I$ such that $F \cap \operatorname{Gr}(f) = \emptyset$ and $m(F_x) \geqq 1 - \varepsilon$ for all $x \in I$.*

Proof. For each element $u \in A$, let $J_u = \{(u, y) : |y - f(u)| < \frac{1}{2}\varepsilon\}$ and $K_u = (\{u\} \times I) \setminus J_u$. Denote by E the closure of the set $D = \cup\{K_u : u \in A\}$. First, we observe that $E \cap \operatorname{Gr}(f) = \emptyset$. Indeed, given any point $(u, f(u)) \in \operatorname{Gr}(f)$, we can choose $\delta > 0$ so small that

$$v \in A \,\&\, |v - u| < \delta \Rightarrow |f(v) - f(u)| < \varepsilon/4 ;$$

then the open rectangle with centre at $(u, f(u))$, width 2δ, and height $\frac{1}{2}\varepsilon$ contains no point of D.

Next, we prove that *for every $x \in \bar{A}$, the set $I \setminus E_x$ is an interval of length at most ε, open relative to I.* Since E_x is closed, it is enough to show that if $y_1, y_2 \in I \setminus E_x$ and $y_1 < y < y_2$ then (i) $y_2 - y_1 < \varepsilon$ and (ii) $y \in I \setminus E_x$. Consider any $u \in A$ with $|u - x| < \delta$, where δ is the smaller of the distances of (x, y_1) and (x, y_2) from E; then $(u, y_1) \in J_u$ and $(u, y_2) \in J_u$, and (i) follows. Moreover $|y - f(u)| < \frac{1}{2}\varepsilon - \min(y_2 - y, y - y_1)$; hence the open rectangle with centre (u, y), width 2δ, and height $2 \min(y_2 - y, y - y_1)$ contains no point of D, and this establishes (ii).

To construct F we adjoin to E a large part of each strip $S = (c, d) \times I$, where (c, d) is an interval of $I \setminus \bar{A}$; namely, the whole of S except for an open "corridor" (with rectilinear edges) joining the open vertical intervals $G = \{c\} \times (I \setminus E_c)$ and $H = \{d\} \times (I \setminus E_d)$. (If $G = H = \emptyset$ we include the whole of S in F; while if, for example, $H = \emptyset$ but $G \neq \emptyset$ then we include the whole of S except for the open triangle joining G to the point (d, z), where (c, z) is the mid-point of G.) It is easy to verify that the resulting set F is closed, and it clearly has the other required properties.

Theorem 2. *Let $f : I \to I$ be such that there is a partition $I = \bigcup_{n=1}^{\infty} A_n$ with each restriction $f \mid A_n$ continuous. Then given $\varepsilon > 0$ there exists a closed set $F \subseteq I \times I$ such that $F \cap \operatorname{Gr}(f) = \emptyset$ and $m(F_x) \geqq 1 - \varepsilon$ for all $x \in I$.*

Proof. Let $\sum \varepsilon_n$ be a convergent series of positive terms with sum less than ε. By Theorem 1 there exists for each n a closed set $F_n \subseteq I \times I$ such that $F_n \cap \operatorname{Gr}(f \mid A_n) = \emptyset$ and $m[(F_n)_x] \geqq 1 - \varepsilon_n$ for all $x \in I$. The set $F = \cap F_n$ has the required properties.

Theorem 3. *There exists a function $f : I \times I$ of Baire class 1 such that I cannot be partitioned into countably many sets A_n with each restriction $f \mid A_n$ continuous.*

Proof. In view of Theorem 2, it is sufficient for f to have the property that $F \cap \operatorname{Gr}(f) \neq \emptyset$ for every closed set $F \subseteq I \times I$ which satisfies $F_x \neq \emptyset$ for all $x \in I$. It is known [3] that there exists a function with G_δ graph having the stated property; this is not quite enough, but the example constructed explicitly in [4] is lower semicontinuous and therefore in the first Baire class.

Note added 13 January 1973. In a paper by L. KELDYSH (Sur les fonctions premières mesurables B, Dokl. Akd. Nauk SSSR (N.S.) 5 (1934), 192–197) it was shown that for every α there exists a function $f : I \to I$ of Baire class α, such that I cannot be partitioned into countably many sets A_n with each restriction f/A_n of class less than α, thereby answering a question of P. S. NOVIKOV, who had already proved the result stated above as Theorem 3.

References

[1] *K. Karták:* Problém 2, Čas. pěst. mat. *91* (1966), p. 104.

[2] *I. Vrkoč:* Remark about the relation between measurable and continuous functions. Čas. pěst. mat. *96* (1971), 225–228.

[3] *E. Michael:* G_δ sections and compact-covering maps. Duke Math. J. *36* (1969), 125–127.

[4] *Roy O. Davies:* A non-Prokhorov space. Bull. London Math. Soc. *3* (1971), 341–342.

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