Štefan Schwabik

On a modified sum integral of Stieltjes type

Časopis pro pěstování matematiky, Vol. 98 (1973), No. 3, 274--277

Persistent URL: http://dml.cz/dmlcz/117810

Časopis pro pěstování matematiky, roč. 98 (1973), Praha

ON A MODIFIED SUM INTEGRAL OF STIELTJES TYPE

ŠTEFAN SCHWABIK, Praha

(Received January 27, 1972)

Let $[a, b]$ be a bounded interval on the real line, $-\infty < a < b < +\infty$. Given a positive function $\delta : [a, b] \to (0, +\infty)$, we consider finite sequences of numbers $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$ such that

(1) $$a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b\,,$$

(2) $$\alpha_{j-1} \leqq \tau_j \leqq \alpha_j\,, \quad j = 1, 2, \ldots, k\,,$$

(3) $$|\alpha_j - \tau_j| \leqq \delta(\tau_j)\,, \quad |\alpha_{j-1} - \tau_j| \leqq \delta(\tau_j)\,, \quad j = 1, 2, \ldots, k\,.$$

The set of all subdivisions A of $[a, b]$ satisfying (1), (2) and (3) with a given $\delta : [a, b] \to (0, +\infty)$ we denote by $\mathscr{A}(\delta)$.

Further, replacing (2) by the condition

(2*) $$\alpha_0 \leqq \tau_1 < \alpha_1\,, \quad \alpha_{j-1} < \tau_j < \alpha_j\,, \quad j = 2, 3, \ldots, k-1\,, \quad \alpha_{k-1} < \tau_k \leqq \alpha_k$$

we denote the set of all A satisfying (1), (2*) and (3) with a given $\delta : [a, b] \to (0, +\infty)$ by $\mathscr{A}^*(\delta)$.

In [2] it was proved that $\mathscr{A}(\delta) \neq \emptyset$ for any $\delta : [a, b] \to (0, +\infty)$ (cf. Lemma 1,1,1 in [2]). The proof is based on choosing a finite covering of $[a, b]$ by intervals of the form $(\tau - \delta(\tau), \tau + \delta(\tau))$ where $\tau \in [a, b]$. By the same argument we can prove that $\mathscr{A}^*(\delta) \neq \emptyset$ for any $\delta : [a, b] \to (0, +\infty)$.

Definition 1. *The function $f : [a, b] \to R$ is K-integrable (K^*-integrable) on $[a, b]$ with respect to $g : [a, b] \to R$ if there exists a number I such that to every $\varepsilon > 0$ there is such a $\delta : [a, b] \to (0, +\infty)$ that*

$$|K(A) - I| < \varepsilon$$

provided $A \in \mathscr{A}(\delta)$ ($A \in \mathscr{A}^*(\delta)$) where

$$K(A) = \sum_{j=1}^{k} f(\tau_j)\,(g(\alpha_j) - g(\alpha_{j-1}))$$

for $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$.

The number I *(if it exists) will be denoted by* $K \int_a^b f \, dg$ $(K^* \int_a^b f \, dg)$ *and will be called the Kurzweil integral (the modified Kurzweil integral) of* f *with respect to* g *on* $[a, b]$.

Remark. The concept of the K-integral was introduced and studied for the first time by J. Kurzweil in [2], it is used in [2] and in a number of other papers to study ordinary differential equations.

In [2] and [4] it is shown that if g is a function of bounded variation on $[a, b]$, i.e. $g \in BV(a, b)$, then the usual Perron-Stieltjes integral $P.S. \int_a^b f \, dg$ (cf. [3]) is equivalent to the integral $K \int_a^b f \, dg$.

In [4] we studied further the relation between $K \int_a^b f \, dg$ and the Young σ-integral $Y \int_b^a f \, dg$ for $g \in BV(a, b)$ (for the Young integral see also [1]). In this direction we have obtained that for $g \in BV(a, b)$ the existence of $Y \int_a^b f \, dg$ does not in general imply the existence of $K \int_a^b f \, dg$ (cf. Sec 3 in [4]). In this note we prove that the modified Kurzweil integral includes the Young σ-integral, i.e. the following theorem holds:

Theorem 1. *Let* $f : [a, b] \to R$, *let* $g : [a, b] \to R$ *be of bounded variation on* $[a, b]$ $(g \in BV(a, b))$. *Then if the Young* σ*-integral* $Y \int_a^b f \, dg$ *exists then also the modified Kurzweil integral* $K^* \int_a^b f \, dg$ *exists and both integrals are equal.*

Proposition 1. *If* $f : [a, b] \to R$, $g \in BV(a, b)$ *and* $K \int_a^b f \, dg$ *exists then* $K^* \int_a^b f \, dg$ *exists and both integrals are equal.*

Proof. It is easy to see that if $A \in \mathscr{A}^*(\delta)$ for some $\delta : [a, b] \to (0, +\infty)$ then also $A \in \mathscr{A}(\delta)$ and the proposition is an easy consequence of Def. 1.

Proposition 2. *If* $f : [a, b] \to R$, $g \in BV(a, b)$ *such that* $g(a) = g(t+) = g(t-) = g(b)$ *for all* $t \in (a, b)$ *then* $K^* \int_a^b f \, dg$ *exists and equals zero.*

Proof. Without any loss of generality we can suppose that $g(a) = 0$. Indeed our proposition evidently holds for $g(t) = \text{const.}$ by definition and therefore the additivity of the integral yields that in the case $g(a) \neq 0$ it is sufficient to consider the function $\tilde{g}(t) = g(t) - g(a)$ for which we have $\tilde{g}(a) = 0$.

Since g is a function of bounded variation there exists a countable set $N = \{t_1, \ldots, t_m, \ldots\} \subset (a, b)$ such that $g(t) = 0$ for $t \in [a, b] - N$ and $g(t) \neq 0$ for $t \in N$. Moreover, we have $\operatorname{var}_a^b g = 2 \sum_{t \in N} |g(t)| < +\infty$. Given now an arbitrary $\varepsilon > 0$, we define for f, g and ε a function $\delta : [a, b] \to (0, +\infty)$ in the following way:

If $\tau \in N$, i.e. $\tau = t_m$ for some $m = 1, 2, \ldots$, then there is a $\delta(\tau) > 0$ such that

$$|g(t)| < \varepsilon \,.\, 2^{-m-1}[|f(\tau)| + 1]^{-1}$$

for $0 < |t - \tau| < \delta(\tau)$. This is a consequence of the existence of limits $g(\tau-)$, $g(\tau+)$ for all $\tau \in (a, b)$ and our assumption $g(\tau+) = g(\tau-) = 0$ for all $\tau \in (a, b)$. For $\tau \in N$ let $\delta(\tau)$ be the positive number given above.

If $\tau \in [a, b] - N$ then we define the set

$$H_l = \{t \in [a, b] - N ; \quad l \leqq |f(t)| < l + 1\}$$

for all $l = 0, 1, 2, \ldots$ Evidently $\bigcup_{l=0}^{\infty} H_l = [a, b] - N$ and $H_l \cap H_m = \emptyset$ for $l \neq m$.

Further we determine for all $l = 0, 1, \ldots$ a set $N_l \subset N$ such that

$$\sum_{t \in N - N_l} 2|g(t)| < \varepsilon(l + 1)^{-1} . 2^{-l} .$$

This is obviously possible since the series $\sum_{t \in N} |g(t)|$ converges. If $\tau \in [a, b] - N$ then there exists a uniquely determined integer $l \geqq 0$ such that $\tau \in H_l$ and we define

$$\delta(\tau) = \tfrac{1}{2}\varrho(\tau, N_l) > 0$$

where ϱ is the Euclidean distance on the real line. This $\delta(\tau)$ is positive since $\tau \notin N_l$. By definition we have $[\tau - \delta(\tau), \tau + \delta(\tau)] \cap N_l = \emptyset$ for all $\tau \in H_l$.

Now let $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_k, \alpha_k\}$ be arbitrary and let us consider the corresponding sum $K(A)$. We have

$$|K(A)| = |\sum_{j=1}^{k} f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1}))| \leqq \sum_{j=1}^{k} |f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1}))| .$$

If $\tau_j \in N$, i.e., $\tau_j = t_m$ for some $m = 1, 2, \ldots$ then

$$|f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1}))| \leqq |f(t_m)| (|g(\alpha_j)| + |g(\alpha_{j-1})|) \leqq$$
$$\leqq |f(t_m)| . 2\varepsilon(|f(t_m)| + 1)^{-1} . 2^{-m-1} < \varepsilon/2^m ,$$

since $A \in \mathscr{A}^*(\delta)$ implies $0 < |\alpha_j - t_m| < \delta(t_m)$ and $0 < |\alpha_{j-1} - t_m| < \delta(t_m)$. If $\tau_j \notin N$ then there is an integer $l \geqq 0$ such that $\tau_j \in H_l$ and we have $|f(\tau_j)| \leqq l + 1$. Hence

$$|f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1}))| \leqq (l + 1)|g(\alpha_j) - g(\alpha_{j-1})| \leqq (l + 1) \operatorname{var}_{\alpha_{j-1}}^{\alpha_j} g$$

and for the sum $S_l = \sum_{\tau_j \in H_l} |f(\tau_j)(g(\alpha_j) - g(\alpha_{j-1}))|$ of all absolute values of summands in $K(A)$ with $\tau_j \in H_l$ we can give the estimate

$$S_l \leqq (l + 1) \sum_{\tau_j \in H_l} \operatorname{var}_{\alpha_{j-1}}^{\alpha_j} g \leqq (l + 1) \sum_{t \in N \cap M_l} 2|g(t)|$$

where $M_l = \bigcup_{\tau_j \in H_l} [\alpha_{j-1}, \alpha_j]$. Let us mention that $M_l \cap N_l = \emptyset$ since $M_l \subset \bigcup_{\tau_j \in H_l} [\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)]$ and $[\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)] \cap N_l = \emptyset$ for any $\tau_j \in H_l$. Hence $N \cap M_l \subset N - N_l$ and we have

$$S_l \leqq (l + 1) \sum_{t \in N - N_l} 2|g(t)| < (l + 1)\, \varepsilon \,.\, (l + 1)^{-1} \,.\, 2^{-l} = \varepsilon \,.\, 2^{-l}$$

Therefore we have

$$|K(A)| < \varepsilon\big(\sum_{m=1}^{\infty} 2^{-m} + \sum_{l=0}^{\infty} 2^{-l}\big) = 3\varepsilon$$

and the proposition follows immediately from Def. 1.

Proof of Theorem 1. Let us define the set

$$N_S = \{t \in (a, b);\, g(t+) = g(t-),\, g(t) \neq g(t-)\}$$

and the function $g_S(t) = 0$, $t \in [a, b] - N_S$, $g_S(t) = g(t)$ for $t \in N_S$. We put $g_R = g - g_S$.

Since $Y \int_a^b f \, dg$ exists by assumption and the existence of $Y \int_a^b f \, dg_S$ and also the equality $Y \int_a^b f \, dg_S = 0$ follows from Proposition 1,1 in [4] the integral $Y \int_a^b f \, dg_R$ exists. Using Theorem 3,1 form [4] we obtain that $K \int_a^b f \, dg_R$ exists and Proposition 1 yields the existence of $K^* \int_a^b f \, dg_R$ and the equality $K^* \int_a^b f \, dg_R = K \int_a^b f \, dg_R = Y \int_a^b f \, dg_R$. By Prop. 2 we obtain the existence of $K^* \int_a^b f \, dg_S$ and $K^* \int_a^b f \, dg_S = 0$. Thus the integral $K^* \int_a^b f \, dg$ exists and

$$K^* \int_a^b f \, dg = K^* \int_a^b f \, dg_S + K^* \int_a^b f \, dg_R = Y \int_a^b f \, dg_R = Y \int_a^b f \, dg \,.$$

References

[1] *Hildebrandt T. H.:* Introduction to the Theory of Integration. Academic Press, New York, London, 1963.

[2] *Kurzweil J.:* Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter. Czech. Math. J. *7* (*82*), (1957), 418–449.

[3] *Saks S.:* Theory of the Integral. Monografie Matematyczne, Warszawa, Lwow 1937.

[4] *Schwabik Š.:* On the relation between Young's and Kurzweil's concept of Stieltjes integral, Čas. pěst. mat. *98* (1973), 237–251.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).