Časopis pro pěstování matematiky

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Alternating connectivity of tournaments

Časopis pro pěstování matematiky, Vol. 96 (1971), No. 4, 346--352

Persistent URL: http://dml.cz/dmlcz/117732

Časopis pro pěstování matematiky, roč. 96 (1971), Praha

ALTERNATING CONNECTIVITY OF TOURNAMENTS

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(Received February 2, 1970)

This paper continues to investigate the concepts introduced in [2] in the case of tournaments. A tournament is a digraph in which any two different vertices u, v are joined exactly by one directed edge (either $\overrightarrow{uv}$, or $\overrightarrow{vu}$) and no loops exist. The concepts of $(+-)$-path, $(-+)$-path, $(+-)$-connectivity, $(-+)$-connectivity and alternating connectivity were defined in [2].

Theorem 1. *Let a tournament T with the vertex set V have a source u and no sink. Then T is $(+-)$-connected, but not $(-+)$-connected. The equivalence classes of the relation of being $(-+)$-connected are $\{u\}$ and $V \dot{-} \{u\}$.*

Remark. A tournament can have at most one source and at most one sink.

Proof. Let v, w be two vertices of T. As T has no sink, there exist vertices v', w' so that $\overrightarrow{vv'}, \overrightarrow{ww'}$ are edges of T. As u is a source, there exist edges $\overrightarrow{uv'}, \overrightarrow{uw'}$. Thus $P = [v, \overrightarrow{vv'}, v', \overleftarrow{v'u}, u, \overrightarrow{uw'}, w', \overleftarrow{w'w}, w]$ is a $(+-)$-path between v and w. As the vertices v, w were chosen arbitrarily, the tournament T is $(+-)$ connected. The source u forms an equivalence class of the relation of being $(-+)$-connected, because it cannot be joined by a $(-+)$-path with any other vertex; the first edge of such a path would be incoming into u which is impossible. If v, w are two vertices of T both different from u, then there exist edges $\overrightarrow{uv}, \overrightarrow{uw}$ and $P' = [v, \overleftarrow{vu}, u, \overrightarrow{uw}, w]$ is a $(-+)$-path between v and w. Thus $V \dot{-} \{u\}$ is an equivalence class of the relation of being $(-+)$-connected.

Theorem 1'. *Let a tournament T with the vertex set V have a sink u and no source. Then T is $(-+)$-connected, but not $(+-)$-connected. The equivalence classes of the relation of being $(+-)$-connected are $\{u\}$ and $V \dot{-} \{u\}$.*

Proof is dual to that of Theorem 1.

Theorem 2. *Let a tournament T with the vertex set V have a source u and a sink v. Then the equivalence classes of the relation of being $(+-)$-connected are $\{v\}$ and*

$V \dot{-} \{v\}$ *and the equivalence classes of the relation of being* $(-+)$*-connected are* $\{u\}$ *and* $V \dot{-} \{u\}$.

Proof is analogous to the proof of Theorem 1.

Theorem 3. *Let* T *be a tournament without a sink which is not strongly connected. Then* T *is* $(+-)$*-connected.*

Proof. The reduced graph R [1] of the tournament T is evidently an acyclic tournament. An acyclic tournament is evidently also transitive. Thus the vertices of R, i.e. the quasicomponents of T, are totally ordered so that for two quasicomponents Q_1, Q_2 we have $Q_1 \prec Q_2$ if and only if $Q_1 \neq Q_2$ and there exists and edge in T outgoing from a vertex of Q_1 and incoming into a vertex of Q_2. (As T is a tournament, from any vertex of Q_1 an edge goes into any vertex of Q_2.) Assume that there exists no greatest element in this ordering and consider two vertices u and v of T. Let Q_1 and Q_2 be the quasicomponent of T containing u and v respectively. There exists a quasicomponent Q_3 such that $Q_1 \prec Q_3$, $Q_2 \prec Q_3$. Choose a vertex w of Q_3. There exist edges $\overrightarrow{uw}$, $\overrightarrow{vw}$ and $P = [v, \overrightarrow{vw}, w, \overleftarrow{wu}, u]$ is a $(+-)$-path between u and v. Now assume that the above defined order has the greatest element; let this quasicomponent be Q_0. Consider again two vertices u and v. If none of them is in Q_0, the proof is the same as in the preceding case. Let u be in Q_0 and v in some $Q_1 \neq Q_0$. If Q_0 consists of a single vertex, this vertex is a sink; this is excluded by the assumption. Thus Q_0 is a strongly connected subtournament of T with more than one vertex; therefore there exists an edge $\overrightarrow{uw}$ such that w is contained also in Q_0. As $Q_1 \neq Q_0$, we have $Q_1 \prec Q_0$ and there exists also the edge $\overrightarrow{vw}$. Then $P = [u, \overrightarrow{uw}, w, \overleftarrow{wv}, v]$ is a $(+-)$-path between u and v. Now let both u and v be in Q_0. As Q_0 is a strongly connected subtournament of T, there exist vertices w, x in Q_0 such that $\overrightarrow{uw}$, $\overrightarrow{vx}$ are edges of T. If $w = x$, the proof is finished. If $w \neq x$, we choose a vertex y not belonging to Q_0. There exist edges $\overrightarrow{yw}$, $\overrightarrow{yx}$ in T and $P = [u, \overrightarrow{uw}, w, \overleftarrow{wy}, y, \overrightarrow{yx}, x, \overleftarrow{xv}, v]$ is a $(+-)$-path between u and v.

Theorem 3'. *Let* T *be a tournament without a source which is not strongly connected. Then* T *is* $(-+)$*-connected.*

Proof is dual to that of Theorem 3.

Before presenting the last theorem we shall prove some lemmas.

Lemma 1. *Let* T *be a tournament which is not acyclic. Then* T *contains at least one cycle of the length three.*

Proof. As T is not acyclic, we may choose a cycle C_1 in it. If the length of C_1 is three, the proof is finished. Assume that this length is $l_1 > 3$. Let $u_1, \ldots, u_{l_1}$ be the vertices of C_1 and $\overrightarrow{u_i u_{i+1}}$ for $i = 1, \ldots, l_1 - 1$ and $\overrightarrow{u_{l_1} u_1}$ be the edges of C_1. Consider the vertices u_1 and u_3. As T is a tournament, it contains either the edge $\overrightarrow{u_1u_3}$, or the edge $\overrightarrow{u_3u_1}$. In the second case the vertices u_1, u_2, u_3 with the edges $\overrightarrow{u_1u_2}, \overrightarrow{u_2u_3}, \overrightarrow{u_3u_1}$ form a cycle of the length three. In the first case there exists a cycle C_2 of the length $l_2 = l_1 - 1$ with the vertices $u_1, u_3, \ldots, u_{l_1}$. If $l_2 = 3$, the proof is finished; if not, we repeat the procedure with C_2 instead of C_1. In this manner we proceed until we obtain a cycle of the length three, which occurs after at most $l_1 - 3$ steps.

Lemma 2. *Let T be a tournament with the vertex set V without sources and sinks. Let $u \in V$ be such a vertex that $\{u\}, V \dot{-} \{u\}$ are equivalence classes of the relation of being $(+-)$-connected. Then the outdegree of u in T is 1 and the indegree of the vertex v such that $\overrightarrow{uv}$ is in T is also 1. The equivalence classes of the relation of being $(-+)$-connected are $\{v\}, V \dot{-} \{v\}$.*

Proof. The outdegree of u cannot be zero, because T does not contain sinks. Assume that there exist two vertices v_1, v_2 such that $v_1 \neq v_2$ and $\overrightarrow{uv_1}$ and $\overrightarrow{uv_2}$ are edges of T. As T is a tournament, the vertices v_1 and v_2 must be joined by an edge. Without any loss of generality let this edge be $\overrightarrow{v_1v_2}$. Let w be an arbitrary vertex of $V \dot{-} \{u\}$. As the set $V \dot{-} \{u\}$ is an equivalence class of the relation of being $(+-)$-connected, the vertices v_1 and w are $(+-)$-connected. There exists a $(+-)$-path $P = [v_1, \ldots, w]$ between v_1 and w. The path $P_2 = [u, \overrightarrow{uv_2}, v_2, \overleftarrow{v_2v_1}, v_1, \ldots, w]$ is a $(+-)$-path between u and w and the vertices u and w are $(+-)$-connected, which is a contradiction with the assumption that $\{u\}$ and $V \dot{-} \{u\}$ are the equivalence classes of the relation of being $(+-)$-connected. We have proved that the outdegree of u must be one. Let v be the terminal vertex of the unique edge outgoing from u. Assume that there exists a vertex $x \in V \dot{-} \{u\}$ such that $\overrightarrow{xv}$ is in T. Then $P_3 = [u, \overrightarrow{uv}, v, \overleftarrow{vx}, x]$ is a $(+-)$-path between u and x and x is $(+-)$-connected with u, which is again a contradiction. Thus also the indegree of v must be one. The vertex v is $(-+)$-connected with no vertex except itself, because any $(-+)$-path from v can only have the form $[v, \overleftarrow{vu}, u, \overrightarrow{uv}, v, \ldots, v]$. Thus $\{v\}$ is an equivalence class of the relation of being $(+-)$-connected. Now let a, b be two vertices of $V \dot{-} \{v\}$. As T is without sinks, there exist vertices a', b' of V such that $\overrightarrow{a'a}, \overrightarrow{b'b}$ are edges of T. If $a' = u$ or $b' = v$, then according to the above proved $a = v$ or $b = v$ respectively, which was excluded. Thus $a' \in V \dot{-} \{u\}$, $b' \in V \dot{-} \{u\}$ and these two vertices are $(+-)$-connected. Let $P_4 = [a', \ldots, b']$ be a $(+-)$-path between a' and b'. Then $P_5 = [a, \overleftarrow{aa'}, a', \ldots, b', \overrightarrow{b'b}, b]$ is a $(-+)$-path between a and b and these two vertices are $(-+)$-connected. As a, b were chosen arbitrarily from $V \dot{-} \{u\}$, this set is an equivalence class of the relation of being $(-+)$-connected in T.

Lemma 2′. *Let T be a tournament with the vertex set V without sources and sinks. Let $v \in V$ be such a vertex that $\{v\}, V \dot{-} \{v\}$ are equivalence classes of the relation of being $(-+)$-connected. Then the indegree of u in T is 1 and the outdegree of the vertex u such that $\overrightarrow{uv}$ is in T is also 1. The equivalence classes of the relation of being $(+-)$-connected are $\{u\}, V \dot{-} \{u\}$.*

Lemma 3. *Let T be a tournament with the vertex set V with at least four vertices without sources and sinks. Let u, v be two of its vertices such that $\{u\}$, $V \dot{-} \{u\}$ are equivalence classes of the relation of being $(+\,-)$-connected and $\{v\}$, $V \dot{-} \{v\}$ are equivalence classes of the relation of being $(-\,+)$-connected in T. Let T_1 a be tournament obtained from T by adding a new vertex w and joining it by exactly one directed edge with any vertex of V so that w is neither a source nor a sink in T_1. Then either T_1 is alternatingly connected or $\{u\}$, $(V \cup \{w\}) \dot{-} \{u\}$ are equivalence classes of the relation of being $(+\,-)$-connected and $\{v\}$, $(V \cup \{w\}) \dot{-} \{v\}$ are equivalence classes of the relation of being $(-\,+)$-connected in T_1.*

Proof. According to Lemmas 2 and 2′ the outdegree of u and the indegree of v are equal to 1 and $\overrightarrow{uv}$ is an edge of T. At first assume that $\overrightarrow{wu}$ and $\overrightarrow{vw}$ are edges of T_1. Then the outdegree of u and the indegree of v also in T_1 are equal to one. Analogously as in the preceding lemmas we can prove that $\{u\}$ is an equivalence class of the relation of being $(+\,-)$-connected and $\{v\}$ is an equivalence class of the relation of being $(-\,+)$-connected also in T_1. Any two vertices of $V \dot{-} \{u\}$ remain $(+\,-)$-connected also in T_1. Now let $x \in V \dot{-} \{u\}$. If $x \neq v$, then $\overrightarrow{xu}$ is in T and also in T_1. The path $P_1 = [x, \overrightarrow{xu}, u, \overleftarrow{uw}, w]$ is a $(+\,-)$-path in T_1 and therefore x and w are $(+\,-)$-connected in T_1. If $x = r$, then for any $x' \in V \dot{-} \{u\}$ the edge $\overrightarrow{xx'}$ is in T. We have $x' \neq v$, thus $x' \in V \dot{-} \{u\}$. The vertex u is also in $V \dot{-} \{v\}$ and the edge $\overrightarrow{wu}$ is in T_1. The vertices x' and u are therefore $(-\,+)$-connected and there exists a $(-\,+)$-path $P_2 = [x', \ldots, u]$ in T and also in T_1. The path $P_3 = [v, \overrightarrow{vx'}, x', \ldots, u, \overleftarrow{uw}, w]$ is a $(+\,-)$-path in T_1 and therefore v and w are $(+\,-)$-connected in T_1. We have proved that $(V \cup \{w\}) \dot{-} \{u\}$ is an equivalence class of the relation of being $(+\,-)$-connected in T_1. Dually we prove that $(V \cup \{w\}) \dot{-} \{v\}$ is an equivalence class of the relation of being $(-\,+)$-connected in T_1. Now assume that $\overrightarrow{uw}$ is an edge of T_1. If $\overrightarrow{vw}$ is also in T_1, then $P_4 = [u, \overrightarrow{uw}, w, \overleftarrow{vw}, v]$ is a $(+\,-)$-path in T_1 and the vertices u, v are $(+\,-)$-connected. Now let x be a vertex of V such that $\overrightarrow{wx}$ is in T_1; such a vertex must exist because w is not a sink. We have $x \neq u$, $x \neq v$. The edge $\overrightarrow{vx}$ is also in T_1, thus $P_5 = [w, \overrightarrow{wx}, x, \overleftarrow{xv}, v]$ is a $(+\,-)$-path in T_1 and the vertices v and w are also $(+\,-)$-connected. As $V \dot{-} \{u\}$ is an equivalence class of the relation of being $(+\,-)$-connected in T and the vertices u and w are both $(+\,-)$-connected with the vertex $v \in V \dot{-} \{u\}$, the set $V \cup \{w\}$ is an equivalence class of the relation of being $(+\,-)$-connected in T_1 and the tournament T_1 is $(+\,-)$-connected. According to [2] it is also $(-\,+)$-connected and thus it is alternatingly connected. If $\overrightarrow{wv}$ is in T_1, the path $P_6 = [u, \overrightarrow{uv}, v, \overleftarrow{vw}, w]$ is a $(+\,-)$-path in T_1 and therefore u and w are $(+\,-)$-connected in T_1. Let $x \in V \dot{-} \{u; v\}$; there exists the edge $\overrightarrow{vx}$. If $\overrightarrow{wx}$ is in T_1, then $P_7 = [v, \overrightarrow{vx}, x, \overleftarrow{xw}, w]$ is a $(+\,-)$-path in T_1 between v and w and these vertices are $(+\,-)$-connected. If $\overrightarrow{xw}$ is in T_1, then $P_8 = [u, \overrightarrow{uw}, w, \overleftarrow{wx}, x]$ is a $(+\,-)$-path between u and x and these vertices are $(+\,-)$-connected. This means that either u or w is $(+\,-)$-connected with some vertex of $V \dot{-} \{u\}$. As $V \dot{-} \{u\}$ is an equivalence class of the relation of being $(+\,-)$-connected, we see that one of the vertices u, w is $(+\,-)$-

connected with all vertices of $V \dot{-} \{u\}$ and so is the other, because u and w are $(+-)$-connected. Thus the tournament T_1 is $(+-)$-connected and also alternatingly connected.

Lemma 4. *Let T be an alternatingly connected tournament with the vertex set V. Let T_1 be a tournament obtained from T by adding a new vertex w and joining it by exactly one directed edge with any vertex of V so that w is neither a source nor a sink in T_1. Then T_1 is also alternatingly connected.*

Proof. It suffices to prove that w is $(+-)$-connected in T_1 with an arbitrary vertex u of T. Both u and w are not sinks; thus there exist vertices u', w' in V such that $\overrightarrow{uu'}$, $\overrightarrow{ww'}$ are edges of T_1. The vertices u' and w' are $(-+)$-connected in T and also in T_1. Thus there exists a path $P_1 = [u', \ldots, w']$. The path $P_2 = [u, \overrightarrow{uu'}, u', \ldots \ldots, w', \overleftarrow{w'w}, w]$ is a $(+-)$-path between u and w in T_1.

Lemma 5. *Let $\{T_\iota\}_{\iota<\alpha}$ be a transfinite sequence of alternatingly connected tournaments of the limit ordinal number α such that for $\iota < \varkappa < \alpha$ the tournament T_ι is a proper subtournament of $T_\varkappa$. Then the tournament $T_\alpha = \bigcup_{\iota<\alpha} T_\iota$ is alternatingly connected.*

Proof. Let u, v be two vertices of T_α. According to the definition there exist ordinal numbers ι, $\varkappa$ less than α such that u is in T_ι and v is in $T_\varkappa$. Let $\lambda = \max(\iota, \varkappa)$. The vertices u, v are both contained in T_λ and are $(+-)$-connected in it. Therefore they are $(+-)$-connected also in T_α whose subtournament T_λ is.

Lemma 6. *Let $\{T_\iota\}_{\iota<\alpha}$ be a transfinite sequence of tournaments without sources and sinks of the limit ordinal number α such that for $\iota < \varkappa < \alpha$ the tournament T_ι is a proper subtournament of $T_\varkappa$. Let u, v be such vertices of T_0 that for any $\iota < \alpha$ the equivalence classes of the relation of being $(+-)$-connected in T_ι are $\{u\}$, $V_\iota \dot{-} \{u\}$ and the equivalence classes of the relation of being $(-+)$-connected in T_ι are $\{v\}$, $V_\iota \dot{-} \{v\}$ where V_ι is the vertex set of T_ι. Then in the tournament $T_\alpha = \bigcup_{\iota<\alpha} T_\iota$ the equivalence classes of the relation of being $(+-)$-connected are $\{u\}$, $V_\alpha \dot{-} \{u\}$ and the equivalence classes of the relation of being $(-+)$-connected are $\{v\}$, $V_\alpha \dot{-} \{v\}$ where V_α is the vertex set of T_α.*

Proof. If x, y are two vertices of $V_\alpha \dot{-} \{u\}$, we prove analogously to the proof of Lemma 5 that they are $(+-)$-connected. Now assume that u and some vertex $x \in V_\alpha$ are $(+-)$-connected in T_α. There exists a $(+-)$-path P between u and x in T_α. Let $V(P)$ be the set of vertices of P and for a given $y \in V_\alpha$ let $\beta(y)$ be the least ordinal number such that $y \in V_{\beta(y)}$; such a number must exist because of the well-ordering of the set of ordinal numbers less than α. Let $\beta(P) = \max_{y \in V(P)} \beta(y)$. As $V(P)$ is a finite set, this maximum exists. The path P is contained in $T_{\beta(P)}$ and therefore $T_{\beta(P)}$ is $(+-)$-connected, which is a contradiction. The rest of the assertion can be proved dually.

Theorem 4. *Let T be a tournament with three vertices. Then only two cases can occur*:

(1) *T is a cycle of the length 3 (Fig. 1a). Then any equivalence class of the relation of being $(+-)$-connected, as well as of the relation of being $(-+)$-connected, consists only of one vertex.*

(2) *T is acyclic (Fig. 1b). Then if u, v, w are vertices of T and $u \prec v \prec w$, then the equivalence classes of the relation of being $(+-)$-connected are $\{u\}$, $\{v, w\}$ and the equivalence classes of the relation of being $(-+)$-connected are $\{u, v\}$, $\{w\}$.*

The assertion is evident.

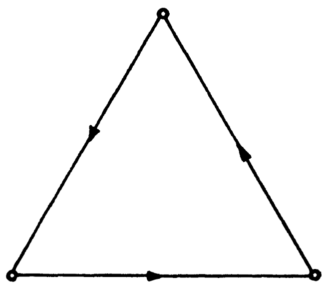

Fig. 1a.

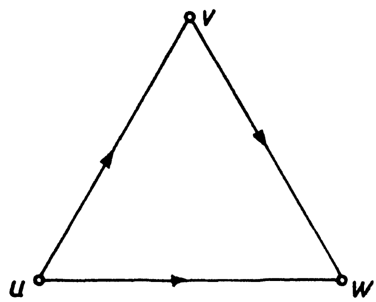

Fig. 1b.

Theorem 5. *Let T be a strongly connected tournament with at least four vertices Then either T is alternatingly connected, or there exist two vertices u, v in T such that the equivalence classes of the relation of being $(+-)$-connected are $\{u\}$, $V \dot{-} \{u\}$ and the equivalence classes of the relation of being $(-+)$-connected are $\{v\}$, $V \dot{-} \{v\}$ where V is the vertex set of T.*

Proof. We shall carry out the proof by the method of transfinite induction. At first we shall investigate tournaments with four vertices. Let T be such a tournament. If a tournament is strongly connected, it is not acyclic. Therefore according to Lemma 1 it contains a cycle of the length 3. Consider the vertex of T not belonging to this cycle. It is neither a source nor a sink, because of the strong connectivity of T. Thus either its indegree is 1 and its outdegree is 2, or its indegree is 2 and its outdegree is 1. We see that in both these cases we obtain a tournament isomorphic to the tournament on Fig. 2. In this tournament the equivalence classes of the relation of being $(+-)$-connected are $\{u\}$, $V \dot{-} \{u\}$ and the equivalence classes of the relation of being $(-+)$-connected are $\{v\}$, $V \dot{-} \{v\}$ which can be easily verified. Now let T be a strongly connected tournament with more than four vertices. It contains a cycle C of the length three; let a, b, c be its vertices, $\overrightarrow{ab}$, $\overrightarrow{bc}$, $\overrightarrow{ca}$ its edges. If C does not belong to any subgraph of T isomorphic to the graph on Fig. 2, then for any vertex x of T

not belonging to C either the edges $\overrightarrow{ax}, \overrightarrow{bx}, \overrightarrow{cx}$ or the edges $\overrightarrow{xa}, \overrightarrow{xb}, \overrightarrow{xc}$ exist. If for each vertex x not belonging to C the edges $\overrightarrow{ax}, \overrightarrow{bx}, \overrightarrow{cx}$ exist, the circuit C is a quasi-component of T, which is a contradiction with the assumption that T is strongly connected. The same holds if for each vertex x not belonging to C the edges $\overrightarrow{xa}, \overrightarrow{xb}, \overrightarrow{xc}$

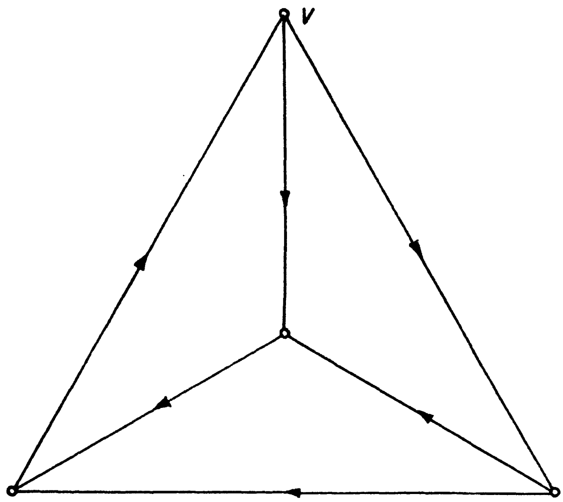

Fig. 2.

exist. Therefore, if X is the set of all vertices x of T not belonging to C such that the edges $\overrightarrow{ax}, \overrightarrow{bx}, \overrightarrow{cx}$ exist and Y is the set of all vertices y of T not belonging to C such that the edges $\overrightarrow{ya}, \overrightarrow{yb}, \overrightarrow{yc}$ exist, then both X and Y are non-empty. As T is strongly connected, there exists at least one $x \in X$ and $y \in Y$ such that $\overrightarrow{xy}$ is in T. Thus a, x, y form a cycle in T and the edges $\overrightarrow{ab}, \overrightarrow{bx}, \overrightarrow{yb}$ exist. The subgraph of T induced by the vertices a, b, x, y is isomorphic to the graph on Fig. 2. We have proved that such a graph is a subgraph of every strongly connected tournament with more than four vertices. Thus we use the transfinite induction according to the number of vertices; this proof follows from Lemmas 3, 4, 5, 6. Obviously if we consider infinite tournaments, the Axiom of Choice is used.

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