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REMARK ABOUT THE RELATION BETWEEN MEASURABLE AND CONTINUOUS FUNCTIONS

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Dedicated to the memory of Prof. VOJTĚCH JARNÍK

This article deals with the problem posed in [1]. Using Luzin's theorem the following result can be easily derived: Let $f(x)$ be a measurable function defined on $J = \langle 0, 1 \rangle$, then there exists countably many disjoint closed sets A_n fulfilling $\mu(J - \bigcup_n A_n) = 0$ (where μ denotes Lebesgue measure) such that the restrictions of f on A_n are continuous for every n. In view of the well-known theorem about the extension of continuous functions there exist continuous functions $f_n(x)$ which are defined on the whole $J = \langle 0, 1 \rangle$ such that $f(x) = f_n(x)$ on A_n. Generally, $J - \bigcup_n A_n \neq 0$. There arises the problem, whether – by another method – it can be obtained $J = \bigcup_n A_n$.

Now we shall formulate the above mentioned problem more precisely [1].

Consider the class of functions f fulfilling the following conditions:

i) *f is real, finite and defined on $(0, 1)$,*

ii) *there exist Lebesgue measurable sets A_i and continuous functions f_i defined on $(0, 1)$, $i = 1, 2, \ldots$ so that $\bigcup_{i=1}^{\infty} A_i = (0, 1)$ and $f_i = f$ on A_i.*

Investigate the properties of the class. Does every measurable function belong to the class?

It will be proved that the last question has the answer in negative even in the stronger form: A function $f(x)$ exists which is defined and measurable on J such that there does not exist countably many disjoint sets A_n (which need not be measurable) fulfilling $J = \bigcup_n A_n$ such that the restriction of f on A_n is continuous for every n.

Several lemmas for the construction and proof will be needed.

Lemma 1. *There exists a discontinuum A, $A \subset J$, $0 \in A$, $1 \in A$ and a function $f(x)$ which is defined on A and $0 \leqq f(x) \leqq 1$ such that the graph of f is not Lebesgue measurable.*

Proof. With respect to [2] there exists a function $f^*(x)$ with nonmeasurable graph. Using elementary modifications and transformations we can assume that the domain D of definition of $f^*(x)$ fulfils $D \subset J$ and $0 \leqq f^*(x) \leqq 1$. If we extend the domain of definition of $f^*(x)$ on the whole J putting $f^*(x) = 0$ on $J - D$, then the graph of $f^*(x)$ remains nonmeasurable. Let μ_e and μ_i denote the outer and inner Lebesgue measure, respectively. Let $\mathrm{Gr}\,\{f\}$ denote the graph of the function f. For every function f

$$\mu_i(\mathrm{Gr}\,\{f\}) = 0 \tag{1}$$

holds. This can be easily proved using Fubini's theorem on every compact subset of $\mathrm{Gr}\,\{f\}$.

Since $\mathrm{Gr}\,\{f^*\}$ is nonmeasurable we have $\mu_e(\mathrm{Gr}\,\{f^*\}) > 0$. We choose the discontinuum A such that $A \subset J$, $0 \in A$, $1 \in A$ and $\mu(J - A) \leqq \mu_e(\mathrm{Gr}\,\{f^*\})/2$. The required function $f(x)$ is defined as follows: $f(x) = f^*(x)$ for $x \in A$.

Obviously $\mathrm{Gr}\,\{f^*\} \subset \mathrm{Gr}\,\{f\} + (J - A) \times J$, since the values of f^* lie in J. With respect to that we obtain

$$\mu_e(\mathrm{Gr}\,\{f\}) \geqq \mu_e(\mathrm{Gr}\,\{f^*\}) - \mu(J - A) \geqq \mu_e(\mathrm{Gr}\,\{f^*\})/2 > 0\,.$$

As (1) is valid for our f the last inequality also implies that the graph of f is nonmeasurable.

Lemma 2. *Let A, B be discontinuums fulfilling $A \subset J$, $B \subset J$, $0 \in A \cap B$, $1 \in A \cap B$, then there exists a homeomorphism $\varphi : J \to J$ which maps A on B.*

Proof. It is well-known that a homeomorphism exists which maps A on B and which fulfils

$$\varphi(x_1) < \varphi(x_2) \quad \text{for} \quad x_1 < x_2\,. \tag{2}$$

This homeomorphism can be extended on the whole J. Denote $G = J - A$, $H = J - B$. The open set G consists of countably many open intervals $G^{(n)} = (p^{(n)}, q^{(n)})$, $p^{(n)} \in A$, $q^{(n)} \in A$. Intervals $(\varphi(p^{(n)}), \varphi(q^{(n)}))$ belong to H. Really, if $y \in B \cap (\varphi(p^{(n)}), \varphi(q^{(n)}))$, then there exists $x \in A$ such that $\varphi(x) = y$. With respect to (2) we obtain $p^{(n)} < x < q^{(n)}$ which is a contradiction with the fact that $G^{(n)}$ is a subset of G. Since the same is valid for the inverse transformation $\varphi^{-1}(y)$, the sum of these intervals covers H. We extend φ such that it is a linear transformation on every $G^{(n)}$. The previous considerations imply that φ maps G onto H and is continuous on G (G is open). It remains to prove that the extension remains continuous on A.

First it will be proved that inequality (2) is valid for all x_1, x_2, $0 \leqq x_1 \leqq 1$, $0 \leqq \leqq x_2 \leqq 1$. If x_1, x_2 are in the same $G^{(n)}$, then (2) follows from the fact that φ is linear on $G^{(n)}$. If $x_1 \in G^{(k)}$, $x_2 \in G^{(l)}$ we take into consideration the points $q^{(k)}$, $p^{(l)}$. As inequality (2) is valid for pairs $x_1, q^{(k)}$; $q^{(k)}, p^{(l)}$; $p^{(l)}, x_2$ it is valid also for x_1, x_2. The cases when $x_1 \in A$, $x_2 \in G$ and $x_1 \in G$, $x_2 \in A$ are similar.

Let $x_0 \in A$ and $x_n \in J$, $x_n \to x_0$ for $n \to \infty$. The sequence x_n can be divided into two parts:

i) $x_n \in A$, then $\varphi(x_n) \to \varphi(x_0)$.

ii) $x_n \in G$, then there exist indices k_n such that $x_n \in G^{(k_n)}$. By definition $G^{(k_n)} = (p^{(k_n)}, q^{(k_n)})$ and we conclude easily that $p^{(k_n)}, q^{(k_n)} \to x_0$ which means $\varphi(p^{(k_n)}) \to \varphi(x_0)$ and $\varphi(q^{(k_n)}) \to \varphi(x_0)$. As $\varphi(x)$ is linear on $G^{(k_n)}$ we obtain that $\varphi(x_n) \to \varphi(x_0)$. The lemma is proved.

Lemma 3. *Let D be a subset of J (it need not be measurable) and $f(x)$ be a function defined and continuous on D, then the graph of the function $f(x)$ is measurable, i.e. $\mu(\mathrm{Gr}\,\{f\}) = 0$.*

Proof. Choose a positive fixed number ε. Since $f(x)$ is continuous on D there exists $\delta(x) > 0$ for every $x \in D$ such that $|y - x| \leqq \delta(x)$ implies $|f(y) - f(x)| \leqq \varepsilon$. For all $x \in D$ we shall consider the set of closed subintervals of $\langle x - \delta(x), x + \delta(x)\rangle$ which have their center at x. Denote by S the set of all such intervals. With respect to Vitali theorem there exists a subset of countably many intervals I_n such that $\mu(D - \bigcup_n I_n) = 0$. Let $g(x)$ be the function with the domain of definition $D - \bigcup_n I_n$ which is defined by $g(x) = f(x)$ on $D - \bigcup_n I_n$. Obviously

$$\mathrm{Gr}\,\{f\} \subset \bigcup_n [I_n \times \langle f(a_n) - \varepsilon, f(a_n) + \varepsilon\rangle] + \mathrm{Gr}\,\{g\}$$

where a_n are centers of I_n. Hence follows

$$\mu_e(\mathrm{Gr}\,\{f\}) \leqq 2\varepsilon \sum \mu(I_n) + \mu_e(\mathrm{Gr}\,\{g\}).$$

As $\bigcup_n I_n \subset J$, I_n are disjoint and the graph of g is measurable we obtain $\mu_e(\mathrm{Gr}\,\{f\}) \leqq \leqq 2\varepsilon$. Since the number ε was arbitrary Lemma 3 is proved.

We pass to the solution of our problem. Let $f(x)$ be the function which is determined in Lemma 1. We define $\bar{f}(x) = f(x)$ for $x \in A$ and $\bar{f}(x) = 0$ for $x \in J - A$. The function $\bar{f}(x)$ has the nonmeasurable graph again and it may be nonzero only on A. Let B be the Cantor discontinuum on J. In view of Lemma 2 there exists a homeomorphism φ from J onto J such that B is transformed on A. Put

$$F(x) = \bar{f}(\varphi(x)). \tag{3}$$

Statement. *The function $F(x)$ which is defined by (3) is measurable and defined on J and there do not exist countable number of disjoint sets C_n fulfilling $J = \bigcup_n C_n$ such that the restrictions of $F(x)$ on C_n are continuous for every n.*

Proof. Notice that $F(x)$ is measurable since $\mu(B) = 0$ and $F(x) = 0$ for $x \in \in J - B$. Assume that such sets exist. Put $F_n(x) = F(x)$ on C_n. Obviously $F_n(x)$ are continuous functions. Further denote $D_n = \varphi(C_n)$. As φ is strictly monotone and maps J onto J the sets D_n are disjoint and $J = \bigcup_n D_n$. Let $\tilde{f}_n(y) = F_n(\varphi^{-1}(y))$ for $y \in D_n$ where $\varphi^{-1}(y)$ is the inverse function to $\varphi(x)$. Since $\varphi^{-1}(y)$ transforms continuously D_n onto C_n and F_n are continuous on C_n, functions $f_n(y)$ are continuous on D_n. With respect to (3) we obtain

$$\tilde{f}(y) = \tilde{f}_n(y) \quad \text{on} \quad D_n .$$

Evidently $\mathrm{Gr}\,\{\tilde{f}\} = \bigcup_n \mathrm{Gr}\,\{\tilde{f}_n\}$. With regard to Lemma 3 functions $\tilde{f}_n$ have the graphs with measure zero so that $\mu_e(\mathrm{Gr}\,\{\tilde{f}\}) = 0$. However, the function $\tilde{f}(x)$ was constructed so that it has nonmeasurable graph. This contradiction implies that Statement is true.

The construction of the function $F(x)$ is only a partial solution of the given problem and there is still a very interesting question. Supposing that the sets A_i are Borel sets, then the function $f(x)$ is a Baire function. It means that under the additional assumption about A_i the investigated class of functions is a subset of the class of Baire functions. The question hinted above is if these two classes are equal or not.

References

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