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A NOTE ON A PAPER BY A. BRANDT AND Y. INTRATOR

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An interesting simple combinatorial method for the assignment problem with three job categories is described in [1]. The purpose of this note is to extend this method to the transportation problem with three origins or three destinations and present some computational experiences.

We shall consider the following problem: *for given numbers v_{ij} and non-negative integers a_i, b_j, where $i = 1, 2, \ldots, m$; $j = 1, 2, 3$ and $\sum a_i = \sum b_j$, to find non-negative values of x_{ij} satisfying the constraints*

$$\sum_{j=1}^{3} x_{ij} = a_i , \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j , \quad j = 1, 2, 3$$

and minimizing the function

$$f(x_{11}, x_{12}, \ldots, x_{m3}) = \sum_{i=1}^{m} \sum_{j=1}^{3} v_{ij} x_{ij}$$

The assignment problem considered in [1] can be regarded as a special case of our problem in which $a_i = 1$ for all $i = 1, 2, \ldots, m$.

Our method is an extention to the one described in [1]. For the reader's convenience, wherever possible, the subsequent notation is identical to the notation used in [1], and for the simplicity we assume henceforth, that the differences

$$\Delta_{kl}(i) = v_{ik} - v_{il} , \quad i = 1, 2, \ldots, m ; \quad (k, l) = (1, 2), (2, 3), (3, 1)$$

satisfy the condition $i \neq j \Rightarrow \Delta_{kl}(i) \neq \Delta_{kl}(j)$. If this condition is not satisfied it is possible to consider the problem with pertubed data as in [1] and modify some formulations and analysis accordingly. The proposed algorithm consists of successive reduction of the original problem to a sufficiently simple problem, which is solved directly. As sufficiently simple problems we understand those in which either at least one of b's is zero or only one of a's is non-zero. Reduction is based on the fact that if for some optimal solution $\|x_{ij}\|$ of the problem with data v_{ij}, a_i, b_j and for i_0, j_0 there is a positive number $d_{i_0 j_0}$ such that $x_{i_0 j_0} \geqq d_{i_0 j_0}$, then we can reduce the problem to the one with data v_{ij}, a'_i, b'_j, where $a'_{i_0} = a_{i_0} - d_{i_0 j_0}$, $b'_{j_0} = b_{j_0} - d_{i_0 j_0}$ and $a'_i = a_i$, $b'_j = b_j$ for $i \neq i_0, j \neq j_0$.

Let us introduce permutations $p_{kl}(i)$ of the set $1, 2, \ldots, m$ such that

$$p_{kl}(i_1) < p_{kl}(i_2) \Leftrightarrow \Delta_{kl}(i_1) < \Delta_{kl}(i_2)$$

and let us define

$$z_{kl}(i) = \max\{0, \min[a_i, b_l - \sum_{\{j \mid p_{kl}(j) > p_{kl}(i)\}} a_j]\}$$

$$z_{kq}(i) = \max\{0, \min[a_i, b_q - \sum_{\{j \mid p_{qk}(j) < p_{qk}(i)\}} a_j]\}$$

for $(k, l, q) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ and $i = 1, 2, \ldots, m$.

Theorem 1. *If $\|x_{ij}\|$ is an optimal solution, then*

$$x_{ik} \leqq a_i - \max[z_{kl}(i), z_{kq}(i)]$$

for all $k = 1, 2, 3$ and $i = 1, 2, \ldots, m$.

Proof. We are to verify the inequalities

$$x_{ik} \leqq a_i - z_{kl}(i), \quad x_{ik} \leqq a_i - z_{kq}(i).$$

If for some $k \in \{1, 2, 3\}$ and some $i_0 \in \{1, 2, \ldots, m\}$ is $x_{i_0 k} > a_{i_0} - z_{kl}(i_0)$, then

$$0 < z_{kl}(i_0) = \min[a_{i_0}, b_l - \sum_{\{i \mid p_{kl}(i) > p_{kl}(i_0)\}} a_i] \leqq$$

$$\leqq b_l - \sum_{\{i \mid p_{kl}(i) > p_{kl}(i_0)\}} a_i \leqq b_l - \sum_{\{i \mid p_{kl}(i) > p_{kl}(i_0)\}} x_{il} =$$

$$= x_{i_0 l} + \sum_{\{i \mid p_{kl}(i) < p_{kl}(i_0)\}} x_{il} < z_{kl}(i_0) + \sum_{\{i \mid p_{kl}(i) < p_{kl}(i_0)\}} x_{il}.$$

Consequently there is an index $i_1 \in \{1, 2, \ldots, m\}$ such that $x_{i_1 l} > 0$ and $p_{kl}(i_1) < p_{kl}(i_0)$. Considering that also $x_{i_0 k} > 0$, we can define another feasible solution x'_{ij} by setting

$$x'_{i_0 k} = x_{i_0 k} - \varepsilon, \quad x'_{i_0 l} = x_{i_0 l} + \varepsilon$$

$$x'_{i_1 k} = x_{i_1 k} + \varepsilon, \quad x'_{i_1 l} = x_{i_1 l} - \varepsilon$$

$$x'_{ij} = x_{ij} \quad \text{for others}$$

where ε is a suitable small positive number. Inasmuch $p_{kl}(i_1) < p_{kl}(i_0)$, the difference

$$\sum_{i=1}^{m} \sum_{j=1}^{3} v_{ij}x'_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{3} v_{ij}x_{ij} = \varepsilon[\Delta_{kl}(i_1) - \Delta_{kl}(i_0)]$$

is negative, so that $\|x_{ij}\|$ is not optimal. If for some $k \in \{1, 2, 3\}$ and some $i_0 \in \in \{1, 2, \ldots, m\}$ is $x_{i_0k} > a_{i_0} - z_{kq}(i_0)$, then

$$z_{kq}(i_0) = \min [a_{i_0}, b_q - \sum_{\{i | p_{qk}(i) < p_{qk}(i_0)\}} a_i]$$

and similar argumentation again leads to contradiction.

Theorem 1 enables us do determine various positive lower bounds requisite to reduction, provided that there is $i \in \{1, 2, \ldots, m\}$ such that

$$(*) \qquad \sum_{k=1}^{3} z_k(i) > a_i$$

where $z_k(i) = \max [z_{kl}(i), z_{kq}(i)]$. Some of these bounds are e.g.

$$x_{ik} \geqq \max \{\min [z_{lq}(i), z_{qk}(i)], \min [z_{lk}(i), z_{qk}(i)], \min [z_{lk}(i), z_{ql}(i)]\}$$

$$x_{iq} \geqq z_k(i) + z_l(i) - a_i, \quad \text{iff} \quad z_k(i) + z_l(i) > a_i$$

$$x_{iq} \geqq z_{lq}(i), \quad \text{if} \quad z_{kl}(i) + z_{lq}(i) > a_i$$

$$x_{iq} \geqq z_{kq}(i), \quad \text{if} \quad z_{kq}(i) + z_{lk}(i) > a_i$$

In order to be able to reduce the problem also in the case when the inequality (*) does not hold for any i, we shall prove the following theorem.

Theorem 2. *If* $\sum_{k=1}^{3} z_k(i) \leqq a_i$ *for all i, then* $b_1 = b_2 = b_3$ *and* $\sum_{k=1}^{3} z_k(i) = a_i, \sum_{i=1}^{m} z_k(i) = b_k$ *for all i and k. If in addition the problem is not sufficiently simple, then* $z_k(i)$ *equals* a_i *or* 0 *for all i and k.*

Proof. It follows directly from the definition of $z_k(i)$ that

$$\max [b_l, b_q] \leqq \sum_{i=1}^{m} z_k(i)$$

for all $(k, l, q) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. In addition

$$\sum_{i=1}^{m} \sum_{k=1}^{3} z_k(i) \leqq \sum_{i=1}^{m} a_i = \sum_{j=1}^{3} b_j$$

so that

$$\max [b_1, b_2] + \max [b_2, b_3] + \max [b_3, b_1] \leqq b_1 + b_2 + b_3$$

which implies $b_1 = b_2 = b_3$. It is also easy to verify that if for some i_0 is $\sum_{k=1}^{3} z_k(i_0) < a_{i_0}$, then $\sum_{i=1}^{m} \sum_{k=1}^{3} z_k(i) < \sum_{i=1}^{m} a_i$, and if for some k_0 is $\sum_{i=1}^{m} z_{k_0}(i) > b_{k_0}$, then $\sum_{k=1}^{3} \sum_{i=1}^{m} z_k(i) > b_1 + b_2 + b_3$, which is impossible since

$$3b \leqq \sum_{k=1}^{3} \sum_{i=1}^{m} z_k(i) = \sum_{i=1}^{m} \sum_{k=1}^{3} z_k(i) \leqq \sum_{i=1}^{m} a_i = 3b$$

where b denotes the common value of b_1, b_2 and b_3. Now let us suppose that the problem is not sufficiently simple and that for some i_0 is $0 < z_1(i_0) < a_{i_0}$. The arguments for other possible cases are virtually the same. In the case under consideration

$$z_1(i_0) = b - \sum_{\{i \mid p_{12}(i) > p_{12}(i_0)\}} a_i = b - \sum_{\{i \mid p_{31}(i) < p_{31}(i_0)\}} a_i \, .$$

If $z_2(i_0) > 0$ (if not, then $z_3(i_0) > 0$ and we can use the analogical arguments), then

$$z_2(i_0) = b - \sum_{\{i \mid p_{23}(i) > p_{23}(i_0)\}} a_i = b - \sum_{\{i \mid p_{12}(i) < p_{12}(i_0)\}} a_i \, .$$

Considering that

$$3b = \sum_{i=1}^{m} a_i = a_{i_0} + \sum_{\{i \mid p_{12}(i) > p_{12}(i_0)\}} a_i + \sum_{\{i \mid p_{12}(i) < p_{12}(i_0)\}} a_i =$$
$$= a_{i_0} + (b - z_1(i_0)) + (b - z_2(i_0))$$

we conclude that $z_3(i_0) = b$. By virtue of the definition of $z_3(i_0)$ this implies

$$\sum_{\{i \mid p_{31}(i) > p_{31}(i_0)\}} a_i = \sum_{\{i \mid p_{23}(i) < p_{23}(i_0)\}} a_i = 0 \, .$$

Inasmuch as, in this case,

$$2b + z_3(i_0) = 3b = \sum_{\{i \mid p_{31}(i) < p_{31}(i_0)\}} a_i + a_{i_0} = \sum_{\{i \mid p_{23}(i) > p_{23}(i_0)\}} a_i + a_{i_0} =$$
$$= a_{i_0} + b - z_1(i_0) = a_{i_0} + b - z_2(i_0)$$

we conclude that also $z_2(i_0) = z_1(i_0) = b$, so that $a_{i_0} = 3b$ which is possible only when the problem is sufficiently simple.

Corollary. *There are i_1, i_2, i_3 such that*

$$\sum_{i=i_1+1}^{m} a_{p_{12}^{-1}(i)} = \sum_{i=i_2+1}^{m} a_{p_{23}^{-1}(i)} = \sum_{i=i_3+1}^{m} a_{p_{31}^{-1}(i)} = b \, .$$

Here p_{kl}^{-1} denotes the inverse to p_{kl}.

In the case under consideration reduction depends on the value

$$\Delta(i_1, i_2, i_3) = \Delta_{12}(p_{12}^{-1}(i_1)) + \Delta_{23}(p_{23}^{-1}(i_2)) + \Delta_{31}(p_{31}^{-1}(i_3))$$

and it is given, by the following rules:

(a) if $\Delta(i_1, i_2, i_3) > 0$, then

$$x_{p_{kl}^{-1}(i_k),l} \geqq \min\left[a_{p_{12}^{-1}(i_1)}, a_{p_{23}^{-1}(i_2)}, a_{31^{-1}(i_3)}\right]$$

for every optimal solution $\|x_{ij}\|$ and $(k, l) = (1, 2), (2, 3), (3, 1)$;

(b) if $\Delta(i_1, i_2, i_3) < 0$, then for every optimal solution $\|x_{ij}\|$ and $(k, l) = (1, 2), (2, 3), (3, 1)$

$$x_{p_{kl}^{-1}(i_k-j),k} = a_{p_{kl}^{-1}(i_k-j)}, \quad j = 0, 1, \ldots, r_k$$

where r_k is defined by the condition $\sum_{j=0}^{r_k} a_{p_{kl}^{-1}(i_k-j)} = b$;

Table 1

Transportation problem

Size m	Procedure [2] in sec	Described procedure in sec
50	2·23	2·53
	4·15	2·72
	2·22	4·34
	4·70	2·15
	1·78	4·11
	4·84	2·39
	2·49	4·01
	1·84	3·31
	6·29	1·75
	4·60	3·06
100	11·13	8·66
	11·60	7·38
	10·43	6·62
	10·29	9·78
	12·28	5·62
	18·59	8·04
	8·95	10·40
	8·66	10·90
	11·58	6·54
	19·52	5·23

Table 2

Assignment problem

Procedure [2] in sec	Described procedure in sec
5·58	2·62
2·04	2·77
0·81	3·00
4·93	2·37
2·28	2·28
3·38	2·22
2·68	2·63
1·94	3·39
4·64	2·66
2·31	2·62
3·33	8·45
10·20	6·78
3·24	8·06
10·89	7·40
10·75	6·90
7·22	7·86
11·84	6·80
10·09	6·51
13·17	6·07
18·50	4·19

(c) if $\Delta(i_1, i_2, i_3) = 0$, then there is an optimal solution $\|x_{ij}\|$ such that

$$x_{p_{kl}^{-1}(i_k),k} = a_{p_{kl}^{-1}(i_k)}, \quad (k, l) = (1, 2), (2, 3), (3, 1).$$

In attempt to verify the efficiency of the approach presented in [1] and here, we wrote a test procedure in ALGOL 60 and carried out a comparition with the procedure presented in [2] on the computer EL-X8 of the Utrecht University Computing Centre. Some results of these experiments are presented in tables 1 and 2. The former concerns transportations problems with three origins and integer a_i, b_j, the latter concerns the special cases corresponding to assignment problems. In both cases the initial data were formed by a random procedure.

Remark. As dr. A. Brandt pointed out (in a personal communication to the authors) it would be desireable, at least from the theoretical point of view, to show that reduction can be organized in such a way that there is a bound to the number of the necessary reductions which depends on m only and not on the largeness of a_i and b_j.

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