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M-POLARS IN LATTICES

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R. D. BYRD [2] introduced the concept of the M-disjointness for lattice ordered groups and studied the properties of M-polars. One of the main results of the paper [2] is the following theorem:

(B) *Let M be a convex l-subgroup of a lattice ordered group G. The system P of all M-polars of G partially ordered by the set-inclusion is a complete Boolean algebra.*

The theorem (B) generalizes the well-known results on polars in K-spaces (KANTOROVIČ-VULICH-PINSKER [7]) and in lattice-ordered groups (ŠIK [8]). The aim of this note is to show that the theorem (B) is a corollary of a more general theorem that is valid for any lattice.

We shall use the standard notations for partially ordered sets and partially ordered groups (cf. [1]). Let G be a lattice ordered group. For $S \subset G$ we put $S^+ = \{x \in S : s \geqq 0\}$. S is said to be convex, if from $s_1, s_2 \in S$, $x \in G$, $s_1 \leqq x \leqq s_2$ it follows $x \in S$. Let M be a convex l-subgroup of G, $S \subset G$. Denote (cf. [2])

$$p(S, M) = \{x \in G : |x| \wedge |s| \in M \text{ for any } s \in S\} . \tag{1}$$

The set $p(S, M)$ is the M-polar (of S). Let $P(M)$ be the system of all M-polars (partially ordered by the set-inclusion). For any set $S \subset G^+$ we put

$$p_0(S, M^+) = \{x \in G^+ : x \wedge s \in M^+ \text{ for any } s \in S\} . \tag{2}$$

Let $P_0(M^+)$ be the system of all sets $p_0(S, M^+)$; this system is partly ordered by the set-inclusion.

1. *The mapping $\varphi(p(S, M)) = p(S, M)^+$ is an isomorphism of the partially ordered set $P(M)$ onto $P_0(M^+)$.*

Proof. From (1) and (2) it follows that $p(S, M)^+ = p_0(S', M^+)$, where $S' = \{|s| : s \in S\}$. Hence φ is a mapping of the system $P(M)$ onto $P_0(M^+)$. If $p(S_1, M) \subset p(S_2, M)$, then, clearly, $p(S_1, M)^+ \subset p(S_2, M)^+$. According to (1) $p(S, M) =$

$= p(S', M)$ for any $S \subset G$. Let $S_1, S_2 \subset G$, $p(S_1, M)^+ \subset p(S_2, M)^+$. Then for $x \in p(S_1, M)$ we have $|x| \in p_0(S'_1, M^+)$, hence $|x| \in p_0(S'_2, M^+)$ and this implies by (1) and (2) $x \in p(S_2, M)$, hence $p(S_1, M) \subset p(S_2, M)$. Since the l-subgroup $p(S, M)$ is generated by $p(S, M)^+$, the mapping φ is one-to-one. This shows that φ is an isomorphism.

Now let L be any lattice and let M be an ideal of L. We shall call M a regular ideal, if there exists a congruence relation Φ on the lattice L such that M is a class of the corresponding partition of the set L (i.e., if for any $m \in M$, $x \in L$ the equivalence $x \equiv m(\Phi) \Leftrightarrow x \in M$ holds). For each subset $S \subset L$ let us put

$$p_0(S, M) = \{x \in L : x \wedge s \in M \text{ for any } s \in S\} .$$

Let M be a regular ideal of L and let $\Phi(M)$ be the least congruence relation on L such that M is a class of the corresponding partition of the set L. Let $\bar{L} = L/\Phi(M)$ be the factor lattice and for $x \in L$ denote by $\bar{x}$ the class of all elements of L that are congruent to $x \bmod \Phi(M)$. If $S \subset L$, let $\bar{S} = \{\bar{s} : s \in S\}$. Clearly M is the least element of the partially ordered set $\bar{L}$. For each $S \subset L$ denote

$$p_0(\bar{S}) = \{\bar{x} \in \bar{L} : \bar{x} \wedge \bar{s} = M \text{ for any } \bar{s} \in \bar{S}\} .$$

Let $P_0(M)$ and $\mathscr{P}_0(M)$ be the system of all sets $p_0(S, M)$, or $p_0(\bar{S})$, respectively; the systems $P_0(M)$ and $\mathscr{P}_0(M)$ are partially ordered by the set inclusion.

2. *Let M be a regular ideal of the lattice L and $S \subset L$. Then*

$$x \in p_0(S, M) \Leftrightarrow \bar{x} \in p_0(\bar{S}) .$$

Proof. Let $x \in p_0(S, M)$, $\bar{s} \in \bar{S}$. Then there exists $s_1 \in \bar{s} \cap S$ and for this element $x \wedge s_1 = m \in M$ holds, hence $\bar{x} \wedge \bar{s} = \bar{x} \wedge \bar{s}_1 = \overline{x \wedge s_1} = M$, and therefore $\bar{x} \in \in p_0(\bar{S})$. Conversely, let $\bar{x} \in p_0(\bar{S})$, $s \in S$. Then we have $\bar{s} \in \bar{S}$, hence $\overline{x \wedge s} = \bar{x} \wedge \bar{s} = = M$, thus $x \wedge s \in M$ and $x \in p_0(S, M)$.

3. *Let M be a regular ideal of the lattice L. The mapping*

$$\varphi(p_0(S, M)) = p_0(\bar{S})$$

is an isomorphism of the partially ordered set $P_0(M)$ onto $\mathscr{P}_0(M)$.

Proof. Clearly φ is a mapping from $P_0(M)$ onto $\mathscr{P}_0(M)$. Let $S_1, S_2 \subset L$, $p_0(S_1, M) \subset p_0(S_2, M)$ and let $\bar{x} \in p_0(\bar{S}_1)$. According to 2 we then have $x \in p_0(S_1, M)$, hence $x \in p_0(S_2, M)$ and $\bar{x} \in p_0(\bar{S}_2)$; therefore $p_0(\bar{S}_1) \subset p_0(\bar{S}_2)$. In a similar manner we can prove that from $p_0(\bar{S}_1) \subset p_0(\bar{S}_2)$ it follows $p_0(S_1, M) \subset p_0(S_2, M)$.

Let $x \in p_0(S_1, M)$, $x \notin p_0(S_2, M)$; hence $\bar{x} \in p_0(\bar{S}_1)$. If $\bar{x} \in p_0(\bar{S}_2)$, then according to 2 $x \in p_0(S_2, M)$, which is a contradiction; therefore φ is one-to-one and thus φ is an isomorphism.

4. $\mathscr{P}_0(M)$ *is a complete Boolean algebra.*

This follows from the Theorem 7, [5] and from the fact that $\{p_0(\bar{X}) : \bar{X} \subset \bar{L}\} = \{p_0(p_0(\bar{Y})) : \bar{Y} \subset \bar{L}\}$ (since obviously $p_0(p_0(p_0(\bar{X}))) = p_0(\bar{X})$ for any $\bar{X} \subset \bar{L}$ holds).

From 3 and 4 we obtain:

5. Theorem. *Let M be a regular ideal of a lattice L. Then $P_0(M)$ is a complete Boolean algebra.*

6. *Each ideal of a distributive lattice is regular.*

This is well-known (cf., e.g., [6], Lemma 1 and Remark 3 on the p. 252).

If G is a lattice ordered group, then the lattice $(G; \leqq)$ is distributive (cf. Birkhoff [1]). Let M be a convex l-subgroup of G. Then M^+ is an ideal of the lattice $(G^+; \leqq)$, hence according to 5 and 6 $P_0(M^+)$ is a complete Boolean algebra, and therefore by 1 $P(M)$ is a complete Boolean algebra, too. Hence we have proved the theorem (B).

By studying the structure of lattice ordered groups the concept of a carrier (Jaffard [4]) is very useful. It is defined by means of disjointness as follows: let G be a lattice ordered group, $a \in G^+$; then the carrier $a^\wedge$ of the element a is the set of all elements $b \in G^+$ such that for any $x \in G^+$ the equivalence

$$b \wedge x = 0 \Leftrightarrow a \wedge x = 0$$

is valid.

Obviously the concept of the carrier can be used for elements of any lattice with zero element (cf. [5]) and, analogously as in the case of M-polars, it can be generalized as follows:

Let M be an ideal of the lattice L. For any $a \in L$ let $a^\wedge(M)$ (the M-carrier of a) be the set of all elements $b \in L$ satisfying

$$b \wedge x \in M \Leftrightarrow a \wedge x \in M$$

for each $x \in L$. Let $E(M)$ be the system of all M-carriers of elements of L. Similarly as in the case of carriers (cf. [4]) we define the partial order $\leqq$ in the set $E(M)$ by the rule: $a^\wedge(M) \geqq b^\wedge(M)$ if and only if $a \wedge x \in M$ implies $b \wedge x \in M$ for any $x \in L$.

7. Theorem. *Let M be an ideal of a distributive lattice L. The partially ordered set $E(M)$ is isomorphic to the partially ordered set of all carriers of the factor lattice $\bar{L} = L/\Phi(M)$.*

Proof. Let $x, y \in L$. From 2 it follows (by putting $S = \{y\}$)

$$(3) \qquad x \wedge y \in M \Leftrightarrow \bar{x} \wedge \bar{y} = M\,.$$

The equivalence (3) implies (since M is the least element of $\bar{L}$)

$$\overline{a^\wedge(M)} = \bar{a}^\wedge$$

for each $a \in L$. Hence the function $\varphi : a^\wedge(M) \to \overline{a^\wedge(M)}$ is a mapping of the set $E(M)$ on the set E consisting of all carriers of the lattice $\bar{L}$. Moreover by (3)

$$a^\wedge(M) \leqq b^\wedge(M) \Leftrightarrow \bar{a}^\wedge \leqq \bar{b}^\wedge$$

holds and this shows that φ is an isomorphism.

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